Research Article

Existence, Uniqueness, and Stability Analysis of Impulsive Neural Networks with Mixed Time Delays

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1. Introduction

As we know, time delay in a system is a common phenomenon that describes the fact that the future state of the system depends not only on the present state but also on the past state and is always unavoidably encountered in many fields such as automatic control, biological chemistry, physical engineer, and neural networks [1–5]. Moreover, time delays can affect the stability of a neural network and create oscillatory and bad dynamic performance [3–5]. Hence, it is significant and necessary to take into account the delay effects on dynamics of neural networks, for example, existence, uniqueness and stability, and so on. To date, neural network models with two categories of time delays, namely, discrete and continuously distributed time delays, have been extensively investigated by many researchers, using some effective approaches; see [6–22] and references therein. For instance, in [6], Kharitonov and Zhabko studied the robust stability of time-delay systems via Lyapunov-Krasovskii functional approach. Wu et al. [9] introduced free-weighting matrix approach and investigated the robust stability problem for time-varying delay systems. Gu introduced the delay decomposition method in [10]. Recently, a special type of time delay, namely, leakage delay (or forgetting delay), is identified and investigated due to its existence in many real systems. In 2007, Gopalsamy [11] proposed the bidirectional associative memory neural networks with constant delay in the leakage term and derived sufficient conditions for existence and stability of equilibrium. Based on this work, Li and Huang [12] investigated the stability of general nonlinear systems with leakage delay, by model transformation, contraction mapping theorem, and degenerate Lyapunov-Krasovskii functional. However, dynamical analysis of neural networks with time delay in leakage term has been little considered due to some theoretical and technical difficulties [23–30]. In fact, time delay in the stabilizing negative feedback term has a tendency to destabilize a system [11] and has great impact on the dynamics of neural networks.

On the other hand, besides delay, impulses are also likely to exist in neural networks. In implementation of electronic networks, the state is subject to instantaneous perturbations and experiences abrupt change at certain moments, which may be caused by switching phenomenon, frequency change, or other sudden noises; that is, it does exhibit impulsive
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effects; see [13–21, 31–35]. Therefore, impulsive perturbations should be taken into account when studying the dynamics of neural networks. Since the existence of delays and impulses is frequently a source of instability, bifurcation, and chaos for dynamical systems, it is significant to study both delay and impulsive effects on dynamical systems [13–21, 25, 26, 32]. In [25], Li et al. investigated the existence, uniqueness, and stability problems of recurrent neural networks with discrete time delay and time delay in the leakage term under impulsive perturbations, while being without distributed delay. Since neural networks usually have a spatial extent, there is a distribution of propagation delays over a period of time. In these circumstances the signal propagation is not instantaneous and cannot be modelled only with discrete delays and a more appropriate way is to incorporate continuously simultaneous and cannot be modelled only with discrete delays these circumstances the signal propagation is not instantaneous, while being without distributed delay. Since time delay and time delay in the leakage term under impulsive effects on dynamical systems [13–21, 25, 26, 32]. Moreover, the proposed stability criteria are given in terms of linear matrix inequalities (LMI) [36] and can be conveniently checked by the LMI toolbox in MATLAB. Finally, an example is given to show the effectiveness and less conservativeness of the obtained results.

Notations. Let $\mathbb{R}$ ($\mathbb{R}^n$) denote the set of (positive) real numbers, $\mathbb{Z}_+$ denote the set of positive integers, and $\mathbb{R}^n$ denote the $n$-dimensional real spaces equipped with the Euclidean norm $\| \cdot \|$. $\mathbb{S}^d > 0$ or $\mathbb{S}^d < 0$ denotes that the matrix $\mathbb{S}^d$ is a symmetric and positive definite or negative definite matrix. The notations $\mathbb{S}^d$ and $\mathbb{S}^{-1}$ mean the transpose of $\mathbb{S}^d$ and the inverse of a square matrix. $\lambda_{\max}(\mathbb{S}^d)$ or $\lambda_{\min}(\mathbb{S}^d)$ denotes the maximum eigenvalue or the minimum eigenvalue of matrix $\mathbb{S}^d$. $I$ denotes the identity matrix with appropriate dimensions and $\Lambda = \{1, 2, \ldots, n\}$. $[\cdot]^*$ denotes the integer function. For any interval $J \subseteq \mathbb{R}$, set $V \subseteq \mathbb{R}^k$ ($1 \leq k \leq n$), $C(J, V) = \{ \varphi : J \rightarrow V$ is continuous}, and $PC^1(J, V) = \{ \varphi : J \rightarrow V$ is continuously differentiable everywhere except at finite number of points $t$, at which $\varphi(t^+), \varphi(t^-), \varphi'(t)$, and $\varphi'(t^-)$ exist and $\varphi(t^-) = \varphi(t), \varphi'(t^+)$ denote the derivative of $\varphi$. For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $[x]^T = (|x_1|, |x_2|, \ldots, |x_n|)^T$ and, for any $Q = (q_{ij})_{n \times n} \in \mathbb{R}^{n \times n}$, $[Q]^T = (q_{ij})_{n \times n}$. For any $t \in \mathbb{R}^+$, $x_1$ is defined by $x_1 = x(t+s)$, $x_n = x(t+s), s \in [-\sigma, 0]$. In addition, the notation $\ast$ always denotes the symmetric block in one symmetric matrix.

2. Preliminaries

Consider the following impulsive neural network model:

$$\dot{x}(t) = -Dx(t - \sigma) + Af(x(t)) + Bg(x(t - \tau(t))) + W \int_{t-\rho(t)}^t K(t-s)h(x(s))\,ds + I, \quad t > 0, t \neq t_k,$$

$$\Delta x(t_k) = x(t_k^+) - x(t_k^-) = I_k(x(t_k^-, t_k^-), k \in \mathbb{Z}_+,$$

where $x(t) = (x_1(t), \ldots, x_n(t))^T$ is the neuron state vector of the neural network; $D = \text{diag}(d_1, \ldots, d_n)$ is a diagonal matrix with $d_i > 0, i \in \Lambda, \Lambda = \{1, 2, \ldots, n\}; A, B$, and $W$ are the connection weight matrix, the delayed weight matrix, and the distributively delayed connection weight matrix, respectively; $I$ is an external input; $\sigma \geq 0$ is a constant which denotes the leakage delay; $\tau(t)$ is a time-varying transmission delay of the neural network; $\rho(t)$ is a time-varying distributed delay of the neural network; $f(x(\cdot)) = (f_1(x_1(\cdot)), \ldots, f_n(x_n(\cdot)))^T$, $g(x(\cdot)) = (g_1(x_1(\cdot)), \ldots, g_n(x_n(\cdot)))^T$, and $h(x(\cdot)) = (h_1(x_1(\cdot)), \ldots, h_n(x_n(\cdot)))^T$ represent the neuron activation functions; $K(\cdot) = \text{diag}(k_1(\cdot), \ldots, k_n(\cdot))$ is the delay kernel function.

