Research Article

Lattice-Valued Convergence Spaces: Weaker Regularity and p -Regularity

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By using some lattice-valued Kowalsky's dual diagonal conditions, some weaker regularities for Jäger's generalized stratified L -convergence spaces and those for Boustique et al's stratified L -convergence spaces are defined and studied. Here, the lattice L is a complete Heyting algebra. Some characterizations and properties of weaker regularities are presented. For Jäger's generalized stratified L -convergence spaces, a notion of closures of stratified L -filters is introduced and then a new p -regularity is defined. At last, the relationships between p -regularities and weaker regularities are established.

Dedicated to the first author's father Zonghua Li on the occasion of his 60th birthday

1. Introduction

In 1954, Kowalsky [1] introduced a diagonal condition (the $\mathbf{K}$ -diagonal condition) to characterize whenever a pretopological convergence space is topological. In 1967, Cook and Fischer [2] defined a stronger diagonal condition (the $\mathbf{F}$ -diagonal condition) which, as they showed therein, is necessary and sufficient for a convergence space to be topological. Furthermore, a dual version of $\mathbf{F}$ (the $\mathbf{DF}$ -diagonal condition) is necessary and sufficient for a convergence space to be regular. Regularity can also be characterized by the requirement that, for each filter $\mathbb{F}$, if $\mathbb{F}$ converges to x then so does $\overline{\mathbb{F}}$ (the closure of $\mathbb{F}$). In [3, 4], by considering a pair of convergence spaces (X, p) and (X, q) , Kent and his coauthors introduced a kind of relative topologicalness (resp., regularity) which was called p -topologicalness (resp., p -regularity). They discussed p -topologicalness (resp., p -regularity) both by neighborhood (resp., closure) of filter [5] and generalized $\mathbf{F}$ (resp., $\mathbf{DF}$)-diagonal condition. When $p = q$, p -topologicalness (resp., p -regularity) is precisely topologicalness (resp., regularity). In 1996, Kent and Richardson defined a weaker regularity by using the duality of

Kowalsky's diagonal condition. They also proved that weaker regularity, regularity, and p -regularity were distinct notions but closely related to each other [6].

In [7], Jäger investigated a kind of lattice-valued convergence spaces, which were called generalized stratified L -convergence spaces. Later, the theory of these spaces was extensively discussed under different lattice context [8–19]. A supercategory of generalized stratified L -convergence spaces, called levelwise stratified L -convergence spaces in this paper, was researched in [20–24]. Indeed, a generalized stratified L -convergence space is precisely a left-continuous levelwise stratified L -convergence space [22].

Lattice-valued $\mathbf{K}$ - and $\mathbf{F}$ -diagonal conditions for generalized stratified L -convergence spaces were studied in [11, 12, 17, 18] and those for levelwise stratified L -convergence spaces were discussed in [18, 23]. Both by lattice-valued $\mathbf{DF}$ -diagonal condition and α -level closures of stratified L -filters, the lattice-valued regularity for generalized stratified L -convergence spaces was presented in [13] and that for levelwise stratified L -convergence spaces was given in [20, 21]. Later, by α -level closures of stratified L -filters, p -regularity for levelwise generalized stratified L -convergence

spaces was studied in [24]. Recently, p -topologicalness and p -regularity for generalized stratified L -convergence spaces and that for level stratified L -convergence spaces were discussed systematically in [25].

In this paper, for generalized stratified L -convergence spaces and levelwise stratified L -convergence spaces, we will discuss some lattice-valued weaker regularities, p -regularities, and their relationships. The content is arranged as follows. Section 2 recalls some basic notions as preliminary. Section 3 presents the definitions, characterizations, and properties of lattice-valued weaker regularities. Section 4 presents a notion of closures of stratified L -filters and a new lattice-valued p -regularity for stratified generalized L -convergence spaces. Also, the relationships between lattice-valued weaker regularities and lattice-valued p -regularities are established.

2. Preliminaries

In this paper, if not otherwise specified, $L = (L, \leq)$ is always a complete lattice with a top element 1 and a bottom element 0, which satisfies the distributive law $\alpha \wedge (\bigvee_{i \in I} \beta_i) = \bigvee_{i \in I} (\alpha \wedge \beta_i)$. A lattice with these conditions is called a complete Heyting algebra or a frame. The operation $\rightarrow : L \times L \rightarrow L$ given by $\alpha \rightarrow \beta = \bigvee \{\gamma \in L : \alpha \wedge \gamma \leq \beta\}$ is called the residuation with respect to $\wedge$. A complete Heyting algebra L is said to be a complete Boolean algebra if it obeys the *law of double negation*: $\forall \alpha \in L, (\alpha \rightarrow 0) \rightarrow 0 = \alpha$.

For a set X , the set L^X of functions from X to L with the pointwise order becomes a complete lattice. Each element of L^X is called an L -set (or a fuzzy subset) of X . For any $\lambda \in L^X$, $\mathcal{K} \subseteq L^X$, and $\alpha \in L$, we denote by $\alpha \wedge \lambda$, $\alpha \rightarrow \lambda$, $\bigvee \mathcal{K}$, and $\bigwedge \mathcal{K}$ the L -sets defined by $(\alpha \wedge \lambda)(x) = \alpha \wedge \lambda(x)$, $(\alpha \rightarrow \lambda)(x) = \alpha \rightarrow \lambda(x)$, $(\bigvee \mathcal{K})(x) = \bigvee_{\mu \in \mathcal{K}} \mu(x)$, and $(\bigwedge \mathcal{K})(x) = \bigwedge_{\mu \in \mathcal{K}} \mu(x)$. Also, we make no difference between a constant function and its value since no confusion will arise. For a crisp subset $A \subseteq X$, let 1_A be the characteristic function; that is $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \notin A$. Clearly, the characteristic function 1_A of a subset $A \subseteq X$ can be regarded as a function from X to L .

Let X be a set. A fuzzy partial order (or an L -partial order) on X [26] is a function $R : X \times X \rightarrow L$ such that (1) $R(a, a) = 1$ for every $a \in X$ (reflexivity); (2) $R(a, b) = R(b, a) = 1$ implies that $a = b$ for all $a, b \in X$ (antisymmetry); (3) $R(a, b) \wedge R(b, c) \leq R(a, c)$ for all $a, b, c \in X$ (transitivity). The pair (X, R) is called an L -partially ordered set.

Let $[L^X] : L^X \times L^X \rightarrow L$ be a function defined by $[L^X](\lambda, \mu) = \bigwedge_{x \in X} (\lambda(x) \rightarrow \mu(x))$; then $[L^X]$ is an L -partial order on L^X . The value $[L^X](\lambda, \mu) \in L$ is interpreted as the degree that λ is contained in μ . In the sequel, we use the symbol $[\lambda, \mu]$ to denote $[L^X](\lambda, \mu)$ for simplicity.

Let $f : X \rightarrow Y$ be an ordinary function. We define $f^\rightarrow : L^X \rightarrow L^Y$ and $f^\leftarrow : L^Y \rightarrow L^X$ [27] by $f^\rightarrow(\lambda)(y) = \bigvee_{f(x)=y} \lambda(x)$ for $\lambda \in L^X$ and $y \in Y$, and $f^\leftarrow(\mu) = \mu \circ f$ for $\mu \in L^Y$.

2.1. Stratified L -(Ultra)filters. A stratified L -filter [27] on a set X is a function $\mathcal{F} : L^X \rightarrow L$ such that for each $\lambda, \mu \in L^X$ and

each $\alpha \in L$, (F1) $\mathcal{F}(0) = 0$, $\mathcal{F}(1) = 1$; (F2) $\mathcal{F}(\lambda) \wedge \mathcal{F}(\mu) = \mathcal{F}(\lambda \wedge \mu)$; (Fs) $\mathcal{F}(\alpha) \geq \alpha$. A stratified L -filter $\mathcal{F}$ is called tight if $\mathcal{F}(\alpha) = \alpha$ for each $\alpha \in L$ [5]. It is proved in [27] that all stratified L -filters are tight if and only if L is a complete Boolean algebra. It is easily seen that for a stratified L -filter $\mathcal{F}$ on X , we have $\forall \lambda \in L^X, \mathcal{F}(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\mu, \lambda])$.

The set $\mathcal{F}_L^s(X)$ of all stratified L -filters on X is ordered by $\mathcal{F} \leq \mathcal{G} \Leftrightarrow \forall \lambda \in L^X, \mathcal{F}(\lambda) \leq \mathcal{G}(\lambda)$. It is shown in [27] that the partially ordered set $(\mathcal{F}_L^s(X), \leq)$ has maximal elements which are called stratified L -ultrafilters. The set of all stratified L -ultrafilters on X is denoted as $\mathcal{U}_L^s(X)$. Let $\mathcal{F} \in \mathcal{F}_L^s(X)$. Then $\mathcal{F}$ is an L -ultrafilter if and only if for all $\lambda \in L^X$ we have $\mathcal{F}(\lambda) = \mathcal{F}(\lambda \rightarrow 0) \rightarrow 0$. A stratified L -filter $\mathcal{F}$ is called a stratified L -prime filter if $\mathcal{F}(\lambda \vee \mu) = \mathcal{F}(\lambda) \vee \mathcal{F}(\mu)$ for each $\lambda, \mu \in L^X$. And when L is a complete Boolean algebra then $\mathcal{F} = \bigwedge_{\mathcal{G} \leq \mathcal{F} \in \mathcal{U}_L^s(X)} \mathcal{G}$ and $\mathcal{F}$ is prime whenever $\mathcal{F}$ is maximal [27].

