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Research Article

On the Oscillation for Second-Order Half-Linear Neutral Delay Dynamic Equations on Time Scales

Quanxin Zhang and Xia Song

Department of Mathematics, Binzhou University, Shandong 256603, China

Correspondence should be addressed to Quanxin Zhang; 3314744@163.com

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We discuss oscillation criteria for second-order half-linear neutral delay dynamic equations on time scales by using the generalized Riccati transformation and the inequality technique. Under certain conditions, we establish four new oscillation criteria. Our results in this paper are new even for the cases of $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

1. Introduction

In recent years, the research results relevant to oscillation of second-order dynamic equations on time scales are emerging, such as [1–7]. The research results of oscillation for the second-order linear, nonlinear, or half-linear dynamic equations can be found in [8–23]. On the basis of the above work, we will study the oscillatory behavior of all solutions of second-order half-linear neutral delay dynamic equation in this paper, which is given as follows:

$$\left(a(t)\Phi\left(z^{\Delta}(t)\right)\right)^{\Delta} + q(t) f(\Phi(x(\tau(t)))) = 0, \quad t \in \mathbb{T}, \quad t \geq t_0, \tag{1}$$

where $\Phi(s) = |s|^{\gamma-2}s$, $z(t) = x(t) + r(t)x(\tau(t))$, $\gamma > 1$. In this paper, we give the following hypotheses.

(H_1) $\mathbb{T}$ is a time scale (i.e., a nonempty closed subset of the real numbers $\mathbb{R}$) which is unbounded above and for $t_0 \in \mathbb{T}$ with $t_0 > 0$; we define the time scale interval of the form $[t_0, \infty)_{\mathbb{T}}$ by $[t_0, \infty)_{\mathbb{T}} = [t_0, \infty) \cap \mathbb{T}$.

(H_2) $a, r, q : \mathbb{T} \to \mathbb{R}$ are positive rd-continuous functions such that $0 < r(t) < 1$.

(H_3) $\tau : \mathbb{T} \to \mathbb{T}$ is a strictly increasing and differentiable function such that

$$\tau(t) \leq t, \quad \lim_{t\to\infty} \tau(t) = \infty, \quad \tau(\mathbb{T}) = \mathbb{T}. \tag{2}$$

(H_4) $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that, for some positive constant L,

$$\frac{f(x)}{x} \geq L \quad \forall x \neq 0. \tag{3}$$

By a solution of (1), we mean a nontrivial real-valued function x satisfying (1) for $t \in \mathbb{T}$. A solution x of (1) is called oscillatory if it is neither eventually positive nor negative; otherwise, it is called nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory. Our attention is restricted on those solutions of (1) which are not eventually identically zero.

The purpose of this paper is to establish the oscillation criteria of Philos [24] for (1). When

$$\int_{t_0}^{\infty} \left[\frac{1}{a(t)}\right]^{1/(\gamma-1)} \Delta t = \infty, \tag{4}$$

the two famous results of Philos [24] about oscillation of second-order linear differential equations are extended to (1) in this paper. At the same time, when

$$\int_{t_0}^{\infty} \left[\frac{1}{a(t)}\right]^{1/(\gamma-1)} \Delta t < \infty, \tag{5}$$

we obtain two criteria of (1) about that each solution is either oscillatory or converges to zero.

The paper is organized as follows. In Section 2, we present some basic definitions and useful results about the theory of calculus on time scales. In Section 3, we give six lemmas. Section 4 introduces the main results of this paper. We establish four new oscillatory criteria when the condition (4) or (5) holds, respectively, for the solutions of (1).

2. Some Preliminaries

On the time scale $\mathbb{T}$ we define the forward and backward jump operators by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\}. \tag{6}$$

A point $t \in \mathbb{T}$ is said to be left-dense if it satisfies $\rho(t) = t$, right-dense if it satisfies $\sigma(t) = t$, left-scattered if it satisfies $\rho(t) < t$, and right-scattered if it satisfies $\sigma(t) > t$. The graininess μ of the time scale is defined by $\mu(t) = \sigma(t) - t$. For a function $f : \mathbb{T} \to \mathbb{R}$, the (delta) derivative is defined by

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \tag{7}$$

if f is continuous at t and t is right-scattered. If t is right-dense, then the derivative is defined by

$$f^{\Delta}(t) = \lim_{s \to t^+} \frac{f(t) - f(s)}{t - s}, \tag{8}$$

provided this limit exists. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at each right-dense point and if there exists a finite left limit at all left-dense points. Denoted by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ the set of rd-continuous functions on $\mathbb{T}$ and $C^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ the set of differentiable function on $\mathbb{T}$, whose derivative is rd-continuous. The derivative f^{Δ} of f, the shift f^{σ} of f, and the graininess μ are related by the formula

$$f^{\sigma} = f + \mu f^{\Delta} \quad \text{where } f^{\sigma} = f \circ \sigma. \tag{9}$$

We will make use of the following product and quotient rules for the derivative of the product fg and the quotient f/g of two differentiable functions f and g:

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)\, g(t) + f(\sigma(t))\, g^{\Delta}(t) \\ &= f(t)\, g^{\Delta}(t) + f^{\Delta}(t)\, g(\sigma(t)),\end{aligned} \tag{10}$$

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)\, g(t) - f(t)\, g^{\Delta}(t)}{g(t)\, g(\sigma(t))} \quad \text{if } gg^{\sigma} \neq 0. \tag{11}$$

For $b, c \in \mathbb{T}$, the Cauchy integral of f^{Δ} is defined by

$$\int_b^c f^{\Delta}(t)\, \Delta t = f(c) - f(b). \tag{12}$$

The integration by parts formula reads

$$\begin{aligned}&\int_b^c f^{\Delta}(t)\, g(t)\, \Delta t \\ &\quad = f(c)\, g(c) - f(b)\, g(b) - \int_b^c f^{\sigma}(t)\, g^{\Delta}(t)\, \Delta t,\end{aligned} \tag{13}$$

and the infinite integral is defined by

$$\int_b^{\infty} f(s)\, \Delta s = \lim_{t \to \infty} \int_b^t f(s)\, \Delta s. \tag{14}$$

For more details, see [8, 9].

3. Several Lemmas

In this section, we present five lemmas that will be needed in the proofs of our results in Section 4. Lemma 1 is the theorem 1.93 of [8]; Lemma 2 is the simple corollary of theorem 1.90 in [8]; Lemma 3 is the theorem 41 in [25]; and Lemma 4 is the theorem 3 in [26].