Throughout this paper, we make the following assumptions.

$$(H_1) \quad \tau(t) \text{ represents the discrete transmission delay with } 0 \leq \tau(t) \leq \tau; \rho(t) \text{ represents the time-varying distributed delay with } 0 \leq \rho(t) \leq \rho, \text{ where } \tau, \rho \text{ are two positive constants.}$$

$$(H_2) \text{ The delay kernels } k_j(\cdot), j \in \Lambda, \text{ are some real valued continuous functions defined on } [0, \rho] \text{ and satisfy}$$

$$\left| k_j(s) \right| \leq |k(s)|, \quad \int_0^\rho |k(s)|\,ds = \kappa, \quad (2)$$

where $\kappa$ is a positive constant.

$$(H_3) \text{ The neuron activation functions } f_j, g_j, \text{ and } h_j, j \in \Lambda, \text{ are continuous on } \mathbb{R} \text{ and satisfy}$$

$$\sigma_j \leq \frac{f_j(u) - f_j(v)}{u - v} \leq \sigma_j^*.$$

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\[
\delta_j \leq \frac{g_j(u) - g_j(v)}{u - v} \leq \delta^*_j, \\
\zeta_j \leq \frac{h_j(u) - h_j(v)}{u - v} \leq \zeta^*_j,
\]

(3)

for any \(u, v \in \mathbb{R}, u \neq v, j \in \Lambda\), where \(s_j, s_j^*, \delta_j, \delta^*_j, \zeta_j, \zeta^*_j\) are some real constants and they may be positive, zero, or negative.

\((H_1)\) \(I_k(\cdot) : \mathbb{R} \times \mathbb{R}^n \rightarrow \mathbb{R}^n, k \in \mathbb{Z}_+,\) are some continuous functions.

\((H_2)\) The impulse times \(t_k\) satisfy \(0 = t_0 < t_1 < \cdots < t_k \rightarrow \infty\) and \(\inf_{k \in \mathbb{Z}_+} |t_k - t_{k-1}| > 0\).

We will consider model (1) with the initial condition

\[
x(s) = \varphi(s), \quad s \in [-\eta, 0],
\]

where \(\eta = \max(\sigma, \tau, \rho)\), and \(\varphi(\cdot) \in PC^1\([-\eta, 0]\), \(\mathbb{R}^n\), whose norm is defined by

\[
\|\varphi\|^2 = \max \left\{ \max_{\eta \leq \theta \leq 0} \sum_{i=1}^{n} |\varphi_i(\theta)|, \max_{-\eta \leq \theta \leq 0} \sum_{i=1}^{n} |\varphi_i(\theta)| \right\}.
\]

(5)

Definition 1 (see [37]). Assume that \(\Omega \subset \mathbb{R}^n\) is a bounded and open set and \(\mathcal{F}(u) : \Omega \rightarrow \mathbb{R}^n\) is a continuous and differentiable function. If \(p \in \mathcal{F}(\partial\Omega)\) and \(I_\varphi(u) \neq 0\) for any \(u \in \mathcal{F}^{-1}(p)\), where \(\partial\Omega\) denotes the boundary of \(\Omega\) and \(I_\varphi\) denotes the Jacobian determinant relative to \(\mathcal{F}\), then the topological degree relative to \(\Omega\) and \(p\) is defined by

\[
\deg(\mathcal{F}, \Omega, p) = \begin{cases} 
\sum_{u \in \mathcal{F}^{-1}(p)} \text{sgn}(I_\varphi(u)), & \mathcal{F}^{-1}(p) \neq \emptyset, \\
0, & \mathcal{F}^{-1}(p) = \emptyset.
\end{cases}
\]

(6)

Remark 2. Generally speaking, the topological degree of \(\mathcal{F}(u)\) relative to \(\Omega\) and \(p\) can be regarded as the algebraic number of solutions of \(\mathcal{F}(u) = p\) in \(\Omega\) if \(\mathcal{F}(\partial\Omega) \neq 0\). For instance, \(\deg(\mathcal{F}, \Omega, 0) = \pm 1\) implies that \(\mathcal{F}(u) = 0\) has at least one solution in \(\Omega\).

Lemma 3 (see [38]). Given any real matrix \(M = M^T > 0\) of appropriate dimension and a vector function \(\omega(\cdot) : [a, b] \rightarrow \mathbb{R}^n\), such that the integrations concerned are well defined, then

\[
\left[ \int_a^b \omega(s) ds \right]^T M \left[ \int_a^b \omega(s) ds \right] \leq (b - a) \int_a^b \omega^T(s) M \omega(s) ds.
\]

(7)

Lemma 4 (see [12]). Given any real matrices \(\Sigma_1, \Sigma_2, \Sigma_3\) of appropriate dimensions and a scalar \(\epsilon > 0\) such that \(0 < \Sigma_3 = \Sigma_1\), then the following inequality holds:

\[
\Sigma_1^T \Sigma_2 + \Sigma_2^T \Sigma_1 \leq \epsilon \Sigma_1^T \Sigma_1 \Sigma_1 + \epsilon^{-1} \Sigma_2^T \Sigma_2 \Sigma_2.
\]

(8)

Lemma 5 (see [27]). Supposing that \(\mathcal{U}, \mathcal{U}_i \geq 0 \quad (i, j = 1, 2)\) are symmetric matrices of appropriate dimensions, \(\alpha \in [0, 1]\) and \(\beta \in [0, 1]\), then \(\mathcal{U} + [(1 - \alpha)\mathcal{U}_{11} + \alpha\mathcal{U}_{12}] + [(1 - \beta)\mathcal{U}_{21} + \beta\mathcal{U}_{22}] < 0\) holds if the following four inequalities \(\mathcal{U} + \mathcal{U}_{11} > 0, \mathcal{U} + \mathcal{U}_{11} + \mathcal{U}_{22} < 0, \mathcal{U} + \mathcal{U}_{11} + \mathcal{U}_{21} < 0, \mathcal{U} + \mathcal{U}_{12} + \mathcal{U}_{21} < 0, \mathcal{U} + \mathcal{U}_{12} + \mathcal{U}_{22} < 0\) hold simultaneously.

For presentation convenience, in the following, we denote

\[
\begin{align*}
\Sigma_1 &= \text{diag}(\sigma_1^* \sigma_2^* \ldots \sigma_n^*) , \\
\Sigma_2 &= \text{diag}\left(\frac{\sigma_1^* + \sigma_2^*}{2}, \ldots, \frac{\sigma_n^* + \sigma_n^*}{2}\right) , \\
\Sigma_3 &= \text{diag}(\delta_1^* \delta_2^* \ldots \delta_n^* , \delta_1^* \delta_2^* \ldots \delta_n^* ) , \\
\Sigma_4 &= \text{diag}(\zeta_1^* \zeta_2^* \ldots \zeta_n^* , \zeta_1^* \zeta_2^* \ldots \zeta_n^* ) , \\
\Sigma_5 &= \text{diag}(\xi_1^* \zeta_2^* \ldots \zeta_n^* , \zeta_1^* \zeta_2^* \ldots \zeta_n^* ) .
\end{align*}
\]

3. Global Existence and Uniqueness of Solution

In this section, by using the contraction mapping theorem, we give a delay-independent sufficient condition to guarantee the global existence and uniqueness of the solution for models (1) and (4).

Theorem 6. Assume that the assumptions \((H_1)-(H_2)\) hold; then the solution \(x = x(t, 0, \varphi)\) of models (1) and (4) exists uniquely on \([-\eta, \infty)\).