For each $\mathcal{F} \in \mathcal{F}_L^s(X)$, it is easily seen that $\mathbb{F}_{\mathcal{F}} = \{A \subseteq X \mid \mathcal{F}(1_A) = 1\}$ is a filter on X . For each $\lambda \in L^X$, take $\iota\lambda = \{x \in X \mid \lambda(x) > 0\}$. Let $\mathbb{F}$ be a filter on X . Then, when L is a linearly order frame or $0 \in L$ is prime ($\alpha \wedge \beta = 0$ implies $\alpha = 0$ or $\beta = 0$), the function $\mathcal{F}_{\mathbb{F}} : L^X \rightarrow L$, defined by $\forall \lambda \in L^X, \mathcal{F}_{\mathbb{F}}(\lambda) = 1$ if $\iota\lambda \in \mathbb{F}$ and $\mathcal{F}_{\mathbb{F}}(\lambda) = 0$ if not so, is a stratified L -filter on X [22]. Also, when L is a linearly order frame or $0 \in L$ is prime, a stratified L -ultrafilter takes values in $\{0, 1\}$ only [10].

Lemma 1 (Jäger [28] for $L = [0, 1]$). *Let L be a linearly order frame or let $0 \in L$ be prime. Then, for each $\mathcal{F} \in \mathcal{U}_L^s(X)$, $\mathbb{F}_{\mathcal{F}}$ is an ultrafilter on X and $\mathcal{F} = \mathcal{F}_{\mathbb{F}_{\mathcal{F}}}$.*

Proof. At first, we check that $\mathbb{F}_{\mathcal{F}}$ is an ultrafilter on X . For each $A \subseteq X$, we assume that $A \notin \mathbb{F}_{\mathcal{F}}$; that is, $\mathcal{F}(1_A) = 0$; then $\mathcal{F}(1_{X-A}) = \mathcal{F}(1_{X-A} \rightarrow 0) \rightarrow 0 = \mathcal{F}(1_A) \rightarrow 0 = 1$. That means $X - A \in \mathbb{F}_{\mathcal{F}}$. By the arbitrariness of A we get that $\mathbb{F}_{\mathcal{F}}$ is an ultrafilter on X . At second, we check $\mathcal{F} \leq \mathcal{F}_{\mathbb{F}_{\mathcal{F}}}$. Note that $\mathcal{F}$ takes values in $\{0, 1\}$ only; thus, it suffices to prove that if $\mathcal{F}(\lambda) = 1$; then $\mathcal{F}_{\mathbb{F}_{\mathcal{F}}}(\lambda) = 1$. Indeed, let $\mathcal{F}(\lambda) = 1$; then $\mathcal{F}(1_{\iota\lambda}) \geq \mathcal{F}(\lambda) = 1$; that is, $\iota\lambda \in \mathbb{F}_{\mathcal{F}}$ and so $\mathcal{F}_{\mathbb{F}_{\mathcal{F}}}(\lambda) = 1$. Therefore, $\mathcal{F} \leq \mathcal{F}_{\mathbb{F}_{\mathcal{F}}}$ and it follows that $\mathcal{F} = \mathcal{F}_{\mathbb{F}_{\mathcal{F}}}$ by the maximality of $\mathcal{F}$. $\square$

The following examples belong to the folklore; we list them here because the notations are needed.

Example 2. (1) For each point x in a set X , the function $[x] : L^X \rightarrow L, [x](\lambda) = \lambda(x)$ is a stratified L -filter on X . In general, $[x]$ is not a stratified L -ultrafilter. But when L is a complete Boolean algebra, then it is so.

(2) Let $\{\mathcal{F}_j \mid j \in J\}$ be a family of stratified L -filters on X ; then $\bigwedge_{j \in J} \mathcal{F}_j$, in particular, $\mathcal{F}_0 = \bigwedge_{j \in J} \mathcal{F}_j$, is a stratified L -filter on X .

(3) Let $f : X \rightarrow Y$ be a function. If $\mathcal{F} \in \mathcal{F}_L^s(X)$, then the function $f^\rightarrow(\mathcal{F}) \in \mathcal{F}_L^s(Y)$, where $f^\rightarrow(\mathcal{F}) : L^Y \rightarrow L$ defined by $\lambda \mapsto \mathcal{F}(\lambda \circ f)$. If $\mathcal{F} \in \mathcal{U}_L^s(X)$, then $f^\rightarrow(\mathcal{F}) \in \mathcal{U}_L^s(Y)$.

There is a natural fuzzy partial order on $\mathcal{F}_L^s(X)$ inherited from L^{L^X} . Precisely, for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, if we let

$[\mathcal{F}_L^s(X)](\mathcal{F}, \mathcal{G}) = [L^X](\mathcal{F}, \mathcal{G}) = \bigwedge_{\lambda \in L^X} (\mathcal{F}(\lambda) \rightarrow \mathcal{G}(\lambda))$, then $[\mathcal{F}_L^s(X)]$ is an L -partially order. For simplicity, we use the symbol $[\mathcal{F}, \mathcal{G}]$ to denote the value $[\mathcal{F}_L^s(X)](\mathcal{F}, \mathcal{G})$ below.

2.2. Lattice-Valued Convergence Spaces

Definition 3. A generalized stratified L -convergence structure [7] on a set X is a function $\lim : \mathcal{F}_L^s(X) \rightarrow L^X$ satisfying (LC1) $\forall x \in X, \limx = 1$; and (LC2) $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X), \mathcal{F} \leq \mathcal{G} \Rightarrow \lim \mathcal{F} \leq \lim \mathcal{G}$. The pair $(X, \lim)$ is called a generalized stratified L -convergence space. If $\lim$ further satisfies the strong axiom (LC2') $\forall \mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X), [\mathcal{F}, \mathcal{G}] \wedge \lim \mathcal{F} \leq \lim \mathcal{G}$, then the pair $(X, \lim)$ is called a strong stratified L -convergence space [8, 15, 16].

A function $f : X \rightarrow X'$ between two generalized stratified L -convergence spaces $(X, \lim), (X', \lim')$ is called continuous if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ and all $x \in X$ we have $\lim \mathcal{F}(x) \leq \lim' f^\rightarrow(\mathcal{F})(f(x))$. The category $SL\text{-}GCS$ has as objects all generalized stratified L -convergence spaces and as morphisms the continuous functions. This category is topological over SET [7, 10]. For a given source $(X \xrightarrow{f_i} (X_i, \lim_i))_{i \in I}$, the initial structure, $\lim$ on X is defined by $\forall \mathcal{F} \in \mathcal{F}_L^s(X), \forall x \in X, \lim \mathcal{F}(x) = \bigwedge_{i \in I} \lim_i f_i^\rightarrow(\mathcal{F})(f_i(x))$.

Definition 4. A collection $\bar{q} = (q_\alpha)_{\alpha \in L}$, where $q_\alpha : \mathcal{F}_L^s(X) \rightarrow \mathcal{P}(X)$, is called a levelwise stratified L -convergence structure on X [20] if it satisfies the following:

- (LL1) $[x] \xrightarrow{q_\alpha} x$ for each $x \in X$;
- (LL2) $\mathcal{G} \geq \mathcal{F} \xrightarrow{q_\alpha} x$ implies $\mathcal{G} \xrightarrow{q_\alpha} x$;
- (LL3) $\mathcal{F} \xrightarrow{q_\alpha} x$ implies $\mathcal{F} \xrightarrow{q_\beta} x$ whenever $\beta \leq \alpha$.

The notation, $\mathcal{F} \xrightarrow{q_\alpha} x$, means that $x \in q_\alpha(\mathcal{F})$. The pair $(X, \bar{q})$ is called a levelwise stratified L -convergence space.

A function $f : X \rightarrow X'$ between two levelwise stratified L -convergence spaces $(X, \bar{q}), (X', \bar{q}')$ is called continuous if for all $\mathcal{F} \in \mathcal{F}_L^s(X)$ all $x \in X$, and all $\alpha \in L$ we have $\mathcal{F} \xrightarrow{q_\alpha} x$ implies $f^\rightarrow(\mathcal{F}) \xrightarrow{q'_\alpha} f(x)$. The category $SL\text{-}LCS$ has as objects all levelwise stratified L -convergence spaces and as morphisms the continuous functions. This category is topological over SET [20, 21]. For a given source $(X \xrightarrow{f_i} (X_i, \bar{q}_i))_{i \in I}$, the initial structure, $\bar{q}$ on X is defined by $\mathcal{F} \xrightarrow{q_\alpha} x \Leftrightarrow \forall i \in I, f_i^\rightarrow(\mathcal{F}) \xrightarrow{q'_\alpha} f_i(x) (\mathcal{F} \in \mathcal{F}_L^s(X), x \in X, \alpha \in L)$.

3. Lattice-Valued Weaker Regularities

In this section, we will present the definitions, characterizations, and properties of lattice-valued weaker regularities.

Let X be a set; a function $\phi : X \rightarrow \mathcal{F}_L^s(X)$ is usually called an L -filter select function on X . We define $\hat{\phi} : L^X \rightarrow L^X$ as $\hat{\phi}(\lambda) : X \rightarrow L, x \mapsto \phi(x)(\lambda)$. Let $\Sigma(X)$ denote the set of

all L -filter select functions on X , and let $\Sigma^*(X)$ be the subset consisting of all $\phi \in \Sigma$ such that $\phi(y) \in \mathcal{U}_L^s(X)$ for all $y \in X$.