Lemma 1. *Assume that $v : \mathbb{T} \to \mathbb{R}$ is strictly increasing and $\widetilde{\mathbb{T}} := v(\mathbb{T})$ is a time scale. Let $w : \widetilde{\mathbb{T}} \to \mathbb{R}$. If $v^{\Delta}(t)$ and $w^{\widetilde{\Delta}}(v(t))$ exist on $\mathbb{T}^k$, where*

$$\mathbb{T}^k = \begin{cases} \mathbb{T} \setminus (\rho(\sup \mathbb{T}), \sup \mathbb{T}], & \text{if } \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if } \sup \mathbb{T} = \infty, \end{cases} \tag{15}$$

then

$$(w \circ v)^{\Delta} = \left(w^{\widetilde{\Delta}} \circ v\right) v^{\Delta}. \tag{16}$$

Lemma 2. *If x is differentiable, then*

$$(x^{\gamma})^{\Delta} = \gamma x^{\Delta} \int_0^1 \left[h x^{\sigma} + (1 - h)\, x\right]^{\gamma - 1} \mathrm{d}h. \tag{17}$$

Lemma 3. *Assume that X and Y are nonnegative real numbers, then*

$$\lambda X Y^{\lambda - 1} - X^{\lambda} \leq (\lambda - 1)\, Y^{\lambda} \quad \forall \lambda > 1, \tag{18}$$

where the equality holds if and only if $X = Y$.

Lemma 4. *Let $a, b \in \mathbb{T}$ with $a < b$. Then for positive rd-continuous functions $f, g : [a, b] \to \mathbb{R}$ we have*

$$\int_a^b |f(s)\, g(s)|\, \Delta s \leq \left(\int_a^b |f(s)|^p \Delta s\right)^{1/p} \left(\int_a^b |g(s)|^q \Delta s\right)^{1/q}, \tag{19}$$

where $p > 1$ and $1/p + 1/q = 1$.

Lemma 5. *Assume that (H_1)–(H_4) and (4) hold. Let $x(t)$ be an eventually positive solution of (1). Then there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that*

$$z^{\Delta}(t) > 0, \quad \left(a(t) \left|z^{\Delta}(t)\right|^{\gamma - 2} z^{\Delta}(t)\right)^{\Delta} < 0. \tag{20}$$

Proof. Suppose that $x(t)$ is an eventually positive solution of (1). There exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that $x(t) > 0$ and $x(\tau(t)) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. From the definition of $z(t)$, we get $z(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$, and at the same time for $t \in [t_1, \infty)_{\mathbb{T}}$, from (1) we get

$$\left(a(t) \left|z^{\Delta}(t)\right|^{\gamma - 2} z^{\Delta}(t)\right)^{\Delta} < 0. \tag{21}$$

Hence, $a(t)|z^{\Delta}(t)|^{\gamma-2}z^{\Delta}(t)$ is decreasing. So, $z^{\Delta}(t)$ is either eventually positive or eventually negative. Therefore, for arbitrary $t \in [t_1, \infty)_{\mathbb{T}}$, we have

$$z^{\Delta}(t) > 0. \tag{22}$$

Otherwise, we assume that (22) is not satisfied, then there exits $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that $z^{\Delta}(t) < 0$ for all $t \in [t_2, \infty)_{\mathbb{T}}$. By (21) we have

$$a(t)\left|z^{\Delta}(t)\right|^{\gamma-2}z^{\Delta}(t) \le a(t_2)\left|z^{\Delta}(t_2)\right|^{\gamma-2}z^{\Delta}(t_2) = -M^{\gamma-1} \tag{23}$$

for $t \in [t_2, \infty)_{\mathbb{T}}$, where $M = [a(t_2)]^{1/(\gamma-1)}|z^{\Delta}(t_2)| > 0$. By (23), we get

$$\left(-z^{\Delta}(t)\right)^{\gamma-1} \ge \frac{M^{\gamma-1}}{a(t)}, \quad t \in [t_2, \infty)_{\mathbb{T}}, \tag{24}$$

that is

$$z^{\Delta}(t) \le -M\left[\frac{1}{a(t)}\right]^{1/(\gamma-1)}, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{25}$$

After integrating the two sides of inequality (25) from t_2 to $t \in [t_2, \infty)_{\mathbb{T}}$, we have

$$z(t) \le z(t_2) - M\int_{t_2}^{t}\left[\frac{1}{a(s)}\right]^{1/(\gamma-1)}\Delta s, \quad t \in [t_2, \infty)_{\mathbb{T}}. \tag{26}$$

Nextly, we find the limits of the two sides of (26) when $t \to \infty$. From (4), we get $\lim_{t\to\infty} z(t) = -\infty$. Therefore, $z(t)$ is eventually negative, which is contradictory to $z(t) > 0$. So the inequality (22) holds. This completes the proof. □

4. Main Results

Firstly, the two famous results of Philos [24] about oscillation of second-order linear differential equations are extended to (1) when condition (4) is satisfied.

Theorem 6. *Assume that (H_1)–(H_4) and (4) hold. Let $H : D_{\mathbb{T}} \equiv \{(t,s) : t \ge s \ge t_0, t, s \in [t_0, \infty)_{\mathbb{T}}\} \to \mathbb{R}$ be rd-continuous function, such that*

$$H(t,t) = 0, \quad t \ge t_0; \qquad H(t,s) > 0, \quad t > s \ge t_0, \quad t, s \in [t_0, \infty)_{\mathbb{T}}, \tag{27}$$

and H has a nonpositive continuous Δ-partial derivative $H^{\Delta_s}(t,s)$ with respect to the second variable and satisfies

$$0 < \inf_{s \ge T_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T_0)}\right] \le \infty, \quad T_0 \in [t_0, \infty)_{\mathbb{T}}, \tag{28}$$

$$-H^{\Delta_s}(t,s) = h(t,s)(H(t,s))^{(\gamma-1)/\gamma}, \quad (t,s) \in D_{\mathbb{T}}, \tag{29}$$

where $h : D_{\mathbb{T}} \to \mathbb{R}$ is a rd-continuous function. If there exist a positive and differentiable function $\delta : \mathbb{T} \to \mathbb{R}$ such that $\delta^{\Delta}(t) \ge 0$ for $t \in [t_0, \infty)_{\mathbb{T}}$, and a real rd-continuous function $\Psi : [t_0, \infty)_{\mathbb{T}} \to \mathbb{R}$ such that

$$\limsup_{t\to\infty}\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{a(\tau(s))}{(\delta(s)\tau^{\Delta}(s))^{\gamma-1}}G_+^{\gamma}(t,s)\Delta s < \infty, \tag{30}$$

$$\int_{T_0}^{\infty}\frac{\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{1/(\gamma-1)}}\left(\frac{\Psi_+(\sigma(s))}{\delta(\sigma(s))}\right)^{\gamma/(\gamma-1)}\Delta s = \infty, \tag{31}$$