Proof. Transform the global existence and uniqueness of solution of the models (1) and (4) into a fixed point problem. Let \(\| \cdot \|^\ast\) be the norm in \(C([0, t_1], \mathbb{R}^n)\) defined by

\[
\|u\|^\ast = \max_{t \in [0, t_1]} \left\{ e^{-\lambda t} \max_{s \in [0, t]} \|u(s)\| \right\}, \quad u \in C\left([0, t_1], \mathbb{R}^n\right),
\]

(10)

where

\[
\lambda = \sqrt{\sum_{j=1}^{n} d_j^2 + \sigma \cdot \sqrt{\sum_{j=1}^{n} b_j^2 + \delta \cdot \sum_{j=1}^{n} \sum_{j=1}^{n} b_j^2}} + \zeta \cdot \kappa, \\
\sigma = \max_{j \in \Lambda} \left\{ |\sigma_j^*|, |\sigma_j^-| \right\}, \quad \delta = \max_{j \in \Lambda} \left\{ |\delta_j^*|, |\delta_j^-| \right\}, \\
\zeta = \max_{j \in \Lambda} \left\{ |\zeta_j^*|, |\zeta_j^-| \right\} .
\]

(11)

Then it is easy to see that \(C([0, t_1], \mathbb{R}^n)\) is a Banach space endowed with the norm \(\| \cdot \|^\ast\). Let \(u \in C([0, t_1], \mathbb{R}^n)\) and
consider the operator \( \mathcal{L}_1 : C([0,t_1], \mathbb{R}^n) \to C([0,t_1], \mathbb{R}^n) \) defined by
\[
(\mathcal{L}_1 u)(t) = \varphi(0) + \int_0^t \left\{ -Du(s-\sigma) + Af(u(s)) + Bg(u(s-\tau(s))) + W \int_0^{\rho(t)} K(\theta) \dot{h}(u(s-\theta)) \, d\theta + I \right\} \, ds,
\]
(13)
where \( u(s) = \varphi(s), s \in [-\eta,0] \).

First we show that \( \mathcal{L}_1 \) is a contraction on \( C([0,t_1], \mathbb{R}^n) \).

Let \( u, v \in C([0,t_1], \mathbb{R}^n) \); we have
\[
\|(\mathcal{L}_1 u)(t) - (\mathcal{L}_1 v)(t)\| \leq \int_0^t \left\| -D[u(s-\sigma) - v(s-\sigma)] \right\| \, ds
\]
\[
+ \int_0^t \left\| A[f(u(s)) - f(v(s))] \right\| \, ds
\]
\[
+ \int_0^t \left\| B[f(u(s-\tau(s))) - f(v(s-\tau(s)))] \right\| \, ds
\]
\[
+ \int_0^t \left\| W \int_0^{\rho(t)} K(\theta) [h(u(s-\theta)) - h(v(s-\theta))] \, d\theta \right\| \, ds
\]
\[
\leq \int_0^t \|D[u(s-\sigma) - v(s-\sigma)]\| \, ds
\]
\[
+ \int_0^t \|A[f(u(s)) - f(v(s))]\| \, ds
\]
\[
+ \int_0^t \|B[f(u(s-\tau(s))) - f(v(s-\tau(s)))]\| \, ds
\]
\[
+ \int_0^t \|W \int_0^{\rho(t)} K(\theta) [h(u(s-\theta)) - h(v(s-\theta))] \, d\theta\| \, ds.
\]
(14)

In view of (H2), we get
\[
\int_0^t \left\| W \int_0^{\rho(t)} K(\theta) [h(u(s-\theta)) - h(v(s-\theta))] \, d\theta \right\| \, ds
\]
\[
\leq \xi \cdot \sum_{i=1}^n \sum_{j=1}^n w_{ij}^2 \int_0^t \left\| K(\theta) \right\| \|u(s) - v(s)\| \, d\theta \, ds
\]
\[
\leq \xi \cdot \sum_{i=1}^n \sum_{j=1}^n w_{ij}^2 \int_0^t \left\| K(\theta) \right\| \|u(s) - v(s)\| \, d\theta \, ds
\]
\[
\leq \xi \cdot \sum_{i=1}^n \sum_{j=1}^n w_{ij}^2 \int_0^t \left\| K(\theta) \right\| \|u(s) - v(s)\| \, d\theta \, ds
\]
\[
\leq \xi \cdot \sum_{i=1}^n \sum_{j=1}^n w_{ij}^2 \int_0^t \left\| K(\theta) \right\| \|u(s) - v(s)\| \, d\theta \, ds
\]
\[
\leq \xi \cdot \sum_{i=1}^n \sum_{j=1}^n w_{ij}^2 \int_0^t \left\| K(\theta) \right\| \|u(s) - v(s)\| \, d\theta \, ds
\]
\[
\leq \xi \cdot \sum_{i=1}^n \sum_{j=1}^n w_{ij}^2 \int_0^t \left\| K(\theta) \right\| \|u(s) - v(s)\| \, d\theta \, ds
\]
\[
\leq \xi \cdot \sum_{i=1}^n \sum_{j=1}^n w_{ij}^2 \int_0^t \left\| K(\theta) \right\| \|u(s) - v(s)\| \, d\theta \, ds
\]
\[
\leq \xi \cdot \sum_{i=1}^n \sum_{j=1}^n w_{ij}^2 \int_0^t \left\| K(\theta) \right\| \|u(s) - v(s)\| \, d\theta \, ds
\]
(15)

Substituting the above inequality to (14), we obtain
\[
\|(\mathcal{L}_1 u)(t) - (\mathcal{L}_1 v)(t)\| \leq \sqrt{\sum_{j=1}^n d_j^2 \int_0^t \max_{r \in [0,t]} \|u(r) - v(r)\| \, ds}
\]
\[
+ \sigma \cdot \sqrt{\sum_{j=1}^n \sum_{i,j} a_{ij}^2 \int_0^t \max_{r \in [0,t]} \|u(r) - v(r)\| \, ds}
\]
\[
+ \delta \cdot \sqrt{\sum_{j=1}^n \sum_{i,j} b_{ij}^2 \int_0^t \max_{r \in [0,t]} \|u(r) - v(r)\| \, ds}
\]
\[
+ \zeta \cdot \sum_{i=1}^n \sum_{j=1}^n w_{ij}^2 \int_0^t \max_{r \in [0,t]} \|u(r) - v(r)\| \, ds
\]
\[
= (\lambda - 1) \int_0^t \max_{r \in [0,t]} \|u(r) - v(r)\| \, ds
\]
(16)
Since $e^{\lambda t}$ is increasing in $t$,
\[
\max_{s \in [0,t]} \mathbb{V} \left((\mathcal{L}_1 u)(s) - (\mathcal{L}_1 v)(s)\right) \leq \frac{\lambda - 1}{\lambda} \mathbb{V}(u - v).
\]
(17)

Then
\[
e^{-\lambda t} \max_{s \in [0,t]} \mathbb{V} \left((\mathcal{L}_1 u)(s) - (\mathcal{L}_1 v)(s)\right) \leq \frac{\lambda - 1}{\lambda} \mathbb{V}(u - v).
\]
(18)

Note the definition of $\| \cdot \|*$; we have
\[
\mathbb{V}( \mathcal{L}_1 u - \mathcal{L}_1 v )^* \leq \frac{\lambda - 1}{\lambda} \mathbb{V}(u - v)^*.
\]
(19)

Thus $\mathcal{L}_1$ is a contraction on $C([0,t_1], \mathbb{R}^n)$, and it has a unique fixed point $u^*_1 \in C([0,t_1], \mathbb{R}^n)$. Thus we get the fact that $u^*_1(t_1)$ exists finitely. It implies that $u^*_1(t_1) + I_1(u^*_1(t_1), u^*_1(t_1))$ also exists finitely, since assumption (H1) holds. Then we replace $u^*_1(t_1)$ with $u^*_1(t_1) + I_1(u^*_1(t_1), u^*_1(t_1))$ and define $\eta_1 = u^*_1(t_1) + I_1(u^*_1(t_1), u^*_1(t_1))$ for later use.