Let $\phi \in \Sigma(X)$. For all $\mathcal{F} \in \mathcal{F}_L^s(X)$, it can be proved that the function $k_L \phi \mathcal{F} : L^X \rightarrow L$, defined by $\forall \lambda \in L^X, k_L \phi \mathcal{F}(\lambda) = \mathcal{F}(\hat{\phi}(\lambda))$, is a stratified L -filter, which is called the L -diagonal filter of $(\phi, \mathcal{F})$ [11, 17]. Then we have the following obvious lemma. It may have appeared in some other places.

Lemma 5. Let $\phi, \sigma \in \Sigma(X)$ or $\Sigma^*(X)$. Then

- (1) $\hat{\phi}(0) = 0, \hat{\phi}(1) = 1$;
- (2) for each $\lambda, \mu \in L^X, \hat{\phi}(\lambda \wedge \mu) = \hat{\phi}(\lambda) \wedge \hat{\phi}(\mu)$;
- (3) $\sigma \leq \phi$ implies $\hat{\sigma} \leq \hat{\phi}$;
- (4) for all $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, then $[\mathcal{F}, \mathcal{G}] \leq [k_L \phi \mathcal{F}, k_L \phi \mathcal{G}]$.
In particular, if $\mathcal{F} \leq \mathcal{G}$ then $k_L \phi \mathcal{F} \leq k_L \phi \mathcal{G}$.

3.1. For Generalized Stratified L -Convergence Spaces. Let $(X, \lim)$ be a generalized stratified L -convergence space. We consider the following axioms.

DLK. For each $\phi \in \Sigma(X)$, we have

$$\forall \mathcal{F} \in \mathcal{F}_L^s(X), \bigwedge_{y \in X} \lim \phi(y)(y) \leq [\lim k_L \phi \mathcal{F}, \lim \mathcal{F}]. \quad (1)$$

DLK'. Taking ϕ as $\forall y \in X, \lim \phi(y)(y) = 1$ in **DLK**.

Replacing $\mathcal{F}_L^s(X)$ by $\mathcal{U}_L^s(X)$ in **DLK** (resp., **DLK'**), we obtain a weaker axiom in symbol **DLK*** (resp., **DLK'***).

Remark 6. The axiom **DLK** is the dual axiom of **LK** which appeared in [11], and the axiom **DLK'** is the dual axiom of **LK'** which appeared in [17].

Definition 7. Let $(X, \lim)$ be a generalized stratified L -convergence space. Then $(X, \lim)$ is called k -regular (resp., k' -regular, k^* -regular, and k'^* -regular) if it satisfies the axiom **DLK** (resp., **DLK'**, **DLK***, and **DLK'***).

Lemma 8 (Li and Jin [25]). Let $\phi \in \Sigma(X)$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$. We define $\mathcal{F}^\phi : L^X \rightarrow L$ as $\mathcal{F}^\phi(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\hat{\phi}(\mu), \lambda])$. Then $\mathcal{F}^\phi$ satisfies (F1), (F2), and (Fs); thus, we say that $\mathcal{F}^\phi$ is nearly a stratified L -filter. If $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$ then $k_L \phi(\mathcal{F}^\phi) \geq \mathcal{F}$.

Lemma 9. Let $\phi \in \Sigma(X)$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$. Then $(k_L \phi \mathcal{F})^\phi \in \mathcal{F}_L^s(X)$ and $(k_L \phi \mathcal{F})^\phi \leq \mathcal{F}$.

Proof. For each $\lambda \in L^X$, we have

$$\begin{aligned} (k_L \phi \mathcal{F})^\phi(\lambda) &= \bigvee_{\mu \in L^X} (k_L \phi \mathcal{F}(\mu) \wedge [\hat{\phi}(\mu), \lambda]) \\ &= \bigvee_{\mu \in L^X} (\mathcal{F}(\hat{\phi}(\mu)) \wedge [\hat{\phi}(\mu), \lambda]) \leq \mathcal{F}(\lambda); \end{aligned} \quad (2)$$

that is, $(k_L \phi \mathcal{F})^\phi \leq \mathcal{F}$. It follows that $(k_L \phi \mathcal{F})^\phi(0) = 0$. From the above lemma we have that $(k_L \phi \mathcal{F})^\phi$ is a stratified L -filter on X . $\square$

By the above two lemmas, we get the following characteristic theorem.

Theorem 10. *Let $(X, \lim)$ be a generalized stratified L -convergence space. Then $(X, \lim)$ is k -regular (resp., k^* -regular) if and only if, for each $\phi \in \Sigma(X)$ (resp., $\phi \in \Sigma^*(X)$), $\bigwedge_{y \in X} \lim \phi(y)(y) \leq [\lim \mathcal{F}, \lim \mathcal{F}^\phi]$ whenever $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$.*

Proof. We prove only for k -regularity. Assume the given condition is satisfied, let $\phi \in \Sigma(X)$ and $\mathcal{F} \in \mathcal{F}_L^s(X)$. By Lemma 9 we have $(k_L \phi \mathcal{F})^\phi \in \mathcal{F}_L^s(X)$ and

$$\begin{aligned} \bigwedge_{y \in X} \lim \phi(y)(y) &\leq [\lim k_L \phi \mathcal{F}, \lim (k_L \phi \mathcal{F})^\phi] \\ &\leq [\lim k_L \phi \mathcal{F}, \lim \mathcal{F}], \end{aligned} \quad (3)$$

and so DLK holds; that is, $(X, \lim)$ is k -regular.

Conversely, let $\mathcal{F} \in \mathcal{F}_L^s(X)$, $\phi \in \Sigma(X)$ with $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$. By Lemma 8, $k_L \phi(\mathcal{F}^\phi) \geq \mathcal{F}$. It follows by DLK that

$$\begin{aligned} [\lim \mathcal{F}, \lim \mathcal{F}^\phi] &\geq [\lim k_L \phi(\mathcal{F}^\phi), \lim \mathcal{F}^\phi] \\ &\geq \bigwedge_{y \in X} \lim \phi(y)(y). \end{aligned} \quad (4)$$

Thus, the requirement is satisfied. $\square$

Corollary 11. *A generalized stratified L -convergence space $(X, \lim)$ is k' -regular (resp., k'^* -regular) if and only if for each $\phi \in \Sigma(X)$ (resp., $\phi \in \Sigma^*(X)$) with $\lim \phi(y)(y) = 1$ for all $y \in X$, we have $\lim \mathcal{F} \leq \lim \mathcal{F}^\phi$ whenever $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$.*

The following theorem considers lattice-valued weaker regularities w.r.t. the initial structures.

Theorem 12. *Let $(X, \lim)$ be the initial structure relative to the source $(X \xrightarrow{f_i} (X_i, \lim_i))_{i \in I}$ with each $f_i : X \rightarrow X_i$ being injective. Then if each $(X_i, \lim_i)$ is k -regular (resp., k' -regular), then the same is true of $(X, \lim)$.*

Proof. We prove only for k -regularity. Let $\phi \in \Sigma(X)$. Fix $i \in I$; define $\phi_i \in \Sigma(X_i)$ as $\phi_i(y) = [y]$ if $y \notin f_i(X)$ and $\phi_i(y) = f_i^{\Rightarrow}(\phi(f_i^{-1}(y)))$ if $y \in f_i(X)$. Then for each $i \in I$, by $\limy = 1$ it follows that

$$\begin{aligned} \bigwedge_{y \in X_i} \lim_i \phi_i(y)(y) &= \bigwedge_{y \in f_i(X)} \lim_i \phi_i(y)(y) \\ &= \bigwedge_{x \in X} \lim_i f_i^{\Rightarrow}(\phi(x))(f_i(x)). \end{aligned} \quad (5)$$

(In particular, if $\forall x \in X, \lim \phi(x)(x) = 1$, then $\forall y \in X_i, \lim_i \phi_i(y)(y) = 1$).

For each $\lambda \in L^{X_i}$ and each $x \in X$, it follows that

$$\begin{aligned} \widehat{\phi}(\lambda \circ f_i)(x) &= \phi(x)(\lambda \circ f_i) = f_i^{\Rightarrow}(\phi(x))(\lambda) \\ &= \phi_i(f_i(x))(\lambda) = \widehat{\phi}_i(\lambda)(f_i(x)). \end{aligned} \quad (6)$$

Hence, $\widehat{\phi}(\lambda \circ f_i) = \widehat{\phi}_i(\lambda) \circ f_i$, and then, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$,

$$\begin{aligned} f_i^{\Rightarrow}(k_L \phi \mathcal{F})(\lambda) &= k_L \phi \mathcal{F}(\lambda \circ f_i) = \mathcal{F}(\widehat{\phi}(\lambda \circ f_i)) \\ &= \mathcal{F}(\widehat{\phi}_i(\lambda) \circ f_i) = f_i^{\Rightarrow}(\mathcal{F})(\widehat{\phi}_i(\lambda)) \\ &= k_L \phi_i(f_i^{\Rightarrow}(\mathcal{F}))(\lambda). \end{aligned} \quad (7)$$

Therefore, $f_i^{\Rightarrow}(k_L \phi \mathcal{F}) = k_L \phi_i(f_i^{\Rightarrow}(\mathcal{F}))$. Then, for each $x \in X$,

$$\begin{aligned} \bigwedge_{y \in X} \lim \phi(y)(y) \wedge \lim k_L \phi \mathcal{F}(x) &= \bigwedge_{y \in X} \bigwedge_{i \in I} \lim_i f_i^{\Rightarrow}(\phi(y))(f_i(y)) \\ &\wedge \bigwedge_{i \in I} \lim_i f_i^{\Rightarrow}(k_L \phi \mathcal{F})(f_i(x)) \\ &= \bigwedge_{i \in I} \bigwedge_{z_i \in X_i} \lim_i \phi_i(z_i)(z_i) \wedge \bigwedge_{i \in I} \lim_i k_L \phi_i(f_i^{\Rightarrow}(\mathcal{F}))(f_i(x)) \\ &\leq \bigwedge_{i \in I} \left(\bigwedge_{z_i \in X_i} \lim_i \phi_i(z_i)(z_i) \wedge \lim_i k_L \phi_i(f_i^{\Rightarrow}(\mathcal{F}))(f_i(x)) \right) \\ &\leq \bigwedge_{i \in I} \lim_i f_i^{\Rightarrow}(\mathcal{F})(f_i(x)) = \lim \mathcal{F}(x). \end{aligned} \quad (8)$$

Here, the last inequality holds because each $(X_i, \lim_i)$ is k -regular. Now, we have proved that $(X, \lim)$ is k -regular. $\square$

The following theorem gives the relationship between types of lattice-valued weaker regularities.