$$\begin{aligned}\limsup_{t\to\infty}&\frac{1}{H(t,T)}\\ &\times\int_{T}^{t}\Bigg[LH(t,s)\delta(s)q(s)(1-r(\tau(s)))^{\gamma-1}\\ &\qquad -\frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\tau^{\Delta}(s))^{\gamma-1}}G_+^{\gamma}(t,s)\Bigg]\Delta s\\ &\ge \Psi(T),\end{aligned} \tag{32}$$

where $G(t,s) = \delta^{\Delta}(s)(H(t,s))^{1/\gamma} - \delta(s)h(t,s)$, $G_+(t,s) = \max\{0, G(t,s)\}$, $\Psi_+(t) = \max\{0, \Psi(t)\}$, $T \in [T_0, \infty)_{\mathbb{T}}$, then (1) is oscillatory on $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality we may assume that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. By the definition of $z(t)$, it follows that

$$\begin{aligned}x(t) &= z(t) - r(t)x(\tau(t))\\ &\ge z(t) - r(t)z(\tau(t))\\ &\ge (1 - r(t))z(t), \quad t \in [t_1, \infty)_{\mathbb{T}}.\end{aligned} \tag{33}$$

Since $\lim_{t\to\infty}\tau(t) = \infty$, there exists $T_0 \in [t_0, \infty)_{\mathbb{T}}$, such that $\tau(t) \ge t_1$, for all $t \in [T_0, \infty)_{\mathbb{T}}$. Then for $t \in [T_0, \infty)_{\mathbb{T}}$, we have

$$x(\tau(t)) \ge (1 - r(\tau(t)))z(\tau(t)). \tag{34}$$

By Lemma 5 and (H_3), we obtain that

$$\frac{1}{z\circ\tau} \ge \frac{1}{z\circ\tau^{\sigma}}, \qquad a\left(z^{\Delta}\right)^{\gamma-1} \ge a^{\sigma}\left(z^{\Delta\sigma}\right)^{\gamma-1} \tag{35}$$

on $[T_0, \infty)_{\mathbb{T}}$ (where $(z^{\Delta})^{\sigma}$ is short hand for $z^{\Delta\sigma}$), and

$$z^{\Delta}\circ\tau \ge \frac{(a^{\sigma})^{1/(\gamma-1)}}{(a\circ\tau)^{1/(\gamma-1)}}z^{\Delta\sigma} \tag{36}$$

holds. Moreover, using Lemmas 2 and 5, it follows that

$$\begin{aligned}&\left[(z\circ\tau)^{\gamma-1}\right]^{\Delta}\\ &\quad = (\gamma-1)(z\circ\tau)^{\Delta}\int_0^1\left[h(z\circ\tau^{\sigma}) + (1-h)(z\circ\tau)\right]^{\gamma-2}\mathrm{d}h\\ &\quad \ge (\gamma-1)(z\circ\tau)^{\Delta}\int_0^1\left[h(z\circ\tau) + (1-h)(z\circ\tau)\right]^{\gamma-2}\mathrm{d}h\\ &\quad = (\gamma-1)(z\circ\tau)^{\gamma-2}(z\circ\tau)^{\Delta}.\end{aligned} \tag{37}$$

In Lemma 1, let $v = \tau, w = z$, and $\mathbb{T}$ is unbounded above by (H_1), so $\mathbb{T}^k = \mathbb{T}$, and $\widetilde{\mathbb{T}} = v(\mathbb{T}) = \tau(\mathbb{T}) = \mathbb{T}$ by (H_3); using Lemma 1, we get

$$(z \circ \tau)^{\Delta} = \left(z^{\Delta} \circ \tau\right) \tau^{\Delta}. \tag{38}$$

Thus

$$\left[(z \circ \tau)^{\gamma-1}\right]^{\Delta} \geq (\gamma - 1)(z \circ \tau)^{\gamma-2}\left(z^{\Delta} \circ \tau\right)\tau^{\Delta}. \tag{39}$$

By the above inequality and the first inequality in (35), we obtain that

$$\frac{\left[(z \circ \tau)^{\gamma-1}\right]^{\Delta}}{(z \circ \tau)^{\gamma-1}} \geq \frac{(\gamma - 1)\left(z^{\Delta} \circ \tau\right)\tau^{\Delta}}{z \circ \tau^{\sigma}} \tag{40}$$

holds on $[T_0, \infty)_{\mathbb{T}}$. Now we define the function W by

$$W = \delta \frac{a\left(z^{\Delta}\right)^{\gamma-1}}{(z \circ \tau)^{\gamma-1}}. \tag{41}$$

Then we have $W > 0$ on $[T_0, \infty)_{\mathbb{T}}$, and

$$\begin{aligned}
W^{\Delta} &\overset{(10)}{=} \frac{\delta}{(z \circ \tau)^{\gamma-1}}\left[a\left(z^{\Delta}\right)^{\gamma-1}\right]^{\Delta} \\
&\quad + a^{\sigma}\left(z^{\Delta\sigma}\right)^{\gamma-1}\frac{(z \circ \tau)^{\gamma-1}\delta^{\Delta} - \delta\left[(z \circ \tau)^{\gamma-1}\right]^{\Delta}}{(z \circ \tau)^{\gamma-1}(z \circ \tau^{\sigma})^{\gamma-1}} \\
&\overset{(1)(H_4)}{\leq} -\frac{Lq\delta(x \circ \tau)^{\gamma-1}}{(z \circ \tau)^{\gamma-1}} \\
&\quad + a^{\sigma}\left(z^{\Delta\sigma}\right)^{\gamma-1}\frac{(z \circ \tau)^{\gamma-1}\delta^{\Delta} - \delta\left[(z \circ \tau)^{\gamma-1}\right]^{\Delta}}{(z \circ \tau)^{\gamma-1}(z \circ \tau^{\sigma})^{\gamma-1}} \\
&\overset{(34)}{\leq} -Lq\delta(1 - r \circ \tau)^{\gamma-1} \\
&\quad + a^{\sigma}\left(z^{\Delta\sigma}\right)^{\gamma-1}\frac{(z \circ \tau)^{\gamma-1}\delta^{\Delta} - \delta\left[(z \circ \tau)^{\gamma-1}\right]^{\Delta}}{(z \circ \tau)^{\gamma-1}(z \circ \tau^{\sigma})^{\gamma-1}} \\
&\overset{(41)}{=} -Lq\delta(1 - r \circ \tau)^{\gamma-1} + \frac{\delta^{\Delta}}{\delta^{\sigma}}W^{\sigma} \\
&\quad - \frac{\delta a^{\sigma}\left(z^{\Delta\sigma}\right)^{\gamma-1}\left[(z \circ \tau)^{\gamma-1}\right]^{\Delta}}{(z \circ \tau)^{\gamma-1}(z \circ \tau^{\sigma})^{\gamma-1}} \\
&\overset{(40)}{\leq} -Lq\delta(1 - r \circ \tau)^{\gamma-1} \\
&\quad + \delta^{\Delta}\frac{W^{\sigma}}{\delta^{\sigma}} - \frac{(\gamma - 1)\delta a^{\sigma}\left(z^{\Delta\sigma}\right)^{\gamma-1}\left(z^{\Delta} \circ \tau\right)\tau^{\Delta}}{(z \circ \tau^{\sigma})^{\gamma}} \\
&\overset{(36)}{\leq} -Lq\delta(1 - r \circ \tau)^{\gamma-1} + \delta^{\Delta}\frac{W^{\sigma}}{\delta^{\sigma}} \\
&\quad - \frac{(\gamma - 1)\delta\tau^{\Delta}\left(a^{\sigma}\right)^{\gamma/(\gamma-1)}\left(z^{\Delta\sigma}\right)^{\gamma}}{(a \circ \tau)^{1/(\gamma-1)}(z \circ \tau^{\sigma})^{\gamma}} \\
&\overset{(41)}{\leq} -Lq\delta(1 - r \circ \tau)^{\gamma-1} + \delta^{\Delta}\frac{W^{\sigma}}{\delta^{\sigma}} \\
&\quad - \frac{(\gamma - 1)\delta\tau^{\Delta}}{(a \circ \tau)^{1/(\gamma-1)}(\delta^{\sigma})^{\gamma/(\gamma-1)}}\left(W^{\sigma}\right)^{\gamma/(\gamma-1)}
\end{aligned} \tag{42}$$