Next we show that $\mathcal{L}_2$ is a contraction on $C([t_1, t_2], \mathbb{R}^n)$. For $u \in C([t_1, t_2], \mathbb{R}^n)$, let
\[
\mathbb{V}(u)^* = \max_{t \in [t_1, t_2]} \left\{ e^{-\lambda(t-t_1)} \max_{s \in [t_1, t]} \mathbb{V}(u(s)) \right\},
\]
(20)

where $\lambda$ is defined in (11). Let $u \in C([t_1, t_2], \mathbb{R}^n)$ and consider the operator $\mathcal{L}_2 : C([t_1, t_2], \mathbb{R}^n) \to C([t_1, t_2], \mathbb{R}^n)$ defined by
\[
(\mathcal{L}_2 u)(t) = \eta_1 + \int_{t_1}^{t} \left\{ -D u(s - \sigma) + A f(u(s)) \\
+ B g(u(s - \tau(s))) \\
+ W \int_{0}^{(t)} K(\theta) h(u(s - \theta)) d\theta + I \right\} ds,
\]
(21)

where $\eta_1 = u^*_1(t_1) + I_1(u^*_1(t_1))$ and
\[
\mathbb{V}(u)^* = \begin{cases} 
\varphi(s), & s \in [-\eta, 0], \\
u_1^*(s), & s \in [0, t_1), \\
u_2^*(s), & s \in [t_1, t_2), \\
\vdots \\
u_n^*(s), & s \in [t_{n-1}, t_n). 
\end{cases}
\]
(22)

By virtue of the definition of $\mathcal{L}_2$, similar to the proof of (19), we get, for $u, v \in C([t_1, t_2], \mathbb{R}^n)$,
\[
\mathbb{V}(\mathcal{L}_2 u - \mathcal{L}_2 v)^* \leq \frac{\lambda - 1}{\lambda} \mathbb{V}(u - v)^*.
\]
(23)

Thus $\mathcal{L}_2$ is a contraction on $C([t_1, t_2], \mathbb{R}^n)$, and it has a unique fixed point $u^*_2 \in C([t_1, t_2], \mathbb{R}^n)$. Moreover, we know that $u^*_2(t_2)$ exists finitely, which implies that $u^*_2(t_2) + I_2(u^*_2(t_2), u^*_2(t_2))$ exists finitely in view of assumption (H2).

Then we replace $u^*_2(t_2)$ with $u^*_2(t_2) + I_2(u^*_2(t_2), u^*_2(t_2))$ and define $\eta_2 = u^*_2(t_2) + I_2(u^*_2(t_2), u^*_2(t_2))$ for later use.

Finally we show that $\mathcal{L}_{n+1}$ is a contraction on $C([t_n, t_{n+1}], \mathbb{R}^n)$. For $u \in C([t_n, t_{n+1}], \mathbb{R}^n)$, let
\[
\mathbb{V}(u)^* = \max_{t \in [t_n, t_{n+1}]} \left\{ e^{-\lambda(t-t_0)} \max_{s \in [t_n, t]} \mathbb{V}(u(s)) \right\},
\]
(24)

where $\lambda$ is defined in (11). Let $u \in C([t_n, t_{n+1}], \mathbb{R}^n)$; then we can similarly consider the operator $\mathcal{L}_{n+1} : C([t_n, t_{n+1}], \mathbb{R}^n) \to C([t_n, t_{n+1}], \mathbb{R}^n)$ defined by
\[
(\mathcal{L}_{n+1} u)(t) = \eta_n + \int_{t_n}^{t} \left\{ -D u(s - \sigma) + A f(u(s)) \\
+ B g(u(s - \tau(s))) \\
+ W \int_{0}^{(t)} K(\theta) h(u(s - \theta)) d\theta + I \right\} ds,
\]
(25)

where $\eta_n = u^*_n(t_n) + I_n(u^*_n(t_n))$ and
\[
\mathbb{V}(u)^* = \begin{cases} 
\varphi(s), & s \in [-\eta, 0], \\
u_1^*(s), & s \in [0, t_1), \\
u_2^*(s), & s \in [t_1, t_2), \\
\vdots \\
u_n^*(s), & s \in [t_{n-1}, t_n). 
\end{cases}
\]
(26)

Then repeating the argument with $\mathcal{L}_{n+1}$ replacing $\mathcal{L}_1$, similar to the proof of (19), we see that, for $u, v \in C([t_n, t_{n+1}], \mathbb{R}^n)$,
\[
\mathbb{V}(\mathcal{L}_{n+1} u - \mathcal{L}_{n+1} v)^* \leq \frac{\lambda - 1}{\lambda} \mathbb{V}(u - v)^*.
\]
(27)

Thus $\mathcal{L}_{n+1}$ is a contraction on $C([t_n, t_{n+1}], \mathbb{R}^n)$, and it has a unique fixed point $u^*_n \in C([t_n, t_{n+1}], \mathbb{R}^n)$.

Continuing in this manner, we construct
\[
\mathbb{V}(u)^* = \begin{cases} 
\varphi(t), & t \in [-\eta, 0], \\
u_1^*(t), & t \in [0, t_1), \\
u_2^*(t), & t \in [t_1, t_2), \\
\vdots \\
u_n^*(t), & t \in [t_{n-1}, t_n), \\
u_{n+1}^*(t), & t \in [t_n, t_{n+1}). 
\end{cases}
\]
(28)

Then $u^*(t)$ is the global solution of models (1) and (4). If $v^*(t)$ is another solution of models (1) and (4), then it is easy to check from the above argument that $u^*(t) = v^*(t)$. Hence, the solution $u^*(t) = u^*(t, 0, \varphi)$ of models (1) and (4) exists uniquely on $[-\eta, \infty)$. This completes the proof.

4. Existence of an Equilibrium Point

In previous sections, we have showed the global existence and uniqueness of solution for models (1) and (4). In this section, without requiring the boundedness, differentiability,
or monotonicity of the activation functions, we establish a delay-independent sufficient condition for the existence of an equilibrium point of model (1). As usual, we denote an equilibrium point of the model (1) by the constant vector \( x^* \in \mathbb{R}^n \), where \( x^* \) satisfies

\[
-Dx^* + Af(x^*) + Bg(x^*) + W\kappa h(x^*) + I = 0,
\]

where \( k = K(t) = \int_{0}^{t} K(s) \, ds \). In this paper, it is assumed that the impulsive function \( I_k(x^*, x^*) = 0 \) for \( k \in \mathbb{Z}_+ \). Hence, to prove the existence of solution of (29), it suffices to show that the following has a solution:

\[
x^* - D^{-1}Af(x^*) - D^{-1}Bg(x^*) - D^{-1}W\kappa h(x^*) - D^{-1}I = 0,
\]

in view of \( D > 0 \).

