Research Article

Two New Types of Fixed Point Theorems in Complete Metric Spaces

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Received 12 May 2014; Accepted 13 June 2014; Published 26 June 2014

Academic Editor: Abdul Latif

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We introduce two new types of fixed point theorems in the collection of multivalued and single-valued mappings in complete metric spaces.

1. Introduction

Let \( T \) be a mapping on a complete (or compact) metric space \((X, d)\). We do not assume richer structure such as convex metric spaces and Banach spaces. There are thousands of theorems which assure the existence of a fixed point of \( T \). We can categorize these theorems into the following four types.

(T1) Leader type [1]: \( T \) has a unique fixed point and \( \{T^n x \} \) converges to the fixed point for all \( x \in X \). Such a mapping is called a Picard operator in [2].

(T2) Unnamed type: \( T \) has a unique fixed point and \( \{T^n x \} \) does not necessarily converge to the fixed point.

(T3) Subrahmanyam type [3]: \( T \) may have more than one fixed point and \( \{T^n x \} \) converges to a fixed point for all \( x \in X \). Such a mapping is called a weakly Picard operator in [3, 4].

(T4) Caristi type [5, 6]: \( T \) may have more than one fixed point and \( \{T^n x \} \) does not necessarily converge to a fixed point.

We know that most of the theorems such as Banach’s [7], Cirić’s [8], Kannan’s [9], Kirk’s [10], Matkowski’s [11], Meir and Keeler’s [12], and Suzuki’s [13, 14] belong to (T1). Also, very recently, Suzuki [15] characterized (T1). Subrahmanyam’s theorem [3] belongs to (T3), and Caristi’s theorem [5, 6] and its generalizations [15–17] belong to (T4).

On the other hand, as far as the authors do know, there are no theorems belonging to (T2); see Kirk’s survey [18]. Also, recently many interesting fixed point theorems are proved in the framework of ordered metric spaces; see [18–35] and others.

In this paper, motivated by the above, we introduce two new types of fixed point theorems in the collection of multivalued and single-valued mappings and will prove them, which belong to (T3).

Let \( (X, d) \) be a metric space, and let \( P_{cl, bd}(X) \) denote the class of all nonempty, closed, and bounded subsets of \( X \). Let \( T : X \rightarrow P_{cl, bd}(X) \) be a multivalued mapping on \( X \). A point \( x \in X \) is called a fixed point of \( T \) if \( x \in Tx \). Set \( \text{Fix}(T) = \{x \in X : x \in Tx\} \).

A famous theorem on multivalued mappings is due to Nadler [36], which extended the Banach contraction principle to multivalued mappings. Many authors have studied the existence and uniqueness of strict fixed points for multivalued mappings in metric spaces; see, for example, [37–44] and references therein.

Let \( H \) be the Hausdorff metric on \( P_{cl, bd}(X) \) induced by \( d \); that is,

\[
H(A, B) := \max \left\{ \sup_{x \in B} d(x, A), \sup_{x \in A} d(x, B) \right\},
\]

where \( A, B \in P_{cl, bd}(X) \).
Denote $\delta(x, A) = \sup\{d(x, y) : y \in A\}$ and $D(x, A) = \inf\{d(x, y) : y \in A\}$, where $A \in P_{\text{cl}}(X)$.

2. Main Results

The following is the first our main results.

Theorem 1. Let $(X, d)$ be a complete metric space and let $T$ be a mapping from $X$ into itself. Suppose that $T$ satisfies the following condition:

$$d(Tx, Ty) \leq \left( \frac{d(x, Tz) + d(z, Ty)}{d(x, Ty) + d(y, Tz)} + 1 \right) d(x, y),$$

for all $x, y \in X$. Then

(a) $T$ has at least one fixed point $x \in X$;
(b) $\{T^n x\}$ converges to a fixed point, for all $x \in X$;
(c) if $x, y$ are two distinct fixed points of $T$, then $d(x, y) \geq 1/2$.