Theorem 13. *Let L be a complete Boolean algebra. Then k -regularity $\Leftrightarrow k^*$ -regularity and k' -regularity $\Leftrightarrow k'^*$ -regularity.*

Proof. We check only the equivalence k -regularity $\Leftrightarrow k^*$ -regularity. The other equivalence is similar. Obviously, k -regularity $\Rightarrow k^*$ -regularity. Conversely, let $(X, \lim)$ be k^* -regular. Note that when L is a complete Boolean algebra, then for every stratified L -filter there exists a stratified L -ultrafilter containing it. Thus, for each $\phi \in \Sigma(X)$, there is some $\phi^* \in \Sigma^*$ such that $\phi(y) \leq \phi^*(y)$ for all $y \in X$. Assume that $\mathcal{F} \in \mathcal{F}_L^s(X)$ with $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$. Then it is easily seen that $\mathcal{F}^{\phi^*} \leq \mathcal{F}^\phi$ and $\mathcal{F}^{\phi^*} \in \mathcal{F}_L^s(X)$. By Theorem 10,

$$\begin{aligned} \bigwedge_{y \in X} \lim \phi(y)(y) &\leq \bigwedge_{y \in X} \lim \phi^*(y)(y) \leq [\lim \mathcal{F}, \lim \mathcal{F}^{\phi^*}] \\ &\leq [\lim \mathcal{F}, \lim \mathcal{F}^\phi]. \end{aligned} \quad (9)$$

Thus, $(X, \lim)$ is k -regular. $\square$

As a consequence, we obtain that when L is a complete Boolean algebra, Theorem 12 holds for k^* -regularity and k'^* -regularity.

Obviously, k -regularity $\Rightarrow k'$ -regularity and k^* -regularity $\Rightarrow k'^*$ -regularity. The following example shows that the reverse inclusions do not hold generally.

Example 14. Let $X = \{x, y\}$ and $L = \{0, \alpha, \beta, 1\}$ with ordering $0 < \alpha, \beta < 1$ and $\alpha \wedge \beta = 0, \alpha \vee \beta = 1$. Then $(L, \wedge)$ becomes a complete Boolean algebra. Obviously, $[x]$ and $[y]$ are all stratified L -ultrafilters on X . Thus, it is easily seen that the function $\lim : \mathcal{F}_L^s(X) \rightarrow L^X$ defined by

$$\begin{aligned} \lim \mathcal{F}(x) &= \begin{cases} 1, & \mathcal{F} = [x]; \\ \alpha, & \mathcal{F} = [y]; \\ 0, & \text{otherwise,} \end{cases} \\ \lim \mathcal{F}(y) &= \begin{cases} 1, & \mathcal{F} = [y]; \\ \beta, & \mathcal{F} = [x]; \\ 0, & \text{otherwise,} \end{cases} \end{aligned} \quad (10)$$

is a generalized stratified L -convergence structure on X .

(1) $(X, \lim)$ satisfies $DLK'(DLK'^*)$. Let $\phi \in \Sigma(X)$ with $\lim \phi(x)(x) = \lim \phi(y)(y) = 1$. Then $\phi(x) = [x], \phi(y) = [y]$. Thus, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$, we have $k_L \phi \mathcal{F} = \mathcal{F}$. Then the axiom DLK' , and thus the axiom DLK'^* holds obviously.

(2) $(X, \lim)$ does not satisfy $DLK(DLK^*)$. Let $\phi \in \Sigma(X)$ be defined by $\phi(x) = \phi(y) = [y]$. Then, for each $\lambda \in L^X$, we have $\hat{\phi}(\lambda) = \lambda(y)$. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$,

$$\begin{aligned} k_L \phi \mathcal{F}(\lambda) &= \mathcal{F}(\hat{\phi}(\lambda)) = \mathcal{F}(\lambda(y)) \stackrel{\text{tight}}{=} \lambda(y) \\ &= [y](\lambda); \end{aligned} \quad (11)$$

that is, $k_L \phi \mathcal{F} = [y]$.

Taking $\mathcal{G} = [x] \wedge [y]$, then $\lim \mathcal{G}(x) = \lim \mathcal{G}(y) = 0$, and $\lim k_L \phi \mathcal{G}(x) = \lim [y](x) = \alpha, \lim k_L \phi \mathcal{G}(y) = \lim y = 1$. It follows that

$$\alpha = \bigwedge_{z \in X} \lim \phi(z)(z) \not\leq 0 = [\lim k_L \phi \mathcal{G}, \lim \mathcal{G}]. \quad (12)$$

It follows that the axiom DLK^* and thus the axiom DLK does not hold.

3.2. For Levelwise Stratified L -Convergence Spaces. Let $(X, \bar{q})$ be a levelwise stratified L -convergence space. We consider the following axioms:

DLLK. For each $\phi \in \Sigma(X)$ and each $\alpha \in L$ with $\forall z \in X, \phi(z) \xrightarrow{q_\alpha} z$. Then $\forall \mathcal{F} \in \mathcal{F}_L^s(X), \forall x \in X, \mathcal{F} \xrightarrow{q_\alpha} x$ whenever $k_L \phi \mathcal{F} \xrightarrow{q_\alpha} x$.

Replacing $\mathcal{F}_L^s(X)$ by $\mathcal{U}_L^s(X)$ in **DLLK**, we obtain a weaker axiom in symbol DLK^* .

Remark 15. The axiom **DLLK** is a special case of the regular axiom (R2) in [23] with $J = X$ and $\psi = id$.

Definition 16. Let $(X, \bar{q})$ be a levelwise stratified L -convergence space. Then $(X, \bar{q})$ is called k -regular (resp., k^* -regular) if it satisfies the axiom **DLK** (resp., **DLK^***).

For k -regularity (k^* -regularity), we have the following characteristic theorem.

Theorem 17. Let $(X, \bar{q})$ be a levelwise stratified L -convergence space. Then $(X, \bar{q})$ is k -regular (resp., k^* -regular) if and only if for each $\mathcal{F} \in \mathcal{F}_L^s(X)$ and each $\phi \in \Sigma(X)$ (resp., $\phi \in \Sigma^*(X)$) and each $\alpha \in L$ with $\forall z \in X, \phi(z) \xrightarrow{q_\alpha} z$, we have that $\mathcal{F} \xrightarrow{q_\alpha} x$ implies $\mathcal{F}^\phi \xrightarrow{q_\alpha} x$ whenever $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$.

Proof. We prove only for k -regularity. Assume the given condition is satisfied; let $\phi \in \Sigma(X)$ satisfy the condition in **DLK** and $k_L \phi \mathcal{F} \xrightarrow{q_\alpha} x$. By Lemma 9 we have $(k_L \phi \mathcal{F})^\phi \in \mathcal{F}_L^s(X)$ and $(k_L \phi \mathcal{F})^\phi \leq \mathcal{F}$. By the given condition, we have $(k_L \phi \mathcal{F})^\phi \xrightarrow{q_\alpha} x$ and then $\mathcal{F} \xrightarrow{q_\alpha} x$. So, the axiom **DLK** holds; that is, $(X, \bar{q})$ is k -regular. Conversely, Let $\phi \in \Sigma(X)$ and $\alpha \in L$ with $\forall z \in X, \phi(z) \xrightarrow{q_\alpha} z$. Suppose that $\mathcal{F} \xrightarrow{q_\alpha} x$ and $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$. By Lemma 8, $k_L \phi(\mathcal{F}^\phi) \geq \mathcal{F}$, so, $k_L \phi(\mathcal{F}^\phi) \xrightarrow{q_\alpha} x$. It follows by **DLK** that $\mathcal{F}^\phi \xrightarrow{q_\alpha} x$ as desired. $\square$

The following theorem shows that k -regular is an initial property relative to any family of injection functions.

Theorem 18. Let $(X, \bar{q})$ be the initial structure relative to the source $(X \xrightarrow{f_i} (X_i, \bar{q}_i))_{i \in I}$ with each $f_i : X \rightarrow X_i$ being injective. If each $(X_i, \bar{q}_i)$ is k -regular, then the same is true of $(X, \bar{q})$.