and then we obtain

$$\begin{aligned}
W^{\Delta}(t) &\leq -Lq(t)\delta(t)(1 - r(\tau(t)))^{\gamma-1} \\
&\quad + \frac{\delta^{\Delta}(t)}{\delta(\sigma(t))}W(\sigma(t)) \\
&\quad - \frac{(\gamma - 1)\delta(t)\tau^{\Delta}(t)}{(a(\tau(t)))^{\lambda-1}(\delta(\sigma(t)))^{\lambda}}(W(\sigma(t)))^{\lambda}
\end{aligned} \tag{43}$$

on $[T_0, \infty)_{\mathbb{T}}$, where $\lambda = \gamma/(\gamma - 1)$. Thus, for every $t, T \in [T_0, \infty)_{\mathbb{T}}$ with $t \geq T \geq T_0$, by (13), we get

$$\begin{aligned}
&\int_T^t LH(t,s)\delta(s)q(s)(1 - r(\tau(s)))^{\gamma-1}\Delta s \\
&\quad \leq H(t,T)W(T) - \int_T^t \left(-H^{\Delta_s}(t,s)\right)W(\sigma(s))\Delta s \\
&\qquad + \int_T^t H(t,s)\frac{\delta^{\Delta}(s)}{\delta(\sigma(s))}W(\sigma(s))\Delta s \\
&\qquad - \int_T^t H(t,s)\frac{(\gamma - 1)\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}\Delta s \\
&\quad \overset{(28)}{=} H(t,T)W(T) \\
&\qquad + \int_T^t \frac{\delta^{\Delta}(s)H(t,s) - \delta(\sigma(s))h(t,s)H^{1/\lambda}(t,s)}{\delta(\sigma(s))} \\
&\qquad\qquad \times W(\sigma(s))\Delta s \\
&\qquad - \int_T^t H(t,s)\frac{(\gamma - 1)\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}\Delta s \\
&\quad \leq H(t,T)W(T) \\
&\qquad + \int_T^t \frac{\delta^{\Delta}(s)H^{(\lambda-1)/\lambda}(t,s) - \delta(s)h(t,s)}{\delta(\sigma(s))} \\
&\qquad\qquad \times H^{1/\lambda}(t,s)W(\sigma(s))\Delta s \\
&\qquad - \int_T^t H(t,s)\frac{(\gamma - 1)\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}\Delta s \\
&\quad \leq H(t,T)W(T) + \int_T^t \frac{G_+(t,s)}{\delta(\sigma(s))}H^{1/\lambda}(t,s)W(\sigma(s))\Delta s \\
&\qquad - \int_T^t H(t,s)\frac{(\gamma - 1)\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}\Delta s,
\end{aligned} \tag{44}$$

where $G(t,s) = \delta^{\Delta}(s)H^{(\lambda-1)/\lambda}(t,s) - \delta(s)h(t,s) = \delta^{\Delta}(s)H^{1/\gamma}(t,s) - \delta(s)h(t,s), G_{+}(t,s) = \max\{0, G(t,s)\}$. So using Lemma 3, let

$$X = \left[H(t,s)\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}\right]^{1/\lambda} W(\sigma(s)),$$
$$Y = \left[\frac{G_{+}(t,s)}{\lambda\delta(\sigma(s))}\left(\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}\right)^{-1/\lambda}\right]^{1/(\lambda-1)}. \tag{45}$$

Using the inequality (18), we have

$$\begin{aligned}&\frac{G_{+}(t,s)}{\delta(\sigma(s))}H^{1/\lambda}(t,s)\,W(\sigma(s))\\&\quad - H(t,s)\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}\\&\quad\le C\left(\frac{G_{+}(t,s)}{\delta(\sigma(s))}\right)^{\lambda/(\lambda-1)}\left(\frac{\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}\right)^{-1/(\lambda-1)},\end{aligned} \tag{46}$$

where $C = (\lambda-1)\lambda^{-\lambda/(\lambda-1)}(\gamma-1)^{-1/(\lambda-1)} = 1/\gamma^{\gamma}$. Thus

$$\begin{aligned}&\frac{G_{+}(t,s)}{\delta(\sigma(s))}H^{1/\lambda}(t,s)\,W(\sigma(s))\\&\quad - H(t,s)\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}\\&\quad\le \frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}}G_{+}^{\gamma}(t,s).\end{aligned} \tag{47}$$

From (44) and (47), we obtain

$$\begin{aligned}&\int_{T}^{t}\left[LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\right.\\&\qquad\left.-\frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}}G_{+}^{\gamma}(t,s)\right]\Delta s\\&\quad\le H(t,T)\,W(T),\end{aligned} \tag{48}$$

that is,

$$\begin{aligned}&\frac{1}{H(t,T)}\int_{T}^{t}\left[LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\right.\\&\qquad\left.-\frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}}G_{+}^{\gamma}(t,s)\right]\Delta s\le W(T).\end{aligned} \tag{49}$$

From condition (32), we have

$$\begin{aligned}&\Psi(T)\le W(T),\quad T\in[T_0,\infty)_{\mathbb{T}},\\&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s\\&\quad\ge\Psi(T).\end{aligned} \tag{50}$$