Proof. Let $x_0 \in X$ be arbitrary and choose a sequence $\{x_n\}$ such that $x_{n+1} = Tx_n$. We have

$$d(x_{n+1}, x_n) = d(Tx_n, Tx_{n-1})$$

$$\leq \left( d(x_n, x_n) + d(x_{n-1}, x_{n+1}) \right)$$

$$= \left( d(x_n, x_n) + d(x_{n-1}, x_{n+1}) \right)$$

$$\leq \left( d(x_n, x_n) + d(x_{n-1}, x_{n+1}) \right)$$

$$d(x_{n-1}, x_n) + d(x_{n-1}, x_n).$$

(3)

Given

$$\beta_n = \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_{n-1}, x_n) + 1},$$

we have

$$d(x_{n+1}, x_n) \leq \beta_n d(x_{n-1}, x_n)$$

$$\leq (\beta_n \beta_{n-1}) d(x_{n-1}, x_{n-2})$$

$$\vdots$$

$$\leq (\beta_n \beta_{n-1} \cdots \beta_1) d(x_1, x_0).$$

(5)

Observe that $(\beta_n)$ is nonincreasing, with positive terms. So $\beta_1 \cdots \beta_n \leq \beta_n^2$ and $\beta_n^2 \rightarrow 0$. It follows that

$$\lim_{n \to \infty} (\beta_1 \beta_2 \cdots \beta_n) = 0.$$

(6)

Thus, it is verified that

$$\lim_{n \to \infty} d(x_{n+1}, x_n) = 0.$$

(7)

Now for all $m, n \in \mathbb{N}$ we have

$$d(x_m, x_n) \leq d(x_m, x_{n+1}) + d(x_{n+1}, x_{n+2})$$

$$+ \cdots + d(x_{m-1}, x_m)$$

$$\leq \left( (\beta_n \beta_{n-1} \cdots \beta_1) + (\beta_{n+1} \beta_n \cdots \beta_1) \right) d(x_1, x_0).$$

(8)

Suppose that $\alpha_k = (\beta_k \beta_{k-1} \cdots \beta_1)$. Since

$$\lim_{k \to \infty} \frac{\alpha_{k+1}}{\alpha_k} = 0$$

$$\sum_{k=1}^{\infty} \alpha_k < \infty.$$ It means that

$$\sum_{k=n}^{m-1} (\beta_k \beta_{k-1} \cdots \beta_1) \rightarrow 0,$$

as $m, n \to \infty$. In other words, $\{x_n\}$ is a Cauchy sequence and so converges to $\hat{x} \in X$.

We claim that $\hat{x}$ is a fixed point.

Note that

$$d(x_{n+1}, T\hat{x}) \leq \left( \frac{d(x_n, T\hat{x}) + d(\hat{x}, T\hat{x})}{d(x_n, T\hat{x}) + d(x_n, T\hat{x}) + 1} \right) d(x_n, \hat{x}).$$

(11)

On taking limit on both sides of (11), we have $d(\hat{x}, T\hat{x}) = 0$. Thus, $T\hat{x} = \hat{x}$.

If there exist two distinct fixed points $\hat{x}, \hat{y} \in X$, then

$$d(\hat{x}, \hat{y}) = d(T\hat{x}, T\hat{y})$$

$$\leq \left[ d(\hat{x}, T\hat{y}) + d(T\hat{x}, \hat{y}) \right] d(\hat{x}, \hat{y})$$

$$= 2[d(\hat{x}, \hat{y})]^2.$$ (12)

Therefore, $d(\hat{x}, \hat{y}) \geq 1/2$ and we find the desired results.

Example 2. Let $X = \{0, 1/2, 1\}$ and let $d : X \times X \to [0, \infty)$ be defined by

$$d\left(0, \frac{1}{2}\right) = 2, \quad d\left(1, \frac{1}{2}\right) = \frac{5}{2}, \quad d(0, 1) = 3,$$

$$d(0, 0) = d\left(\frac{1}{2}, \frac{1}{2}\right) = d(1, 1) = 0,$$

$$d(a, b) = d(b, a), \quad \forall a, b \in X.$$

(13)
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$$(X, d)$$ is a complete metric space. Let $T : X \to X$ be defined by