Proof. Let $\phi \in \Sigma(X)$ and $\alpha \in L$ satisfy $\phi(x) \xrightarrow{q_\alpha} x$ for all $x \in X$. Fix $i \in I$; define $\phi_i \in \Sigma(X_i)$ as $\phi_i(y) = [y]$ if $y \notin f_i(X)$ and $\phi_i(y) = f_i^{-1}(\phi(f_i^{-1}(y)))$ if $y \in f_i(X)$. Then $\phi_i(y) \xrightarrow{q_\alpha} y$ for each $y \in X_i$. Indeed, if $y \notin f_i(X)$, then $\phi_i(y) = [y] \xrightarrow{q_\alpha} y$, and if $y \in f_i(X)$, then there exists an $x \in X$ such that $f_i(x) = y$ and so $\phi_i(y) = f_i^{-1}(\phi(x)) \xrightarrow{q_\alpha} f_i(x) = y$. Let $k_L \phi \mathcal{F} \xrightarrow{q_\alpha} x$. Similar to Theorem 12, we have $f_i^{-1}(k_L \phi \mathcal{F}) = k_L \phi_i(f_i^{-1}(\mathcal{F}))$ for all $i \in I$. Because each f_i is continuous, thus $k_L \phi_i(f_i^{-1}(\mathcal{F})) = f_i^{-1}(k_L \phi \mathcal{F}) \xrightarrow{q_\alpha} f_i(x)$. Then $f_i^{-1}(\mathcal{F}) \xrightarrow{q_\alpha} f_i(x)$ since each $(X_i, \bar{q}_i)$ is k -regular. It follows that $\mathcal{F} \xrightarrow{q_\alpha} x$ by the definition of initial structure. We have proved that $(X, \bar{q})$ is k -regular. $\square$

Theorem 19. Let L be a complete Boolean algebra. Then k -regularity $\Leftrightarrow k^*$ -regularity.

Proof. The proof is similar to Theorem 13 and thus it is omitted. $\square$

As a consequence, we obtain that when L is a complete Boolean algebra, then Theorem 18 holds for k^* -regularity.

The last theorem gives the relationship between k -regularity for generalized stratified L -convergence space and k -regularity for levelwise stratified L -convergence space.

Let $(X, \lim)$ be a generalized stratified L -convergence space. It is proved in [22] that the pair $(X, \bar{q}^{\lim})$, where

$\mathcal{F} \xrightarrow{(q^{\text{lim}})_\alpha} x$ if and only if $\lim \mathcal{F}(x) \geq \alpha$, is a levelwise stratified L -convergence space.

Theorem 20. Let (X, lim) be a generalized stratified L -convergence space. Then (X, lim) is k -regular (resp., k^* -regular) if and only if $(X, \overline{q^{\text{lim}}})$ is k -regular (resp., k^* -regular).

Proof. We prove only for k -regularity. Let (X, lim) be k -regular. Take $\phi \in \Sigma(X)$ and $\alpha \in L$ with $\forall z \in X, \phi(z) \xrightarrow{(q^{\text{lim}})_\alpha} z$; then we have $\alpha \leq \bigwedge_{y \in X} \lim \phi(y)(y)$. Take $\mathcal{F} \in \mathcal{F}_L^s(X)$ with $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$; then we have $\mathcal{F} \xrightarrow{(q^{\text{lim}})_\alpha} x$; that is, $\lim \mathcal{F}(x) \geq \alpha$. By Theorem 10 we obtain $\alpha \leq \bigwedge_{y \in X} \lim \phi(y)(y) \leq [\lim \mathcal{F}, \lim \mathcal{F}^\phi]$. Then $\lim \mathcal{F}(x) \geq \alpha$; that is, $\mathcal{F}^\phi \xrightarrow{(q^{\text{lim}})_\alpha} x$. It follows by Theorem 17 that $(X, \overline{q^{\text{lim}}})$ is k -regular.

Conversely, assume that $(X, \overline{q^{\text{lim}}})$ is k -regular. Let us take $\phi \in \Sigma(X)$ with $\bigwedge_{y \in X} \lim \phi(y)(y) = \alpha$ and take $\mathcal{F} \in \mathcal{F}_L^s(X)$ with $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$. Then if $\lim \mathcal{F}(x) = \beta$ for $x \in X$, we have $\phi(y) \xrightarrow{(q^{\text{lim}})_{\alpha \wedge \beta}} y$ and $\mathcal{F} \xrightarrow{(q^{\text{lim}})_{\alpha \wedge \beta}} x$. It follows by Theorem 17 that $\mathcal{F}^\phi \xrightarrow{(q^{\text{lim}})_{\alpha \wedge \beta}} x$; that is, $\lim \mathcal{F}^\phi(x) \geq \alpha \wedge \beta$. By the arbitrariness of x we note that $\bigwedge_{y \in X} \lim \phi(y)(y) = \alpha \leq [\lim \mathcal{F}, \lim \mathcal{F}^\phi]$. It follows by Theorem 10 that (X, lim) is k -regular. $\square$

4. On the Relationship between Weaker Regularity and p -Regularity

4.1. For Generalized Stratified L -Convergence Spaces. Generally, p -regularity relates to two different generalized stratified L -convergence structures on the same underlying set. Thus, in this section, we add the lowercases p, q as the superscript of lim and use $\text{lim}^p, \text{lim}^q$ to denote different generalized stratified L -convergence structures.

At first, we give the notion of closures of stratified L -filters and then introduce a new p -regularity.

Definition 21. Let (X, lim^p) be a generalized stratified L -convergence space. For each $\lambda \in L^X$, the L -set $\overline{\lambda}_p \in L^X$ defined by

$$\forall x \in X, \quad \overline{\lambda}_p(x) = \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)) \quad (13)$$

is called the closure of λ w.r.t (X, lim^p) .

Lemma 22. Let (X, lim^p) be a generalized stratified L -convergence space. Then for all $\lambda, \mu \in L^X$ and all $\alpha \in L$ we get the following:

- (1) $\lambda \leq \overline{\lambda}_p$;
- (2) $\lambda \leq \mu$ implies $\overline{\lambda}_p \leq \overline{\mu}_p$;
- (3) $\overline{(\beta \wedge \lambda)}_p \geq \beta \wedge \overline{\lambda}_p$ and the equality holds if L is a complete Boolean algebra;

- (4) if L is a complete Boolean algebra, then $\forall x \in X$, $\overline{\lambda}_p(x) = \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda))$, and $\overline{(\lambda \vee \mu)}_p = \overline{\lambda}_p \vee \overline{\mu}_p$.

Proof. (1) For each $x \in X$, by $\text{lim}^px = 1$ we get $\overline{\lambda}_p(x) \geq [x](\lambda) = \lambda(x)$. So, $\lambda \leq \overline{\lambda}_p$. Take $\lambda = 1$ in (1); we obtain $\overline{1}_p = 1$.
 (2) It follows from the property (F2) of stratified L -filters.
 (3) For each $x \in X$ we have

$$\begin{aligned} \overline{(\beta \wedge \lambda)}_p(x) &= \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\beta \wedge \lambda)) \\ &= \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\beta) \wedge \mathcal{F}(\lambda)) \\ &\geq \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{F}(x) \wedge \beta \wedge \mathcal{F}(\lambda)) \\ &= \beta \wedge \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)) \\ &= \beta \wedge \overline{\lambda}_p(x). \end{aligned} \quad (14)$$

When L is a complete Boolean algebra, then $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$, $\mathcal{F}(\beta) = \beta$. So, the “ $\geq$ ” in the above inequality can be replaced by “ $=$ ”. Thus, $\overline{(\beta \wedge \lambda)}_p = \beta \wedge \overline{\lambda}_p$.

(5) Let L be a complete Boolean algebra. That $\overline{\lambda}_p(x) = \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda))$ follows because, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$, there exists an L -ultrafilter $\mathcal{G}$ such that $\mathcal{F} \leq \mathcal{G}$. To prove $\overline{(\lambda \vee \mu)}_p = \overline{\lambda}_p \vee \overline{\mu}_p$, it suffices to check that $\overline{(\lambda \vee \mu)}_p \leq \overline{\lambda}_p \vee \overline{\mu}_p$ since the reverse inequality holds by (2). Indeed, because each stratified L -ultrafilter is prime we have

$$\begin{aligned} \overline{\lambda}_p(x) \vee \overline{\mu}_p(x) &= \left(\bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)) \right) \\ &\quad \vee \left(\bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{G}(x) \wedge \mathcal{G}(\mu)) \right) \\ &= \bigvee_{\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)} ((\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)) \\ &\quad \vee (\text{lim}^p \mathcal{G}(x) \wedge \mathcal{G}(\mu))) \\ &\geq \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} ((\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)) \\ &\quad \vee (\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\mu))) \\ &= \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{F}(x) \wedge (\mathcal{F}(\lambda) \vee \mathcal{F}(\mu))) \\ &= \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\text{lim}^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda \vee \mu)) = \overline{(\lambda \vee \mu)}_p(x). \end{aligned} \quad (15)$$

$\square$

Theorem 23. Let $(X, \lim^p)$ be a generalized stratified L -convergence space. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$, the function $\overline{\mathcal{F}}_p : L^X \rightarrow L$ defined by

$$\forall \lambda \in L^X, \quad \overline{\mathcal{F}}_p(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\overline{\mu}_p, \lambda]) \quad (16)$$

is a stratified L -filter, called the closure of $\mathcal{F}$.