By (44), we have

$$\begin{aligned}&\frac{1}{H(t,T)}\int_{T}^{t}LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s\\&\quad\le W(T)+\frac{1}{H(t,T)}\int_{T}^{t}\frac{G_{+}(t,s)}{\delta(\sigma(s))}H^{1/\lambda}(t,s)\,W(\sigma(s))\,\Delta s\\&\qquad-\frac{1}{H(t,T)}\int_{T}^{t}H(t,s)\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}\\&\qquad\times(W(\sigma(s)))^{\lambda}\Delta s.\end{aligned} \tag{51}$$

In the above inequality, take $T = T_0$, and write

$$\begin{aligned}A(t)&=\frac{1}{H(t,T_0)}\int_{T_0}^{t}\frac{G_{+}(t,s)}{\delta(\sigma(s))}H^{1/\lambda}(t,s)\,W(\sigma(s))\,\Delta s,\\B(t)&=\frac{1}{H(t,T_0)}\\&\quad\times\int_{T_0}^{t}H(t,s)\frac{(\gamma-1)\,\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}\Delta s,\end{aligned} \tag{52}$$

and meanwhile noting that (50), we obtain

$$\begin{aligned}&\liminf_{t\to\infty}[B(t)-A(t)]\\&\quad\le W(T_0)-\limsup_{t\to\infty}\frac{1}{H(t,T_0)}\\&\qquad\times\int_{T_0}^{t}LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s\\&\quad\le W(T_0)-\Psi(T_0)<\infty.\end{aligned} \tag{53}$$

Now we assert that

$$\int_{T_0}^{\infty}\frac{\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}\Delta s<\infty \tag{54}$$

holds. Suppose to the contrary that

$$\int_{T_0}^{\infty}\frac{\delta(s)\,\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}\Delta s=\infty, \tag{55}$$

by (28), there exists a constant $\varepsilon > 0$ such that

$$\inf_{s\ge T_0}\left[\liminf_{t\to\infty}\frac{H(t,s)}{H(t,T_0)}\right]>\varepsilon>0. \tag{56}$$

From (55), there exists a $T \in [T_0, \infty)_{\mathbb{T}}$ for arbitrary real number $M > 0$ such that

$$\int_{T_0}^{t} \frac{\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1}(\delta(\sigma(s)))^{\lambda}}(W(\sigma(s)))^{\lambda}\Delta s \geq \frac{M}{(\gamma-1)\varepsilon}, \tag{57}$$

for $t \in [T, \infty)_{\mathbb{T}}$. By (13), we have

$$\begin{aligned} B(t) &= \frac{1}{H(t,T_0)} \\ &\quad \times \int_{T_0}^{t} \Bigg\{ (\gamma-1) H(t,s) \\ &\qquad \times \Bigg(\int_{T_0}^{s} \frac{\delta(u)\tau^{\Delta}(u)}{(a(\tau(u)))^{\lambda-1}(\delta(\sigma(u)))^{\lambda}} \\ &\qquad\qquad \times (W(\sigma(u)))^{\lambda}\Delta u \Bigg)^{\Delta_s} \Bigg\} \Delta s \\ &= \frac{1}{H(t,T_0)} \\ &\quad \times \int_{T_0}^{t} \Bigg\{ \left[-(\gamma-1) H^{\Delta_s}(t,s)\right] \\ &\qquad \times \int_{T_0}^{\sigma(s)} \frac{\delta(u)\tau^{\Delta}(u)}{(a(\tau(u)))^{\lambda-1}(\delta(\sigma(u)))^{\lambda}} \\ &\qquad\qquad \times (W(\sigma(u)))^{\lambda}\Delta u \Bigg\} \Delta s \\ &\geq \frac{1}{H(t,T_0)} \\ &\quad \times \int_{T}^{t} \Bigg\{ \left[-(\gamma-1) H^{\Delta_s}(t,s)\right] \\ &\qquad \times \int_{T_0}^{s} \frac{\delta(u)\tau^{\Delta}(u)}{(a(\tau(u)))^{\lambda-1}(\delta(\sigma(u)))^{\lambda}} \\ &\qquad\qquad \times (W(\sigma(u)))^{\lambda}\Delta u \Bigg\} \Delta s \\ &\geq \frac{1}{H(t,T_0)} \int_{T}^{t} \left[-(\gamma-1) H^{\Delta_s}(t,s)\right] \frac{M}{(\gamma-1)\varepsilon}\Delta s \\ &= \frac{M}{\varepsilon}\frac{H(t,T)}{H(t,T_0)}. \end{aligned} \tag{58}$$

From (56), there exists a $t_2 \in [T, \infty)_{\mathbb{T}}$ such that $H(t,T)/H(t,T_0) \geq \varepsilon$ for $t \in [t_2, \infty)_{\mathbb{T}}$. So $B(t) \geq M$. Since M is arbitrary, we have

$$\lim_{t\to\infty} B(t) = \infty. \tag{59}$$

Selecting a sequence $\{T_n\}_{n=1}^{\infty} : T_n \in [T_0, \infty)_{\mathbb{T}}$ with $\lim_{n\to\infty} T_n = \infty$ satisfying

$$\lim_{n\to\infty} \left[B(T_n) - A(T_n)\right] = \liminf_{t\to\infty} \left[B(t) - A(t)\right] < \infty, \tag{60}$$

and then there exists a constant $M_0 > 0$ such that

$$B(T_n) - A(T_n) \leq M_0 \tag{61}$$

for sufficiently large positive integer n. From (59), we can easily see

$$\lim_{n\to\infty} B(T_n) = \infty, \tag{62}$$

and (61) implies that

$$\lim_{n\to\infty} A(T_n) = \infty. \tag{63}$$

From (61) and (62), we have

$$\frac{A(T_n)}{B(T_n)} - 1 \geq -\frac{M_0}{B(T_n)} > -\frac{M_0}{2M_0} = -\frac{1}{2}, \tag{64}$$

that is,

$$\frac{A(T_n)}{B(T_n)} > \frac{1}{2} \tag{65}$$

for sufficiently large positive integer n, which together with (63) implies

$$\lim_{n\to\infty} \frac{[A(T_n)]^{\gamma}}{[B(T_n)]^{\gamma-1}} = \lim_{n\to\infty} \left[\frac{A(T_n)}{B(T_n)}\right]^{\gamma-1} A(T_n) = \infty. \tag{66}$$