**Theorem 7.** Assume that the assumption \((H_3)\) holds. Then model (1) has at least one equilibrium point if, for any \( t > 0 \), \( D - [A]^* \Sigma - [B]^* \Delta - [W\kappa]^* \Theta \) is an \( M \)-matrix, where \( \Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n), \sigma_j = \max(\|\delta_j^+\|, \|\delta_j^-\|), \Delta = \text{diag}(\delta_1, \ldots, \delta_n), \delta_j = \max(\|\delta_j^+\|, \|\delta_j^-\|), \text{ and } \Theta = \text{diag}(\zeta_1, \ldots, \zeta_n), \zeta_j = \max(\|\zeta_j^+\|, \|\zeta_j^-\|) \).

**Proof.** From (30), we note that it suffices to prove that the following has at least one solution:

\[
p(x) = x - W_1f(x) - W_2g(x) - W_3h(x) - J' = 0,
\]

where \( W_1 = D^{-1}A, W_2 = D^{-1}B, W_3 = D^{-1}W\kappa, J' = D^{-1}J \).

In order to use topological degree theory, we consider the following homotopic mapping:

\[
P(x, \lambda) = \lambda p(x) + (1 - \lambda) x, \quad \lambda \in [0, 1].
\]

Note that \( D - [A]^* \Sigma - [B]^* \Delta - [W\kappa]^* \Theta \) is an \( M \)-matrix; it can be deduced that \( I - [W_1]^* \Sigma - [W_2]^* \Delta - [W_3]^* \Theta \) is also an \( M \)-matrix. This implies that \( (I - [W_1]^* \Sigma - [W_2]^* \Delta - [W_3]^* \Theta)^{-1} \geq 0 \) and there exists a positive vector \( X_0 \in \mathbb{R}^n \) such that \( (I - [W_1]^* \Sigma - [W_2]^* \Delta - [W_3]^* \Theta)X_0 > 0 \). It then follows that

\[
[P(x, \lambda)]^+ \geq [x]^* - \lambda[W_1]^* [f(x)]^* - \lambda[W_2]^* [g(x)]^* - \lambda[W_3]^* [h(x)]^* - \lambda[J']^* + (1 - \lambda) [x]^* \\
= \left[ x - \lambda W_1 f(x) - \lambda W_2 g(x) - \lambda W_3 h(x) - \lambda J' \right]^* \\
\geq [x]^* - \lambda [W_1 f(x)]^* - \lambda [W_2 g(x)]^* - \lambda [W_3 h(x)]^* - \lambda [J']^* \\
\geq [x]^* - \lambda [W_1 f(x)]^* - \lambda [W_2 g(x)]^* - \lambda [W_3 h(x)]^* - \lambda [J']^*.
\]

It is obvious that set \( \Omega \) is not empty and, for any \( x \in \partial\Omega \), we have

\[
[P(x, \lambda)]^+ \geq (1 - \lambda) [x]^* \\
+ \lambda (I - [W_1]^* \Sigma - [W_2]^* \Delta - [W_3]^* \Theta) X_0 > 0, \quad \lambda \in [0, 1],
\]

which implies that \( P(x, \lambda) \neq 0 \) for all \( x \in \partial\Omega \) and \( \lambda \in [0, 1] \).

By topological degree invariance theory, we obtain

\[
\deg(p(x), \Omega, 0) = \deg(P(x, \lambda), \Omega, 0) \\
= \deg(P(x, 0), \Omega, 0) = 1.
\]

Therefore, from Remark 2, we know that \( p(x) = 0 \) has at least one solution in \( \Omega \). This completes the proof.

---

5. **Global Asymptotic Stability**

It should be noted that Theorem 7 can guarantee the existence of an equilibrium point but not the uniqueness. In this section, we will derive some sufficient conditions to guarantee not only the global asymptotic stability but also the uniqueness of the equilibrium point. For this purpose, the impulsive function \( I_k \) which is viewed as a perturbation of the equilibrium point \( x^* \) of models (1) and (4) without impulses is defined by

\[
x(t_k) - x(t^-_k) = I_k(x(t^-_k), x(t^-_k)) \\
= -J_k \left( x(t^-_k) - x^* - D \int_{t^-_k}^{t_k} (x(u) - x^*) \, du \right), \quad k \in \mathbb{Z}_+,
\]

where \( J_k, k \in \mathbb{Z}_+ \) are some \( n \times n \) real matrices. It is clear that \( I_k(x^*, x^*) = 0, k \in \mathbb{Z}_+ \). Such type of impulses describes
the fact that the encountered instantaneous perturbations depend on not only the state of neurons at impulse times $t_k$ but also the state of neurons in its recent history, which reflects more realistic dynamics. The similar nonlinear impulsive perturbations, which include linear impulsive perturbations and nonimpulsive perturbations as their special cases, have also been investigated by some researchers recently [17–21, 39].

Let $y(t) = x(t) - x^*$; then we rewrite the models (1) and (4) as follows:

$$
\dot{y}(t) = -Dy(t - \sigma) + A\tilde{f}(y(t)) + B\tilde{g}(y(t - \tau(t))) + W\int_{t-\rho(t)}^{t} K(t - s) \tilde{h}(y(s)) ds,
$$

(t > 0, \ t \neq t_k),

$$
\Delta y(t_k) = y(t_k) - y(t^-_k)
$$

$$
= -I_k \left\{ y(t^-_k) - D \int_{t^-_k}^{t_k} y(u) du \right\}, \quad k \in \mathbb{Z}_+,
$$

$$
y(s) = q(s) - x^*, \quad s \in [-\eta, 0],
$$

where $\tilde{f}(y(\cdot)) = f(y(\cdot) + x^*) - f(x^*)$, $\tilde{g}(y(\cdot)) = g(y(\cdot) + x^*) - g(x^*)$ and $\tilde{h}(y(\cdot)) = h(y(\cdot) + x^*) - h(x^*)$. For convenience in our discussion, in the following, we replace $f$ with $\tilde{f}$, replace $g$ with $g$, and replace $h$ with $\tilde{h}$. Then using a simple transformation, model (38) has an equivalent form as follows:

$$
\frac{d}{dt} \left[ y(t) - D \int_{t^-}^{t} y(u) du \right]
$$

$$
= -Dy(t) + Af(y(t)) + Bg(y(t - \tau(t)))
$$

where

$$
\Gamma = \begin{bmatrix}
\Pi_{11} & -Q_1 & \gamma T_{12}^T & -Q_1D & X_1 & DP\Pi & \Pi_{38} & U_3\Sigma_6 & \Pi_{11,10} \\
* & \Pi_{22} & -Q_1D & X_2 & 0 & \gamma Q_1A & \gamma Q_1B & 0 & \Pi_{12,10} \\
* & * & \Pi_{33} & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & \Pi_{35} & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -Q_2 & -DPA & -DPB & 0 & -DPW \\
* & * & * & * & * & -U_1 & 0 & 0 & 0 \\
* & * & * & * & * & * & \Pi_{99} & 0 \\
* & * & * & * & * & * & * & \Pi_{10,10}
\end{bmatrix},
$$