Proof. (F1) That $\overline{\mathcal{F}}_p(1) = 1$ is obvious. By Lemma 22(1) we have

$$\begin{aligned} \overline{\mathcal{F}}_p(\lambda) &= \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\overline{\mu}_p, \lambda]) \\ &\leq \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\mu, \lambda]) \leq \mathcal{F}(\lambda). \end{aligned} \quad (17)$$

Thus, $\overline{\mathcal{F}}_p(0) = 0$. $\square$

(F2) Firstly, note that $\overline{\mathcal{F}}_p(\lambda) \leq \overline{\mathcal{F}}_p(\mu)$ whenever $\lambda \leq \mu$. It follows that $\overline{\mathcal{F}}_p(\lambda \wedge \mu) \leq \overline{\mathcal{F}}_p(\lambda) \wedge \overline{\mathcal{F}}_p(\mu)$. Conversely,

$$\begin{aligned} \overline{\mathcal{F}}_p(\lambda) \wedge \overline{\mathcal{F}}_p(\mu) &= \bigvee_{a \in L^X} (\mathcal{F}(a) \wedge [\overline{a}_p, \lambda]) \wedge \bigvee_{b \in L^X} (\mathcal{F}(b) \wedge [\overline{b}_p, \mu]) \\ &= \bigvee_{a, b \in L^X} (\mathcal{F}(a) \wedge \mathcal{F}(b) \wedge [\overline{a}_p, \lambda] \wedge [\overline{b}_p, \mu]) \\ &\leq \bigvee_{a, b \in L^X} (\mathcal{F}(a \wedge b) \wedge [\overline{(a \wedge b)}_p, \lambda \wedge \mu]) \\ &\leq \bigvee_{c \in L^X} (\mathcal{F}(c) \wedge [\overline{c}_p, \lambda \wedge \mu]) = \overline{\mathcal{F}}_p(\lambda \wedge \mu). \end{aligned} \quad (18)$$

(Fs) For all $\beta \in L$, it follows that $\overline{\mathcal{F}}_p(\beta) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\overline{\mu}_p, \beta]) \geq \mathcal{F}(1) \wedge \beta = \beta$ by $\overline{1}_p = 1$.

It is easily seen that the following lemma holds. We omit the routine proof.

Lemma 24. Let $(X, \lim^p)$ be a generalized stratified L -convergence space. Then, for each $\mathcal{F}, \mathcal{G} \in \mathcal{F}_L^s(X)$, $[\mathcal{F}, \mathcal{G}] \leq [\overline{\mathcal{F}}_p, \overline{\mathcal{G}}_p]$.

Definition 25. Let $(X, \lim^p, \lim^q)$ be a pair of generalized stratified L -convergence spaces. Then $(X, \lim^q)$ is called p -regular if and only if, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$, we have $\lim^q \mathcal{F} \leq \lim^q \overline{\mathcal{F}}_p$.

Remark 26. When $L = \{0, 1\}$, a generalized stratified L -convergence space reduces to a convergence space. It is easily seen that $\overline{\mathcal{F}}_p$ is precisely the filter generated by $\{\overline{A} : A \in \mathbb{F}\}$ as a filterbasis [29]. And the p -regularity reduces to the corresponding crisp notion in [3].

The following theorem shows that p -regularity is preserved under initial constructions.

Theorem 27. Let $\{(X_i, \lim^{q_i}, \lim^{p_i})\}_{i \in I}$ be pairs of generalized stratified L -convergence spaces with each $\lim^{q_i}$ being p_i -regular. If $\lim^q$ (resp., $\lim^p$) is the initial structure on X relative to the source $(X \xrightarrow{f_i} (X_i, \lim^{q_i}))_{i \in I}$ (resp., $(X \xrightarrow{f_i} (X_i, \lim^{p_i}))_{i \in I}$), then $(X, \lim^q)$ is p -regular.

Proof. At first, we check below that for each $i \in I$ and each $\lambda_i \in L^{X_i}$ we have $\overline{(f_i^{\leftarrow}(\lambda_i))}_p \leq f_i^{\leftarrow}(\overline{(\lambda_i)}_{p_i})$. Indeed, for each $x \in X$,

$$\begin{aligned} \overline{(f_i^{\leftarrow}(\lambda_i))}_p(x) &= \bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X)} (\lim^p \mathcal{G}(x) \wedge \mathcal{G}(f_i^{\leftarrow}(\lambda_i))) \\ &= \bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X)} \left(\left(\bigwedge_{j \in I} \lim^{p_j} f_j^{\rightarrow}(\mathcal{G})(f_j(x)) \right) \wedge \mathcal{G}(f_i^{\leftarrow}(\lambda_i)) \right) \\ &\leq \bigvee_{\mathcal{G} \in \mathcal{F}_L^s(X)} (\lim^{p_i} f_i^{\rightarrow}(\mathcal{G})(f_i(x)) \wedge f_i^{\rightarrow}(\mathcal{G})(\lambda_i)) \\ &\leq \bigvee_{\mathcal{G}_i \in \mathcal{F}_L^s(X_i)} (\lim^{p_i} \mathcal{G}_i(f_i(x)) \wedge \mathcal{G}_i(\lambda_i)) \\ &= f_i^{\leftarrow}(\overline{(\lambda_i)}_{p_i})(x). \end{aligned} \quad (19)$$

It follows that, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$ and each $\lambda_i \in L^{X_i}$,

$$\begin{aligned} f_i^{\rightarrow}(\overline{\mathcal{F}}_p)(\lambda_i) &= \overline{\mathcal{F}}_p(f_i^{\leftarrow}(\lambda_i)) = \bigvee_{\mu \in L^X} ([\overline{\mu}_p, f_i^{\leftarrow}(\lambda_i)] \wedge \mathcal{F}(\mu)) \\ &\geq \bigvee_{\mu_i \in L^{X_i}} ([\overline{(f_i^{\leftarrow}(\mu_i))}_p, f_i^{\leftarrow}(\lambda_i)] \wedge \mathcal{F}(f_i^{\leftarrow}(\mu_i))) \\ &\geq \bigvee_{\mu_i \in L^{X_i}} ([f_i^{\leftarrow}(\overline{(\mu_i)}_{p_i}), f_i^{\leftarrow}(\lambda_i)] \wedge \mathcal{F}(f_i^{\leftarrow}(\mu_i))) \\ &\geq \bigvee_{\mu_i \in L^{X_i}} ([\overline{(\mu_i)}_{p_i}, \lambda_i] \wedge f_i^{\rightarrow}(\mathcal{F})(\mu_i)) \\ &= \overline{(f_i^{\rightarrow}(\mathcal{F}))}_{p_i}(\lambda_i). \end{aligned} \quad (20)$$

Thus, $f_i^{\rightarrow}(\overline{\mathcal{F}}_p) \geq \overline{(f_i^{\rightarrow}(\mathcal{F}))}_{p_i}$ for all $i \in I$. It follows by each $(X_i, \lim^{q_i})$ being p_i -regular that

$$\begin{aligned} \lim^q \overline{\mathcal{F}}_p(x) &= \bigwedge_{i \in I} \lim^{q_i} f_i^{\rightarrow}(\overline{\mathcal{F}}_p)(f_i(x)) \\ &\geq \bigwedge_{i \in I} \lim^{q_i} \overline{(f_i^{\rightarrow}(\mathcal{F}))}_{p_i}(f_i(x)) \\ &\geq \bigwedge_{i \in I} \lim^{q_i} f_i^{\rightarrow}(\mathcal{F})(f_i(x)) = \lim^q \mathcal{F}(x). \end{aligned} \quad (21)$$

Thus, $(X, \lim^q)$ is p -regular. $\square$

When $L = \{0, 1\}$, Kent and Richardson [6] studied the relationships between weaker regularities and p -regularity. Now we discuss them for the general case.

Definition 28. A generalized (strong) stratified L -convergence space $(X, \lim^q)$ is called

(i) a (strong) L -Kent convergence space [10] if $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$, $\forall x \in X$, $\lim^q \mathcal{F}(x) \leq \lim^q(\mathcal{F} \wedge [x])(x)$;

(ii) pretopological [11] if $\forall \mathcal{F} \in \mathcal{F}_L^s(X)$, $\forall x \in X$, $\lim^q \mathcal{F}(x) = [\mathcal{U}_q(x), \mathcal{F}]$, where $\mathcal{U}_q(x)$, defined by $\forall \lambda \in L^X$, $\mathcal{U}_q(x)(\lambda) = \bigwedge_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\lim^q \mathcal{F}(x) \rightarrow \mathcal{F}(\lambda))$, is called the stratified neighborhood L -filter of x w.r.t. $\lim^q$, and when $(X, \lim^q)$ is a strong stratified L -convergence space, then $(X, \lim^q)$ is pretopological if and only if it satisfies $\lim^q \mathcal{U}_q(x)(x) = 1$ for all $x \in X$ [17];

(iii) ultrapretopological if it is pretopological and for each $x \in X$, there exists a stratified L -ultrafilter $\mathcal{F}_x$ such that $\mathcal{U}_q(x) = [x] \wedge \mathcal{F}_x$;

(iv) topological [11] if there exists a stratified L -topology $\mathcal{T}$ such that $\forall \lambda \in L^X$, $\forall x \in X$, we have $\mathcal{U}_q(x)(\lambda) = \text{int}(\lambda)(x)$, where $\text{int}(\lambda) = \bigvee_{\mu \in \mathcal{T}} (\mu \wedge [\mu, \lambda])$ is called the interior of λ w.r.t. $\mathcal{T}$ [11, 30].