On the other hand, by Lemma 4, we obtain

$$\begin{aligned} A(T_n) &= \frac{1}{H(T_n,T_0)} \int_{T_0}^{T_n} \frac{G_+(T_n,s)}{\delta(\sigma(s))} H^{1/\lambda}(T_n,s) W(\sigma(s)) \Delta s \\ &= \int_{T_0}^{T_n} \Bigg\{ \left[\frac{(\gamma-1)H(T_n,s)\delta(s)\tau^{\Delta}(s)}{H(T_n,T_0)}\right]^{(\gamma-1)/\gamma} \\ &\qquad \times \frac{W(\sigma(s))}{[a(\tau(s))]^{1/\gamma}\delta(\sigma(s))} \Bigg\} \\ &\quad \times \Bigg\{ \frac{[a(\tau(s))]^{1/\gamma} G_+(T_n,s)}{H(T_n,T_0)} H^{(\gamma-1)/\gamma}(T_n,s) \\ &\qquad \times \left[\frac{(\gamma-1)H(T_n,s)\delta(s)\tau^{\Delta}(s)}{H(T_n,T_0)}\right]^{(1-\gamma)/\gamma} \Bigg\} \Delta s \\ &\leq \Bigg\{ \int_{T_0}^{T_n} \frac{(\gamma-1) H(T_n,s)\delta(s)\tau^{\Delta}(s)}{H(T_n,T_0)} \\ &\qquad \times \left[\frac{W(\sigma(s))}{(a(\tau(s)))^{1/\gamma}\delta(\sigma(s))}\right]^{\gamma/(\gamma-1)} \Delta s \Bigg\}^{(\gamma-1)/\gamma} \end{aligned}$$

$$\times \left\{ \int_{T_0}^{T_n} \frac{a(\tau(s)) G_+^{\gamma}(T_n, s)}{H^{\gamma}(T_n, T_0)} H^{\gamma-1}(T_n, s) \times \left[\frac{(\gamma-1) H(T_n, s) \delta(s) \tau^{\Delta}(s)}{H(T_n, T_0)} \right]^{1-\gamma} \Delta s \right\}^{1/\gamma}$$

$$= [B(T_n)]^{(\gamma-1)/\gamma} \left\{ \frac{(\gamma-1)^{1-\gamma}}{H(T_n, T_0)} \times \int_{T_0}^{T_n} a(\tau(s)) G_+^{\gamma}(T_n, s) \times \left[\delta(s) \tau^{\Delta}(s)\right]^{1-\gamma} \Delta s \right\}^{1/\gamma}. \tag{67}$$

The above inequality shows that

$$\frac{[A(T_n)]^{\gamma}}{[B(T_n)]^{\gamma-1}} \leq \frac{(\gamma-1)^{1-\gamma}}{H(T_n, T_0)} \int_{T_0}^{T_n} \frac{a(\tau(s))}{(\delta(s) \tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(T_n, s) \Delta s. \tag{68}$$

Hence, (66) implies

$$\lim_{n \to \infty} \frac{1}{H(T_n, T_0)} \int_{T_0}^{T_n} \frac{a(\tau(s))}{(\delta(s) \tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(T_n, s) \Delta s = \infty, \tag{69}$$

which contradicts (30). Therefore (54) holds. Noting $\Psi(T) \leq W(T)$ for $T \in [T_0, \infty)_{\mathbb{T}}$, by using (54), we obtain

$$\int_{T_0}^{\infty} \frac{\delta(s) \tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1} (\delta(\sigma(s)))^{\lambda}} (\Psi_+(\sigma(s)))^{\lambda} \Delta s \leq \int_{T_0}^{\infty} \frac{\delta(s) \tau^{\Delta}(s)}{(a(\tau(s)))^{\lambda-1} (\delta(\sigma(s)))^{\lambda}} (W(\sigma(s)))^{\lambda} \Delta s < \infty, \tag{70}$$

which is contradicting with (31). This completes the proof. □

Remark 7. From Theorem 6, we can obtain different conditions for oscillation of all solutions of (1) with different choices of $\delta(t)$ and $H(t, s)$. For example, $H(t, s) = (t - s)^m$ or $H(t, s) = (\ln((t+1)/(s+1)))^m$.

Theorem 8. *Assume that* (H_1)–(H_4), (4), (28)-(29), *and* (31) *hold. Suppose that* H, h, δ, *and* Ψ *are defined in Theorem 6. Assume that*

$$\liminf_{t \to \infty} \frac{1}{H(t, T_0)} \int_{T_0}^{t} LH(t, s) \delta(s) q(s) (1 - r(\tau(s)))^{\gamma-1} \Delta s < \infty, \tag{71}$$

$$\liminf_{t \to \infty} \frac{1}{H(t, T)} \times \int_{T}^{t} \left[LH(t, s) \delta(s) q(s) (1 - r(\tau(s)))^{\gamma-1} - \frac{a(\tau(s))}{\gamma^{\gamma} (\delta(s) \tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t, s) \right] \Delta s \geq \Psi(T), \tag{72}$$

where $T \in [T_0, \infty)_{\mathbb{T}}$, $G(t, s) = \delta^{\Delta}(s)(H(t, s))^{1/\gamma} - \delta(s) h(t, s)$, $G_+(t, s) = \max\{0, G(t, s)\}$. *Then* (1) *is oscillatory on* $[t_0, \infty)_{\mathbb{T}}$.

Proof. Assume that (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality we may assume that there exists a $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. So $z(t) > 0$ and there exists a $T_0 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$x(\tau(t)) \geq (1 - r(\tau(t))) z(\tau(t)) \tag{73}$$

for $t \in [T_0, \infty)_{\mathbb{T}}$. Define the function W by

$$W = \delta \frac{a(z^{\Delta})^{\gamma-1}}{(z \circ \tau)^{\gamma-1}}, \quad t \in [T_0, \infty)_{\mathbb{T}}. \tag{74}$$

We proceed as in the proof of Theorem 6 to obtain (44) and (47), so that

$$\frac{1}{H(t, T)} \times \int_{T}^{t} \left[LH(t, s) \delta(s) q(s) (1 - r(\tau(s)))^{\gamma-1} - \frac{a(\tau(s))}{\gamma^{\gamma} (\delta(s) \tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t, s) \right] \Delta s \leq W(T). \tag{75}$$

Hence, (72) implies

$$\Psi(T) \le W(T), \quad T \in [T_0, \infty)_{\mathbb{T}}; \tag{76}$$

$$\begin{aligned} &\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s \\ &\quad \ge \Psi(T); \end{aligned} \tag{77}$$

$$\begin{aligned} &\Psi(T) \\ &\quad \le \liminf_{t\to\infty} \frac{1}{H(t,T)} \\ &\qquad \times \int_T^t \Bigg[LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1} \\ &\qquad\qquad - \frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s) \Bigg] \Delta s \\ &\quad \le \liminf_{t\to\infty} \frac{1}{H(t,T)} \\ &\qquad \times \int_T^t LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s \\ &\qquad - \liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{a(\tau(s))}{\gamma^{\gamma}(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\,\Delta s. \end{aligned} \tag{78}$$

From the above inequality and (71), we have

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_T^t \frac{a(\tau(s))}{(\delta(s)\,\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\,\Delta s < \infty. \tag{79}$$