$$
\chi_1 = \text{col} \{ X_1, X_2, O, O, X_3, O, O, O, O, O \},
$$

$$
\chi_2 = \text{col} \{ X_4, X_5, O, O, O, O, O, O, O, X_6 \},
$$

$$
\Pi_{11} = -PD - DP + \sigma^2 Q_2 + Q_3 - U_1\Sigma_1 - U_2\Sigma_5,
$$

$$
\Pi_{12} = PA + Q_1A + U_1\Sigma_2,
$$

$$
\Pi_{13} = PB + Q_1B,
$$

$$
\Pi_{1,10} = PW + Q_1W + X_2,
$$

$$
\Pi_{2,10} = \gamma Q_1W + X_5,
$$

$$
\Pi_{33} = \tau T_{11} - \gamma T_{12}^T - \gamma T_{12} - U_2\Sigma_3,
$$

$$
\Pi_{55} = X_3 + X_3^T,
$$

$$
\Pi_{99} = \rho^2 K(0) ZK(0) - U_3,
$$

$$
\Pi_{10,10} = X_6 + X_6^T.
$$

Theorem 8. Under the conditions in Theorem 7, model (1) has a unique equilibrium point which is globally asymptotically stable if there exist a constant $\gamma > 0$, an $n \times n$ inverse matrix $Q_1$, six $n \times n$ matrices $P > 0, Q_2 > 0, Q_3 > 0, Z > 0, T_{11} > 0$, and $T_{22} > 0$, three $n \times n$ diagonal matrices $U_1 > 0, U_2 > 0, \ U_3 > 0$, and real matrices $T_{12}, X_l (l = 1, \ldots, 6)$ with appropriate dimension such that

$$
\begin{bmatrix}
T_{11} & T_{12} \\
* & T_{22}
\end{bmatrix} > 0,
$$

$$
\begin{bmatrix}
P & (I - I_k)^T P
\end{bmatrix} \geq 0, \quad k \in \mathbb{Z}_+,
$$

$$
\begin{bmatrix}
\Xi \chi_j \\
* & -Z
\end{bmatrix} < 0, \quad j = 1, 2,
$$

where

$$
\chi_1 = \text{col} \{ X_1, X_2, O, O, X_3, O, O, O, O, O \},
$$

$$
\chi_2 = \text{col} \{ X_4, X_5, O, O, O, O, O, O, O, X_6 \},
$$

$$
\Pi_{11} = -PD - DP + \sigma^2 Q_2 + Q_3 - U_1\Sigma_1 - U_2\Sigma_5,
$$

$$
\Pi_{12} = PA + Q_1A + U_1\Sigma_2,
$$

$$
\Pi_{13} = PB + Q_1B,
$$

$$
\Pi_{1,10} = PW + Q_1W + X_2,
$$

$$
\Pi_{2,10} = \gamma Q_1W + X_5,
$$

$$
\Pi_{33} = \tau T_{11} - \gamma T_{12}^T - \gamma T_{12} - U_2\Sigma_3,
$$

$$
\Pi_{55} = X_3 + X_3^T,
$$

$$
\Pi_{99} = \rho^2 K(0) ZK(0) - U_3,
$$

$$
\Pi_{10,10} = X_6 + X_6^T.
$$
Proof. Consider the following Lyapunov-Krasovskii functional as
\[
V(t, y(t)) = V_1(t, y(t)) + V_2(t, y(t)) + V_3(t, y(t)) + V_4(t, y(t)) + V_5(t, y(t)),
\]
where
\[
\begin{align*}
V_1(t, y(t)) &= \left[ y(t) - D \int_{t-\sigma}^{t} y(u) \, du \right]^T P \left[ y(t) - D \int_{t-\sigma}^{t} y(u) \, du \right], \\
V_2(t, y(t)) &= \sigma \int_{t-\sigma}^{t} \int_{t-\tau}^{t} y^T(u) Q_2 y(u) \, du \, ds, \\
V_3(t, y(t)) &= \int_{t-\tau}^{t} \int_{t-\tau}^{t} \left( y(u - \tau(u)) \right)^T \begin{bmatrix} T_{11} & T_{12} \\ T_{12} & T_{22} \end{bmatrix} \left( y(u - \tau(u)) \right) \, ds \, du, \\
V_4(t, y(t)) &= \gamma^2 \int_{t-\tau}^{t} \int_{t-\tau}^{t} h^T(y(s)) K(t-s) ZK(t-s) \, ds \, du, \\
V_5(t, y(t)) &= \rho \int_{t-\rho}^{t} \int_{t-\rho}^{t} h^T(y(s)) K(t-s) ZK(t-s) \, ds \, du.
\end{align*}
\]
Calculating the upper right derivative of $V$ along the trajectories of model (39) at the continuous interval $[t_{k-1}, t_k)$, $k \in \mathbb{Z}_+$, we obtain
\[
\begin{align*}
D^+ V_1 &= 2 \left[ y(t) - D \int_{t-\sigma}^{t} y(u) \, du \right]^T P \left[ -D y(t) + A f(y(t)) + B g(y(t - \tau(t))) \right] \\
&+ W \int_{t-\rho}^{t} K(t-s) h(y(s)) \, ds \\
&= -2 y^T(t) PDy(t) + 2 y^T(t) PAf(y(t)) \\
&+ 2 y^T(t) PBg(y(t - \tau(t))) \\
&+ 2 y^T(t) P W \int_{t-\rho}^{t} K(t-s) h(y(s)) \, ds \\
&+ 2 y^T(t) DP D \int_{t-\sigma}^{t} y(u) \, du \\
&- 2 \left[ \int_{t-\sigma}^{t} y(u) \, du \right]^T DPAf(y(t)) \\
&- 2 \left[ \int_{t-\sigma}^{t} y(u) \, du \right]^T DPBg(y(t - \tau(t))) \\
&- 2 \left[ \int_{t-\sigma}^{t} y(u) \, du \right]^T DPW \int_{t-\rho}^{t} K(t-s) h(y(s)) \, ds.
\end{align*}
\]
It follows from Lemma 3 that
\[
\begin{align*}
D^+ V_2 &= \sigma^2 y^T(t) Q_2 y(t) \\
&- \sigma \int_{t-\sigma}^{t} y^T(u) Q_2 y(u) \, du + y^T(t) Q_3 y(t) \\
&- y^T(t-\sigma) Q_3 y(t-\sigma) \\
&\leq \sigma^2 y^T(t) Q_2 y(t) - \left[ \int_{t-\sigma}^{t} y(u) \, du \right]^T Q_2 \left[ \int_{t-\sigma}^{t} y(u) \, du \right] \\
&+ y^T(t) Q_3 y(t) - y^T(t-\sigma) Q_3 y(t-\sigma),
\end{align*}
\]
\[
\begin{align*}
D^+ V_3 &= \int_{t-\tau}^{t} \int_{t-\tau}^{t} h^T(y(s)) K(t-s) ZK(t-s) \, ds \, du \\
&\leq y^T(t-\tau(t)) \left[ \tau T_{11} - 2 \gamma T_{12} \right] y(t-\tau(t)) \\
&+ 2 \gamma y^T(t) T_{12} y(t - \tau(t)) \\
&- 2 \gamma y^T(t) T_{12} y(t - \tau(t)) \\
&+ y^2 \int_{t-\tau}^{t} y^T(s) T_{22} y(s) \, ds, \\
D^+ V_4 &= y^2 \gamma y^T(t) T_{22} y(t) - y^2 \int_{t-\tau}^{t} y^T(t + u) T_{22} y(t + u) \, du \\
&= y^2 \gamma y^T(t) T_{22} y(t) - y^2 \int_{t-\tau}^{t} y^T(s) T_{22} y(s) \, ds, \\
D^+ V_5 &= \rho^2 h^T(y(t)) K(0) ZK(0) h(y(t))
\end{align*}
\]
\[
- \frac{\rho}{\rho - \rho(t)} \xi_2(t)^T Z \xi_2(t) - \frac{\rho}{\rho(t)} \xi_1(t)^T Z \xi_1(t) \\
\leq 2 \xi^T(t) \chi_1 \xi_2(t) + \frac{\rho(t)}{\rho} \xi_1(t)^T Z \chi_1^{-1} \chi_2 \xi(t) \\
+ \frac{\rho(t)}{\rho} \xi_1(t)^T \chi_2 \xi_1(t),
\]
(51)