Proposition 29. Let $(X, \lim^q)$ be a strong stratified L -Kent convergence space which is p -regular relative to every ultrapretopological generalized stratified L -convergence structure $\lim^p \leq \lim^q$. Then $(X, \lim^q)$ is k^{r*} -regular.

Proof. Let $\phi \in \Sigma^*(X)$ with $\forall y \in X$, $\lim^q \phi(y)(y) = 1$. Let $\lim^p$ be the ultrapretopological generalized stratified L -convergence structure defined by $\forall y \in X$, $\mathcal{U}_p(y) = \phi(y) \wedge [y]$. From $\phi(y) \geq \mathcal{U}_p(y)$ we have $\lim^p \phi(y)(y) = 1$. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$ with $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$, it follows that for each $\lambda \in L^X$, $\bar{\lambda}^p(y) = \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\lim^p \mathcal{F}(y) \wedge \mathcal{F}(\lambda)) \geq \phi(y)(\lambda)$, which means $\bar{\lambda}_p \geq \hat{\phi}(\lambda)$. Thus,

$$\begin{aligned} \bar{\mathcal{F}}_p(\lambda) &= \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\bar{\mu}_p, \lambda]) \\ &\leq \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\hat{\phi}(\mu), \lambda]) = \mathcal{F}^\phi(\lambda); \end{aligned} \quad (22)$$

that is, $\bar{\mathcal{F}}_p \leq \mathcal{F}^\phi$. Because $(X, \lim^q)$ is a strong L -Kent convergence space, then it follows that $\lim^q \mathcal{U}_p(y) = \lim^q(\phi(y) \wedge [y])(y) \geq \lim^q \phi(y)(y) = 1$, and so

$$\forall \mathcal{G} \in \mathcal{F}_L^s(X), \forall y \in X,$$

$$\lim^p \mathcal{G}(y) = [\mathcal{U}_p(y), \mathcal{G}] = \lim^q \mathcal{U}_p(y) \wedge [\mathcal{U}_p(y), \mathcal{G}]$$

$$\stackrel{(LC2')}{\leq} \lim^q \mathcal{G}(y). \quad (23)$$

That is, $\lim^p \leq \lim^q$. It follows by the assumption that $(X, \lim^q)$ is p -regular. Thus $\lim^q \mathcal{F}^\phi(x) \geq \lim^q \bar{\mathcal{F}}_p(x) \geq \lim^q \mathcal{F}(x)$. By Theorem 17 we know that $(X, \lim^q)$ is k^{r*} -regular. $\square$

It is easily seen that when L is a complete Boolean algebra, then the above proposition holds for k' -regularity.

Lemma 30. Let $(X, \lim^q)$ be a topological generalized stratified L -convergence space and let $\mathcal{T}$ be the stratified L -topology corresponding to $\lim^q$. Then $\mathcal{F} \geq \mathcal{U}_q(x)$ if and only if $\mathcal{F}(\mu) \geq \mathcal{U}_q(x)(\mu)$ for all $\mu \in \mathcal{T}$.

Proof. We need only to check the sufficiency. Note that to for each $\mu \in L^X$, $\mathcal{U}_q(x)(\mu) = \text{int}(\mu)(x)$ and $\mathcal{U}_q(x)(\mu) = \text{int}(\mu)(x) = \mu(x)$ if $\mu \in \mathcal{T}$ [11, 30]. It follows that, for each $\lambda \in L^X$,

$$\begin{aligned} \mathcal{F}(\lambda) &= \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\mu, \lambda]) \\ &\geq \bigvee_{\mu \in \mathcal{T}} (\mathcal{F}(\mu) \wedge [\mu, \lambda]) \geq \bigvee_{\mu \in \mathcal{T}} (\mathcal{U}_q(x)(\mu) \wedge [\mu, \lambda]) \\ &= \bigvee_{\mu \in \mathcal{T}} (\mu(x) \wedge [\mu, \lambda]) = \text{int}(\lambda)(x) = \mathcal{U}_q(x)(\lambda). \end{aligned} \quad (24) \quad \square$$

Theorem 31. Let L be a linearly order frame or let $0 \in L$ be prime. A topological generalized stratified L -convergence space $(X, \lim^q)$ is k^{r*} -regular if and only if it is p -regular for every ultrapretopological generalized stratified L -convergence structure $\lim^p \leq \lim^q$.

Proof. Note that a topological generalized stratified L -convergence space is natural a strong stratified L -Kent convergence space [17]. Then the sufficiency follows by Proposition 29. Thus, we prove only the necessity. Let $(X, \lim^q)$ be k^{r*} -regular and let $\lim^p$ be an arbitrary ultrapretopological generalized stratified L -convergence structure with $\lim^p \leq \lim^q$. Then, for each $y \in X$, there exists a $\mathcal{H}_y \in \mathcal{U}_L^s(X)$ such that $\mathcal{U}_p(y) = \mathcal{H}_y \wedge [y]$. Obviously, $\lim^p \mathcal{H}_y(y) \geq \lim^p \mathcal{U}_p(y)(y) = 1$ and then $\lim^q \mathcal{H}_y(y) = 1$ by $\lim^p \leq \lim^q$.

Let $\phi \in \Sigma^*(X)$ be defined by $\phi(y) = \mathcal{H}_y$, for all $y \in X$. Then $\lim^q \phi(y)(y) = 1$ for each $y \in X$. For each $\lambda \in \mathcal{T}$, we check below $[\bar{\lambda}_p, \hat{\phi}(\lambda)] = 1$. Here, $\mathcal{T}$ is the stratified L -topology corresponding to $\lim^q$. For each $\phi(y) \in \mathcal{U}_L^s(X)$, it follows by Lemma 1 that $\phi(y)_{\mathbb{F}_{\phi(y)}} = \phi(y)$; that is,

$$\hat{\phi}(\lambda)(y) = \phi(y)(\lambda) = \begin{cases} 1, & \text{if } \lambda \in \mathbb{F}_{\phi(y)}; \\ 0, & \text{if } \lambda \notin \mathbb{F}_{\phi(y)}. \end{cases} \quad (25)$$

Note that $[\bar{\lambda}_p, \hat{\phi}(\lambda)] = \bigwedge_{y \in \iota(\bar{\lambda}_p)} (\bar{\lambda}_p(y) \rightarrow \phi(y)(\lambda))$. For each $y \in \iota(\bar{\lambda}_p)$, it follows that $\bar{\lambda}_p(y) = \bigvee_{\mathcal{F} \in \mathcal{F}_L^s(X)} (\lim^p \mathcal{F}(x) \wedge \mathcal{F}(\lambda)) > 0$, which means that there exists an $\mathcal{F}_y \in \mathcal{F}_L^s(X)$ such that $\lim^p \mathcal{F}_y(y) > 0$ and $\mathcal{F}_y(\lambda) > 0$. Thus, $\mathcal{F}_y(1_{\lambda}) \geq \mathcal{F}_y(\lambda) > 0$. Fix $y \in \iota(\bar{\lambda}_p)$; we have $y \in \iota\lambda$ or $y \in X - \iota\lambda$.

Case 1. $y \in \iota\lambda$; that is, $\lambda(y) > 0$. Because $(X, \lim^q)$ is topological, then $\lambda(y) = \mathcal{U}_q(y)(\lambda) > 0$. From $\lim^q \phi(y)(y) = 1$,

we get $\phi(y) \geq \mathcal{U}_q(y)$ and then $\phi(y)(\lambda) > 0$; indeed, $\phi(y)(\lambda) = 1$ since $\phi(y) \in \mathcal{U}_L^s(X)$ takes values in $\{0, 1\}$.

Case 2. $y \in X - \iota\lambda$; that is, $\lambda(y) = 0$. We assume that $\phi(y)(\lambda) \neq 1$; it follows by equality (25) that $\iota\lambda \notin \mathbb{F}_{\phi(y)}$. Because $\mathbb{F}_{\phi(y)}$ is an ultrafilter on X , then $X - \iota(\lambda) \in \mathbb{F}_{\phi(y)}$ and so $\phi(y)(1_{X-\iota\lambda}) = 1$. As we have known $\lim^p \mathcal{F}_y(y) > 0$ and $(X, \lim^p)$ is ultrapretopological; hence, $\lim^p \mathcal{F}_y(y) = [\mathcal{U}_p(y), \mathcal{F}_y] > 0$, then by $\mathcal{U}_p(y)(1_{X-\iota\lambda}) = \phi(y)(1_{X-\iota\lambda}) \wedge [y](1_{X-\iota\lambda}) = 1$ it follows that $\mathcal{F}_y(1_{X-\iota\lambda}) > 0$. Now,

$$0 = \mathcal{F}_y(1_{\iota\lambda} \wedge 1_{X-\iota\lambda}) \geq \mathcal{F}_y(1_{\iota\lambda}) \wedge \mathcal{F}_y(1_{X-\iota\lambda}) > 0. \quad (26)$$

A contradiction! Thus, if $y \in X - \iota\lambda$, then $\phi(y)(\lambda) = 1$.

Combining Cases 1 and 2 we get that if $y \in \iota(\bar{\lambda}_p)$ then $\hat{\phi}(\lambda)(y) = 1$. It follows immediately that $[\bar{\lambda}_p, \hat{\phi}(\lambda)] = 1$.