Therefore, there exists a sequence $\{T_n\}_{n=1}^{\infty} : T_n \in [T_0, \infty)_{\mathbb{T}}$ with $\lim_{n\to\infty} T_n = \infty$ such that

$$\lim_{n\to\infty} \frac{1}{H(T_n,T)} \int_T^{T_n} \frac{a(\tau(s))}{(\delta(s)\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\,\Delta s < \infty. \tag{80}$$

Definitions of $A(t)$ and $B(t)$ are as in Theorem 6. From (44), and noting (77), we have

$$\begin{aligned} &\limsup_{t\to\infty} [B(t) - A(t)] \\ &\quad \le W(T_0) - \liminf_{t\to\infty} \frac{1}{H(t,T_0)} \\ &\qquad \times \int_{T_0}^t LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s \\ &\quad \le W(T_0) - \Psi(T_0) < \infty. \end{aligned} \tag{81}$$

For the above sequence $\{T_n\}_{n=1}^{\infty}$,

$$\lim_{n\to\infty} [B(T_n) - A(T_n)] \le \limsup_{t\to\infty} [B(t) - A(t)] < \infty. \tag{82}$$

We proceed by reduction to absurdity to obtain (54). The rest of the proof is similar to that of Theorem 6 and hence is omitted. This completes the proof. □

If (4) is not satisfied, that is, if the condition (5) holds, we can obtain the following result.

Theorem 9. *Assume that (H_1)–(H_4),* (5)*, and* (28)–(32) *hold. Suppose that H, h, δ, and Ψ are defined in Theorem 6. Assume that*

$$\int_{t_0}^{\infty} \left(\frac{1}{a(t)} \int_{t_0}^t q(s)[1-r(\tau(s))]^{\gamma-1}\Delta s \right)^{1/(\gamma-1)} \Delta t = \infty \tag{83}$$

holds. Then every solution $x(t)$ of (1) *is either oscillatory or converges to zero on $[t_0, \infty)_{\mathbb{T}}$.*

Proof. As the proof of Theorem 6, assume that (1) has a nonoscillatory solution $x(t)$ on $[t_0, \infty)_{\mathbb{T}}$. Without loss of generality we may assume that there exists $t_1 \in [t_0, \infty)_{\mathbb{T}}$, such that $x(t) > 0$ and $x[\tau(t)] > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. So $z(t) > 0$ and there exists $t_2 \in [t_1, \infty)_{\mathbb{T}}$ such that

$$x(\tau(t)) \ge (1 - r(\tau(t)))\, z(\tau(t)) \tag{84}$$

for $t \in [t_2, \infty)_{\mathbb{T}}$. In the proof of Lemma 5, we find that $z^{\Delta}(t)$ is either eventually positive or eventually negative. Thus, we shall distinguish the following two cases:

(I) $z^{\Delta}(t) > 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$;

(II) $z^{\Delta}(t) < 0$ for $t \in [t_2, \infty)_{\mathbb{T}}$.

Case (I). When $z^{\Delta}(t)$ is eventually positive, the proof is similar to that of the proof of Theorem 6, and we can obtain that (1) is oscillatory.

Case (II). When $z^{\Delta}(t)$ is eventually negative, $z(t)$ is decreasing and $\lim_{t\to\infty} z(t) =: b \ge 0$ exists. Therefore, there exists $T_0 \in [t_2, \infty)_{\mathbb{T}}$, such that

$$z(\tau(t)) > z(t) > z(\sigma(t)) \ge b \ge 0, \tag{85}$$

for $t \in [T_0, \infty)_{\mathbb{T}}$. Define the function $u(t) = a(t)|z^{\Delta}(t)|^{\gamma-2} z^{\Delta}(t) = -a(t)|z^{\Delta}(t)|^{\gamma-1}$. Equations (1) and (85) yield

$$\begin{aligned} u^{\Delta}(t) &= -q(t)\, f\left[(x(\tau(t)))^{\gamma-1}\right] \\ &\le -Lb^{\gamma-1} q(t)\,(1-r(\tau(t)))^{\gamma-1}, \quad t \in [T_0, \infty)_{\mathbb{T}}. \end{aligned} \tag{86}$$

The inequality (86) is the assumed inequality of [8, Theorem 6.1] (see also [27, Lemma 1]). All assumptions of [8, Theorem 6.1] are satisfied as well. Hence the conclusion of [8, Theorem 6.1] holds, that is,

$$\begin{aligned} u(t) &\le u(T_0) - Lb^{\gamma-1} \int_{T_0}^t q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s \\ &< -Lb^{\gamma-1} \int_{T_0}^t q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s \end{aligned} \tag{87}$$

for all $t \in [T_0, \infty)_{\mathbb{T}}$, and thus

$$\int_{T_0}^{l} z^{\Delta}(t)\,\Delta t < -bL^{1/(\gamma-1)} \times \int_{T_0}^{l} \left(\frac{1}{a(t)} \int_{T_0}^{t} q(s)(1 - r(\tau(s)))^{\gamma-1} \Delta s \right)^{1/(\gamma-1)} \Delta t \tag{88}$$

for all $l \in [T_0, \infty)_{\mathbb{T}}$. Assuming $b > 0$ and using (83) in (88), we can get $\lim_{l\to\infty} z(l) = -\infty$, and this is a contradiction to the fact that $z(t) > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$. Thus $b = 0$, that is $\lim_{t\to\infty} z(t) = 0$. Then, it follows from $(1 - r(t))z(t) \le x(t) \le z(t)$ that $\lim_{t\to\infty} x(t) = 0$ holds. This completes the proof. □

Using the same method as in the proofs of Theorems 8 and 9, we can easily obtain the following result.

Theorem 10. *Assume that* (H_1)–(H_3), (5), (28)-(29), (31), (71)-(72), *and* (83) *hold. Suppose that* H, h, δ, *and* Ψ *are defined in Theorem 8. Then every solution* $x(t)$ *of* (1) *is either oscillatory or converges to zero on* $[t_0, \infty)_{\mathbb{T}}$.

Remark 11. The theorems in this paper are new even for the cases of $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$.