\[
T(0) = 0, \quad T\left(\frac{1}{2}\right) = \frac{1}{2}, \quad T(1) = 0
\]

\[
d(T(0), T(1)) = d(0, 0) = 0,
\]

\[
d\left(T(0), T\left(\frac{1}{2}\right)\right) = d\left(0, \frac{1}{2}\right) = 2,
\]

\[
d\left(T(1), T\left(\frac{1}{2}\right)\right) = d\left(0, \frac{1}{2}\right) = 2,
\]

and we have

\[
d\left(T(0), T\left(\frac{1}{2}\right)\right) = d\left(0, \frac{1}{2}\right) = 2
\]

\[
\leq \left(\frac{d(0, T(1/2)) + d(1/2, T(0))}{d(0, T(0)) + d(1/2, T(1/2)) + 1}\right) \times d\left(0, \frac{1}{2}\right) = 8
\]

(14)

and also

\[
d\left(T(1), T\left(\frac{1}{2}\right)\right) = d\left(0, \frac{1}{2}\right) = 2
\]

\[
\leq \left(\frac{d(1, T(1/2)) + d(1/2, T(1))}{d(1, T(1)) + d(1/2, T(1/2)) + 1}\right) \times d\left(1, \frac{1}{2}\right) = \left(\frac{5/2 + 2}{4}\right) \times \frac{5}{2} = \frac{45}{16}.
\]

(15)

Therefore, $T$ satisfies all the conditions of Theorem 1. Also, $T$ has two distinct fixed points $\{0, 1/2\}$ and $d(0, 1/2) = 2 \geq 1/2$.

Example 3. Let $X = [0, 2 - \sqrt{3}]$ be endowed with Euclidean metric and let $T : X \to X$ be defined by

\[
T(x) = \begin{cases} 
0 & 0 \leq x < 2 - \sqrt{3} \\
2 - \sqrt{3} & x = 2 - \sqrt{3}.
\end{cases}
\]

(17)

Then we claim that $T$ satisfies all the conditions of Theorem 1. If $x = 2 - \sqrt{3}$ and $0 \leq y < 2 - \sqrt{3}$, we have

\[
\begin{align*}
|T(x) - T(y)| & = (|x - T(x)| + |y - T(y)| + 1) \\
& = (2 - \sqrt{3})(|y| + 1) = (2 - \sqrt{3})(y + 1) \\
& \leq (2 - \sqrt{3} - y)^2 - (2 - \sqrt{3})(2 - \sqrt{3} - y) \\
& = (|x - T(y)| + |y - T(x)|) |x - y|.
\end{align*}
\]

(18)

Thus,

\[
|T(x) - T(y)| \leq \left(\frac{|x - T(y)| + |y - T(x)|}{|x - T(x)| + |y - T(y)| + 1}\right) |x - y|.
\]

(19)

Similar argument holds for the other conditions.

Remark 4. Note that in (2) the ratio

\[
\frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1}
\]

might be greater or less than 1 and has not introduced an upper bound. Note that if, for every $x, y \in X$, $d(x, y) < 1/2$, then we have

\[
d(x, Ty) + d(y, Tx) \\
\leq 2d(x, y) + d(x, Tx) + d(y, Ty)
\]

(20)

\[
< d(x, Tx) + d(y, Ty) + 1.
\]

(21)

It means that

\[
\frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} < 1,
\]

(22)

and thus Theorem 1 is a special case of Banach contraction principle. Therefore, when $(X, d)$ is a complete metric space such that, for all $x, y \in X$, $d(x, y) \geq 1/2$, Theorem 1 is valuable because (20) might be greater than 1. Example 2 shows this note precisely.

The following is the second in our main results.

Theorem 5. Let $(X, d)$ be a complete metric space and let $T$ be a multivalued mapping from $X$ into $P_{cl, bd}(X)$. Let $T$ satisfy the following:

\[
H(T_x, T_y) \leq \left(\frac{D(x, Ty) + D(y, Tx)}{\delta(x, Tx) + \delta(y, Ty) + 1}\right) d(x, y),
\]

(23)

for all $x, y \in X$. Then $T$ has a fixed point $\hat{x} \in X$.