where
\[
\xi(t) = \begin{pmatrix}
y(t), \dot{y}(t), y(t - \tau(t)), y(t - \sigma), \xi_2(t), \\
\int_{t - \sigma}^t y(s) ds, f(y(t)), g(y(t - \tau(t)))
\end{pmatrix}^T.
\]

Moreover, for any \( n \times n \) diagonal matrices, \( U_1 > 0, U_2 > 0, \) and \( U_3 > 0, \) the following inequality holds by the methods in [40]:
\[
\begin{align*}
\{ & \begin{bmatrix}
y(t) \\
f(y(t))
\end{bmatrix}^T \\
\begin{bmatrix}
-U_1 & U_1 \\
0 & -U_2
\end{bmatrix} \\
\begin{bmatrix}
y(t) \\
g(y(t - \tau(t)))
\end{bmatrix}
\end{align*}
\]
+ \[
\begin{bmatrix}
y(t - \tau(t)) \\
h(y(t))
\end{bmatrix}
\] \( \geq 0. \)
(53)

Combining (45)–(53), one may deduce that
\[
D' V \leq \xi^T(t) \Xi(t) \xi(t), \quad t \in [t_{k-1}, t_k), \quad k \in \mathbb{Z}^+,
\]
(54)

where
\( \Xi(t) = \Xi + \frac{\rho(t)}{\rho} X_1 Z^{-1} X_1^T + \frac{\rho(t)}{\rho} X_2 Z^{-1} X_2^T, \)

\( \Xi = 
\begin{bmatrix}
\Pi_{11} & -Q_1 & y_{12}^T & -Q_3 & D_P & D_X & \Pi_{17} & \Pi_{18} & U_3 & U_6 & \Pi_{10,11} \\
\star & \Pi_{22} & 0 & -y_{12} & X_2 & 0 & \gamma_{12} & 0 & 0 & 0 & 0 \\
\star & \star & \Pi_{33} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & -Q_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & -Q_2 & -D_P & -D_B & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & -U_1 & 0 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & -U_2 & 0 & 0 & 0 & 0 \\
\star & \star & \star & \star & \star & \star & \star & \star & \star & \star & \star \\
\end{bmatrix}, \quad (55) 

\( \xi(t) = (y(t), \dot{y}(t), y(t - \tau(t)), y(t - \sigma), \xi_2(t), \int_{t-\sigma}^t y(s) ds, f(y(t)), g(y(t - \tau(t))), h(g(t)), \xi_1(t) \) \text{ ds})^T. \)

From Lemma 5 we obtain that \( \Xi(t) < 0 \) if the following inequalities hold simultaneously:

\( \Xi + X_1 Z^{-1} X_1^T < 0, \quad \Xi + X_2 Z^{-1} X_2^T < 0. \quad (56) \)

Based on the well-known Schur complements [36], we get the fact that (56) is equivalent to (42). Therefore, \( \Xi(t) < 0 \), \( t \in [t_{k-1}, t_k], k \in \mathbb{Z}_+ \).

For arbitrary \( t > 0 \), without loss of generality, we set \( t \in [t_{n-1}, t_n] \), for some \( n \in \mathbb{Z}_+ \). Then integrating inequality (54) at each interval \( [t_{k-1}, t_k], 1 \leq k \leq n - 1, \) and \( [t_{n-1}, t_n] \), we derive

\[
V(t_1) \leq V(0) + \int_0^{t_1} \xi^T(u) \Xi(t) \xi(u) du, \\
V(t_2) \leq V(t_1) + \int_{t_1}^{t_2} \xi^T(u) \Xi(t) \xi(u) du, \\
\vdots \\
V(t_{n-1}) \leq V(t_{n-2}) + \int_{t_{n-2}}^{t_{n-1}} \xi^T(u) \Xi(t) \xi(u) du, \\
V(t) \leq V(t_{n-1}) + \int_{t_{n-1}}^{t} \xi^T(u) \Xi(t) \xi(u) du, 
\]

which implies that

\[
V(t) \leq V(0) + \int_0^{t} \xi^T(u) \Xi(t) \xi(u) du + \sum_{0=t_{k-1}}^{t} [V(t_k) - V(t_{k-1})], \quad t \geq 0. \quad (58) 
\]

Now, in order to analyze (58), we consider the change of \( V \) at impulse times \( t_k, k \in \mathbb{Z}_+ \).
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Thus, we can deduce that
\[ V(t_k) \leq V(t_k), \quad k \in \mathbb{Z}_+. \]

Substituting the above inequality to (58), it yields
\[ V(t) - \int_0^t \xi^T(u) \Xi(t) \xi(u) du \leq V(0), \quad t \geq 0. \] (63)

Applying Lemma 3 and (63), we have
\[
\left\| D \int_{t-\sigma}^t y(u) du \right\|^2
\leq \lambda_{\max} \left( D^2 \right) \left\[ \int_{t-\sigma}^t y(s) ds \right\]^T \left[ \int_{t-\sigma}^t y(s) ds \right]
\leq \lambda_{\max} \left( D^2 \right) \left[ \int_{t-\sigma}^t y(s) ds \right]^T Q_3 \left[ \int_{t-\sigma}^t y(s) ds \right]
\leq \lambda_{\max} \left( D^2 \right) V_2(t) \leq \sigma \lambda_{\max} \left( D^2 \right) V(t)
\]
\[
\leq \sigma \lambda_{\max} \left( D^2 \right) V(0), \quad t \geq 0.
\] (64)

Similarly,
\[
\left\| y(t) \right\| \leq \sigma \lambda_{\max} \left( D^2 \right) \left\| y(0) \right\| < \infty, \quad t \geq 0.
\] (65)