Example. Consider a second-order half-linear delay 2-difference equation

$$\left[\frac{1}{t^2} \left| z^{\Delta}(t) \right| z^{\Delta}(t) \right]^{\Delta} + \frac{1}{t^3} \left| x\left(\frac{t}{2}\right) \right| x\left(\frac{t}{2}\right) = 0, \quad t \in \overline{2^{\mathbb{Z}}}, \quad t \ge t_0 := 2, \tag{89}$$

where $z(t) = x(t) + (1/2)x(t/2)$. Here, we have

$$a(t) = \frac{1}{t^2}, \qquad r(t) = \frac{1}{2}, \qquad q(t) = \frac{1}{t^3}, \qquad f(u) = u, \qquad \tau(t) = \frac{t}{2}, \qquad \gamma = 3. \tag{90}$$

Then $\mathbb{T} = \overline{2^{\mathbb{Z}}}$ is unbounded above, $\sigma(t) = 2t$, and $\mu(t) = t$. Conditions (H_1)–(H_3) are clearly satisfied, and (H_4) holds with $L = 1$. Next, we have

$$\int_2^t \left[\frac{1}{a(s)} \right]^{1/(\gamma-1)} \Delta s = \int_2^t \left[s^2 \right]^{1/2} \Delta s \longrightarrow \infty \quad \text{as } t \longrightarrow \infty. \tag{91}$$

Hence (4) is satisfied. Now let $H(t,s) = (t-s)^2$, then

$$H^{\Delta_s}(t,s) = \frac{(t-2s)^2 - (t-s)^2}{s} = \frac{(2t-3s)\cdot(-s)}{s} = -(2t-3s) < 0 \quad \forall t > s \ge t_0 := 2. \tag{92}$$

Since

$$-H^{\Delta_s}(t,s) = 2t - 3s = \frac{2t-3s}{(t-s)^{4/3}} \left[(t-s)^2 \right]^{2/3} = \frac{2t-3s}{(t-s)^{4/3}} [H(t,s)]^{(\gamma-1)/\gamma}, \tag{93}$$

let $h(t,s) = (2t-3s)/(t-s)^{4/3}$; then condition (28) holds. We have

$$0 < \inf_{s\ge T_0} \left[\liminf_{t\to\infty} \frac{H(t,s)}{H(t,T_0)} \right] = \inf_{s\ge T_0} \left[\liminf_{t\to\infty} \frac{(t-s)^2}{(t-T_0)^2} \right] = 1 < \infty \quad \forall T_0 \in [t_0, \infty)_{\mathbb{T}} \tag{94}$$

so condition (28) holds. Let $\delta(t) = t$ as $t \ge 2$, then $\delta^{\Delta}(t) = 1$ for all $t \in [t_0, \infty)_{\mathbb{T}}$, and

$$G(t,s) = \delta^{\Delta}(s)(H(t,s))^{1/\gamma} - \delta(s)h(t,s) = H^{1/3}(t,s) - \frac{s(2t-3s)}{H^{2/3}(t,s)} < H^{1/3}(t,s), \tag{95}$$

for all $t > s \ge 2$. Hence

$$\begin{aligned} \int_{T_0}^{t} \frac{a(\tau(s))}{(\delta(s)\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\,\Delta s &< \int_{T_0}^{t} \frac{(s/2)^{-2}}{(s\cdot(1/2))^2} \left(H^{1/3}(t,s) \right)^3 \Delta s = 16 \int_{T_0}^{t} \frac{(t-s)^2}{s^4} \Delta s \\ &= 16\left[-\frac{8}{7t} + \frac{8}{3t} - \frac{2}{t} \right] - 16\left[-\frac{8t^2}{7T_0^3} + \frac{8t}{3T_0^2} - \frac{2}{T_0} \right]. \end{aligned} \tag{96}$$

We get

$$\begin{aligned} \limsup_{t\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} \frac{a(\tau(s))}{(\delta(s)\tau^{\Delta}(s))^{\gamma-1}} G_+^{\gamma}(t,s)\,\Delta s &\le \limsup_{t\to\infty} \left(16\left[-\frac{8}{7t} + \frac{8}{3t} - \frac{2}{t} \right] - 16\left[-\frac{8t^2}{7T_0^3} + \frac{8t}{3T_0^2} - \frac{2}{T_0} \right] \right) \\ &\quad \times \left((t-T_0)^2 \right)^{-1} = \frac{128}{7} \frac{1}{T_0^3} < \infty, \end{aligned} \tag{97}$$

and thus condition (30) holds. Let $\Psi(t) = 1/4t$, then

$$\begin{aligned} \int_{T_0}^{\infty} \frac{\delta(s)\tau^{\Delta}(s)}{(a(\tau(s)))^{1/(\gamma-1)}} \left(\frac{\Psi_+(\sigma(s))}{\delta(\sigma(s))} \right)^{\gamma/(\gamma-1)} \Delta s &= \int_{T_0}^{\infty} \frac{s\cdot(1/2)}{\left((s/2)^{-2}\right)^{1/2}} \left(\frac{1/8s}{2s} \right)^{3/2} \Delta s = \int_{T_0}^{\infty} \frac{s^2}{4} \left(\frac{1}{16s^2} \right)^{3/2} \Delta s \\ &= \frac{1}{256} \int_{T_0}^{\infty} \frac{1}{s} \Delta s = \frac{1}{256} \frac{\ln s}{\ln 2} \Big|_{T_0}^{\infty} = \infty, \end{aligned} \tag{98}$$

that is, condition (31) holds. Since

$$\begin{aligned}
&\int_T^t LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s \\
&\quad= \frac{1}{4}\int_T^t \frac{(t-s)^2\cdot s}{s^3}\Delta s = \frac{1}{4}\int_T^t \left(\frac{t^2}{s^2}-\frac{2t}{s}+1\right)\Delta s \\
&\quad= \frac{1}{4}\left[-\frac{2t^2}{s}-\frac{2t\ln s}{\ln 2}+s\right]_T^t \\
&\quad= \frac{1}{4}\left[-2t-\frac{2t\ln t}{\ln 2}+t\right]-\frac{1}{4}\left[-\frac{2t^2}{T}-\frac{2t\ln T}{\ln 2}+T\right],
\end{aligned} \tag{99}$$

then

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\Delta s \\
&\quad= \frac{1}{2T}.
\end{aligned} \tag{100}$$

Moreover, (97) implies

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \frac{a(\tau(s))}{\gamma^\gamma(\delta(s)\tau^\Delta(s))^{\gamma-1}}G_+^\gamma(t,s)\,\Delta s \le \frac{128}{63}\frac{1}{T^3}. \tag{101}$$

Thus, when T is enough large, we have

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,T)} \\
&\quad\times\int_T^t\left[LH(t,s)\,\delta(s)\,q(s)\,(1-r(\tau(s)))^{\gamma-1}\right. \\
&\qquad\left.-\frac{a(\tau(s))}{\gamma^\gamma(\delta(s)\tau^\Delta(s))^{\gamma-1}}G_+^\gamma(t,s)\right]\Delta s \\
&\quad\ge \frac{1}{2T}-\frac{128}{63}\frac{1}{T^3}\ge\frac{1}{4T}=\Psi(T);
\end{aligned} \tag{102}$$

so (32) is satisfied. By Theorem 6, (89) is oscillatory on $[t_0,\infty)_{\mathbb{T}}$. Similarly, conditions (71) and (72) are satisfied as well. By Theorem 8, we can also obtain that (89) is oscillatory. But the other known results cannot be applied in (89).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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