Proof. Let $x_0 \in X$ and $x_1 \in T_x$. For each $0 < h_1 < 1$ one can choose $x_2 \in T_{x_1}$ such that

\[
d(x_1, x_2) < H(T_{x_0}, T_{x_1}) + \left(1 - \frac{1}{h_1}\right) H(T_{x_0}, T_{x_1})
\]

(24)

\[
= \frac{1}{h_1} H(T_{x_0}, T_{x_1}).
\]

(25)

For each $0 < h_n < 1$ we can choose $x_{n+1} \in T_{x_n}$ such that

\[
d(x_n, x_{n+1}) < H(T_{x_{n-1}}, T_{x_n}) + \left(1 - \frac{1}{h_n}\right) H(T_{x_0}, T_{x_1})
\]

\[
= \frac{1}{h_n} H(T_{x_0}, T_{x_1}).
\]

Specifically if

\[
h_n = \sqrt{\frac{d(x_{n-1} + x_{n+1})}{d(x_{n-1} + x_n) + d(x_n + x_{n+1}) + 1}}
\]

(26)
then
\[ d(x_n, x_{n+1}) \leq \sqrt{\beta_n d(x_{n-1}, x_n)} \leq \beta_n d(x_{n-1}, x_n). \]

Therefore,
\[ d(x_{n+1}, x_n) \leq \beta_n d(x_{n-1}, x_n) \]
\[ \vdots \]
\[ \leq (\beta_n \beta_{n-1} \cdots \beta_1) d(x_1, x_0). \]

It can easily be seen that
\[ \lim_{n \to \infty} \beta_n = 0. \]

Thus, it is easily verified that
\[ d(x_{n+1}, x_n) \rightarrow 0. \]

Now for all \( m, n \in \mathbb{N} \) we have
\[ d(x_m, x_n) \leq d(x_m, x_{m-1}) + d(x_{m-1}, x_{m-2}) + \cdots + d(x_1, x_0) \]
\[ \leq (\beta_m \beta_{m-1} \cdots \beta_1) d(x_1, x_0) \]
\[ \leq \sum_{k=0}^{m-1} (\beta_k \beta_{k-1} \cdots \beta_1) d(x_1, x_0). \]

Suppose that \( a_k = (\beta_k \beta_{k-1} \cdots \beta_1). \) Since
\[ \lim_{k \to \infty} \frac{a_{k+1}}{a_k} = 0, \]
\[ \sum_{k=1}^{\infty} a_k < \infty. \]

It means that
\[ \sum_{k=n}^{m-1} (\beta_k \beta_{k-1} \cdots \beta_1) \rightarrow 0, \]
as \( m, n \to \infty. \) In other words, \( \{x_n\} \) is a Cauchy sequence and so converges to \( \hat{x} \in X. \) We claim that \( \hat{x} \) is a fixed point. Consider
\[ D(\hat{x}, T \hat{x}) \leq d(\hat{x}, x_{n+1}) + D(x_{n+1}, T \hat{x}) \]
\[ \leq H(Tx_n, T \hat{x}) + d(\hat{x}, x_{n+1}) \]
\[ \leq \left( \frac{D(\hat{x}, x_{n+1}) + D(x_{n+1}, T \hat{x})}{\delta(\hat{x}, T \hat{x}) + \delta(x_{n+1}, T x_{n+1}) + 1} \right) \times d(x_{n+1}, \hat{x}) + d(\hat{x}, x_{n+1}) \]
\[ \leq [D(\hat{x}, x_{n+1}) + D(x_{n+1}, T \hat{x})] \]
\[ \times d(x_{n+1}, \hat{x}) + d(\hat{x}, x_{n+1}). \]

On taking limit on both sides of (31) we have \( D(\hat{x}, T \hat{x}) = 0. \)

It means that \( \hat{x} \in T \hat{x}. \)

\[ \square \]

**Remark 6.** Note that Theorem 5 is a generalization of Theorem 1 because by taking \( Fx = \{Tx\} \) and applying Theorem 5 for \( F \) we obtain Theorem 1.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**References**