Hence, it can be obtained that
\[
\left\| y(t) \right\| \leq \left\| D \int_{t-\sigma}^t y(u) du \right\| + \left\| y(t) - D \int_{t-\sigma}^t y(u) du \right\|
\leq \sigma \lambda_{\max} \left( D^2 \right) \frac{V(0)}{\lambda_{\min}(P)} < \infty, \quad t \geq 0.
\] (66)

where
\[
V(0) = \left[ y(0) - D \int_0^\sigma y(u) du \right]^T P \left[ y(0) - D \int_0^\sigma y(u) du \right]
+ \sigma \int_0^\sigma y^T(s) Q_3 y(s) ds
+ \int_0^\sigma y^T(s) T_{22} y(s) ds
+ \rho \int_0^\sigma h^T(s) K(-s) ZK(-s) h(y(s)) ds du
\leq \left\{ 2 \lambda_{\max}(P) \left( 1 + \sigma_2^2 \lambda_{\max}(Q_2) \right) + \frac{1}{2} \sigma^2 \lambda_{\max}(Q) \right\} \left\| y \right\|_\eta^2 < \infty.
\] (67)

So the solution \( y(t) \) of models (1) and (4) is uniformly bounded on \([0, \infty)\). Thus, considering the continuity of activation function \( f \) (i.e., \( H_2 \)), it can be deduced from system (38) that there exists some constant \( M > 0 \) such that \( \| y(t) \| \leq M, t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+ \). It implies that, \( |y_j(t)| \leq M, t \in [t_{k-1}, t_k), k \in \mathbb{Z}_+ \), and \( i \in \Lambda \), where \( y_j \) denotes the right hand derivative of \( y \) at impulsive times.

Finally, we can prove that \( \| y(t) \| \to 0 \) as \( t \to \infty \), which is similar to the corresponding proof in the literature [25]. Here we omit it. Therefore, the zero solution of (38) or (39) is globally asymptotically stable, which implies that models (1) and (4) have a unique equilibrium point which is globally asymptotically stable. This completes the proof.

When there is no leakage delay, that is, \( \sigma = 0 \), models (1) and (4) become
\[
x(t) = -Dx(t) + Af(x(t)) + Bg(x(t))
+ W \int_{t-\rho(t)}^t K(s) h(x(s)) ds + I,
\]
\[
t > 0, \quad t \neq t_k, \Delta x(t_k)
= x(t_k) - x(t_k^-), \quad k \in \mathbb{Z}_+, s \in [-\eta, 0].
\] (68)

For model (68), we have the following result by Theorem 8.

**Corollary 9.** Under the conditions in Theorem 7, model (68) has a unique equilibrium point which is globally asymptotically stable if there exist a constant \( \gamma > 0 \), an \( n \times n \) inverse matrix
Consider the following recurrent neural networks model:

\[
\dot{x}(t) = -D x(t - \sigma) + Af(x(t)) + Bg(x(t - \tau(t))) + W \int_{t-\rho(t)}^{t} K(t-s) h(x(s)) \, ds + I, \quad t > 0, \quad t \neq k_i,
\]

\[
\Delta x(t_k) = x(t_k) - x(t_{k-1}) = I_k(x(t_k), x_{t_k}), \quad k \in \mathbb{Z}_+,
\]

where \( f = g = h = |s|, \sigma = 0.08, \tau(t) = 0.09 - 0.01[\sin t]^+, \rho(t) = 0.08 - 0.02[\sin t]^+, K(s) = e^{-s}, I = (0, 0)^T, t_k = 0.1k, \) and \( I_k = \text{diag}(0.5, 0.5), k \in \mathbb{Z}_+ \), and parameter matrices \( D, A, B, \) and \( W \) are given as follows:

\[
D = \begin{bmatrix} 9 & 0 \\ 0 & 8 \end{bmatrix}, \quad A = \begin{bmatrix} 1.5 & -0.3 \\ -0.1 & 0.2 \end{bmatrix}, \quad B = \begin{bmatrix} -0.6 & 0.4 \\ 1.7 & -1.8 \end{bmatrix}, \quad W = \begin{bmatrix} 0.2 & 0.3 \\ -0.5 & 0.15 \end{bmatrix},
\]

In this case, we know that \( \sigma^- = \delta^- = \zeta^- = -1, \sigma^+ = \delta^+ = \zeta^+ = 1, \sigma = 0.08, \tau = 0.1, \) and \( \rho = 0.1. \) Let \( \gamma = 2.6; \) via Matlab.
LMI toolbox, the feasible solution for the LMIs in Theorem 8 is derived as follows:

\[ P = \begin{bmatrix} 0.0017 & 0 \\ 0 & 0.0026 \end{bmatrix}, \quad Q_1 = \begin{bmatrix} 0.0030 & 0.0001 \\ 0.0001 & 0.0037 \end{bmatrix}, \]

\[ Q_2 = \begin{bmatrix} 0.8011 & 0.0002 \\ 0.0002 & 0.8061 \end{bmatrix}, \quad Q_3 = \begin{bmatrix} 0.5982 & 0.0009 \\ 0.0009 & 0.6026 \end{bmatrix}, \]

\[ Z = \begin{bmatrix} 0.8887 & 0 \\ 0 & 0.8887 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 1.0046 & 0 \\ 0 & 1.0046 \end{bmatrix}, \]

\[ U_2 = \begin{bmatrix} 0.8793 & 0 \\ 0 & 0.8793 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 1.0117 & 0 \\ 0 & 1.0117 \end{bmatrix}, \]

\[ T_{11} = \begin{bmatrix} 1.3262 & -0.0001 \\ -0.0001 & 1.3260 \end{bmatrix}, \quad T_{12} = \begin{bmatrix} 0.0388 & 0 \\ 0 & 0.0387 \end{bmatrix}, \]

\[ T_{22} = \begin{bmatrix} 0.0096 & 0.0002 \\ 0.0002 & 0.0113 \end{bmatrix}, \]

\[ X_1 = 10^{-3} \begin{bmatrix} 0.0534 & 0.0807 \\ -0.1311 & 0.0400 \end{bmatrix}, \]

\[ X_2 = 10^{-3} \begin{bmatrix} 0.1117 & 0.1773 \\ -0.3554 & 0.1106 \end{bmatrix}, \]

\[ X_3 = \begin{bmatrix} -0.3176 & 0 \\ 0 & -0.3176 \end{bmatrix}, \quad X_4 = \begin{bmatrix} -0.0004 & -0.0007 \\ 0.0012 & -0.0004 \end{bmatrix}, \]

\[ X_5 = \begin{bmatrix} -0.0014 & -0.0022 \\ 0.0044 & -0.0014 \end{bmatrix}, \]

\[ X_6 = \begin{bmatrix} -0.3177 & 0 \\ 0 & -0.3177 \end{bmatrix}. \]

(74)

Hence, from Theorem 8, the unique equilibrium point \( x^* = (0, 0)^T \) of system (72) is globally asymptotically stable.

### 7. Conclusion

In this paper, we have investigated a class of impulsive neural networks with mixed time delays and generalized activation functions. Firstly, by using the contraction mapping theorem, we have given a sufficient condition to guarantee the global existence and uniqueness of the solution for the addressed neural networks. Then, a delay-independent sufficient condition for existence of the equilibrium point and some delay-dependent sufficient conditions for stability have been derived, respectively, by using topological degree theory and suitable Lyapunov-Krasovskii functional. The obtained results require neither the boundedness, monotonicity, and differentiability of the activation functions nor the differentiability of time-varying delay. Finally, an example has been given to show the effectiveness and less conservativeness of the obtained results. In the future, we will do some further research on impulsive neural network models with leakage time-varying delay and continuously distributed delay.

### Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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