Next we prove that $k_L \phi(\overline{\mathcal{U}_q(x)}_p) \geq \mathcal{U}_q(x)$. By Lemma 30, we need only to check that $k_L \phi(\overline{\mathcal{U}_q(x)}_p)(\lambda) \geq \mathcal{U}_q(x)(\lambda)$ for all $\lambda \in \mathcal{T}$. Indeed,

$$\begin{aligned} k_L \phi(\overline{\mathcal{U}_q(x)}_p)(\lambda) &= \overline{\mathcal{U}_q(x)}_p(\hat{\phi}(\lambda)) \\ &= \bigvee_{\mu \in L^X} (\mathcal{U}_q(x)(\mu) \wedge [\bar{\mu}_p, \hat{\phi}(\lambda)]) \\ &\geq \mathcal{U}_q(x)(\lambda) \wedge [\bar{\lambda}_p, \hat{\phi}(\lambda)] \\ &= \mathcal{U}_q(x)(\lambda). \end{aligned} \quad (27)$$

Then, for each $\mathcal{F} \in \mathcal{F}_L^s(X)$,

$$\begin{aligned} \lim^q \mathcal{F}(x) &= [\mathcal{U}_q(x), \mathcal{F}] \leq [\overline{\mathcal{U}_q(x)}_p, \overline{\mathcal{F}}_p] \\ &\leq [k_L \phi(\overline{\mathcal{U}_q(x)}_p), k_L \phi(\overline{\mathcal{F}}_p)] \\ &\leq [\mathcal{U}_q(x), k_L \phi(\overline{\mathcal{F}}_p)] \\ &= \lim^q k_L \phi(\overline{\mathcal{F}}_p)(x) \\ &\leq \lim^q \overline{\mathcal{F}}_p(x), \end{aligned} \quad (28)$$

where the first and the second equalities hold by the pretopologicalness of $(X, \lim^q)$, the first inequality holds by Lemma 24, the second inequality holds by Lemma 5(4), and the last inequality holds because $(X, \lim^q)$ is k'^* -regular. Then it follows that $(X, \lim^q)$ is p -regular. $\square$

Remark 32. To prove that Theorem 31 holds for k' -regularity, it seems that L must be a complete Boolean algebra. If we further assume that L is linearly ordered or $0 \in L$ is prime then $L = \{0, 1\}$. Thus, we guess that Theorem 31 holds for k' -regularity only if $L = \{0, 1\}$.

4.2. For Levelwise Stratified L -Convergence Spaces

Definition 33 (see [31]). Let $(X, \bar{p})$ be a levelwise stratified L -convergence space. For each $\lambda \in L^X$, the L -set $\bar{\lambda}_p^\alpha \in L^X$ defined by

$$\begin{aligned} \forall x \in X, \quad \bar{\lambda}_p^\alpha(x) &= \bigvee_{\mathcal{F} \in c_p^\alpha(x)} \mathcal{F}(\lambda), \\ c_p^\alpha(x) &= \left\{ \mathcal{F} \in \mathcal{F}_L^s(X) : \mathcal{F} \xrightarrow{p_\alpha} x \right\} \end{aligned} \quad (29)$$

is called α -level closure of λ w.r.t. $(X, \bar{p})$.

It is easily seen that α -level closures of L -sets have similar properties to closures of L -sets. We do not list them but use them directly.

In [20], Boustique and Richardson modified Jäger's definition [11] and introduced a notion of α -level closures of stratified L -filters. In [25], we give an equivalent characterization of Boustique and Richardson's definition. This characterization seems more simple and more intuitive. Thus, we use it as the definition of α -level closures of stratified L -filters.

Definition 34. Let $(X, \bar{p})$ be a levelwise stratified L -convergence space. For each $\alpha \in L$ and each $\mathcal{F} \in \mathcal{F}_L^s(X)$, it is easily seen that the function $\overline{\mathcal{F}}_p^\alpha : L^X \rightarrow L$, defined by $\forall \lambda \in L^X, \overline{\mathcal{F}}_p^\alpha(\lambda) = \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\bar{\mu}_p^\alpha, \lambda])$, is a stratified L -filter; then $\overline{\mathcal{F}}_p^\alpha$ is called the α -level closure of $\mathcal{F}$ w.r.t. $(X, \bar{p})$.

Definition 35 (see [24]). Let $(X, \bar{p}, \bar{q})$ be a pair of levelwise stratified L -convergence spaces. Then $(X, \bar{q})$ is called p -regular if, for each $\alpha \in L$ and each $\mathcal{F} \in \mathcal{F}_L^s(X)$, we have $\overline{\mathcal{F}}_p^\alpha \xrightarrow{q_\alpha} x$ whenever $\mathcal{F} \xrightarrow{q_\alpha} x$.

It is proved in [25] that p -regularity is preserved under initial constructions. Now, we look at the relationships between weaker regularities and p -regularity.

Definition 36. A levelwise stratified L -convergence space $(X, \bar{q})$ is called

- (i) an L -Kent convergence space if $[x] \wedge \mathcal{F} \xrightarrow{q_\alpha} x$ whenever $\mathcal{F} \xrightarrow{q_\alpha} x$;
- (ii) pretopological [23] if $\mathcal{F} \xrightarrow{q_\alpha} x$ if and only if $\mathcal{F} \geq \mathcal{U}_q^\alpha(x) = \bigwedge \{ \mathcal{F} \mid \mathcal{F} \xrightarrow{q_\alpha} x \}$;
- (iii) ultrapretopological if, for each $x \in X$ and each $\alpha \in L$, there exists a stratified L -ultrafilter $\mathcal{F}_x$ such that $\mathcal{U}_q^\alpha(x) = [x] \wedge \mathcal{F}_x$;
- (iv) topological [23] if there exists a stratified L -topology $\mathcal{T}_\alpha$ for each $\alpha \in L$ such that $\forall \lambda \in L^X, \forall x \in X$, we have $\mathcal{U}_q^\alpha(x)(\lambda) = \text{int}^\alpha(\lambda)(x)$, where $\text{int}^\alpha(\lambda)$ is the interior of λ w.r.t. $\mathcal{T}_\alpha$.

Proposition 37. Let $(X, \bar{q})$ be a levelwise stratified L -Kent convergence space which is p -regular relative to every ultrapre-topological levelwise stratified L -convergence structure $\bar{p} \geq \bar{q}$. Here for $\bar{p} \geq \bar{q}$, we mean that $\mathcal{F} \xrightarrow{p_\alpha} x$ implies $\mathcal{F} \xrightarrow{q_\alpha} x$. Then $(X, \bar{q})$ is k^* -regular.

Proof. Let $\phi \in \Sigma^*(X)$ and $\alpha \in L$ with $\forall y \in X, \phi(y) \xrightarrow{q_\alpha} y$. Let $\bar{p}$ be the ultrapre-topological levelwise stratified L -convergence structure defined by $\forall \alpha \in L, \forall y \in X, \mathcal{U}_p^\alpha(y) = \phi(y) \wedge [y]$. From $\phi(y) \geq \mathcal{U}_p^\alpha(y)$ we have $\phi(y) \xrightarrow{p_\alpha} y$. For each $\mathcal{F} \in \mathcal{F}_L^s(X)$ such that $\mathcal{F}^\phi \in \mathcal{F}_L^s(X)$ and $\mathcal{F} \xrightarrow{q_\alpha} x$, it follows that for each $\lambda \in L^X$, $\bar{\lambda}_p^\alpha(y) = \bigvee_{\mathcal{F} \in c_p^\alpha(y)} \mathcal{F}(\lambda) \geq \phi(y)(\lambda)$, which means $\bar{\lambda}_p^\alpha \geq \hat{\phi}(\lambda)$. Thus,

$$\begin{aligned} \bar{\mathcal{F}}_p^\alpha(\lambda) &= \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\bar{\mu}_p^\alpha, \lambda]) \\ &\leq \bigvee_{\mu \in L^X} (\mathcal{F}(\mu) \wedge [\hat{\phi}(\mu), \lambda]) = \mathcal{F}^\phi(\lambda); \end{aligned} \quad (30)$$

that is, $\bar{\mathcal{F}}_p^\alpha \leq \mathcal{F}^\phi$. Because $(X, \bar{q})$ is an L -Kent convergence space, then it follows by $\phi(y) \xrightarrow{q_\alpha} y$ that $\mathcal{U}_p^\alpha(y) = \phi(y) \wedge [y] \xrightarrow{q_\alpha} y$. Thus, $\bar{p} \geq \bar{q}$; then $(X, \bar{q})$ is p -regular by the assumption. It follows that $\bar{\mathcal{F}}_p^\alpha \xrightarrow{q_\alpha} x$ and then $\mathcal{F}^\phi \xrightarrow{q_\alpha} x$ by $\bar{\mathcal{F}}_p^\alpha \leq \mathcal{F}^\phi$. By Theorem 17 we know that $(X, \bar{q})$ is k^* -regular. $\square$

It is easily seen that when L is a complete Boolean algebra, then the above proposition holds for k -regularity.

Lemma 38. Let $(X, \bar{q})$ be a topological levelwise stratified L -convergence space and let $\mathcal{T}_\alpha$ ($\alpha \in L$) be the stratified L -topologies corresponding to $\bar{q}$. Then $\mathcal{F} \geq \mathcal{U}_q^\alpha(x)$ if and only if $\mathcal{F}(\mu) \geq \mathcal{U}_q^\alpha(x)(\mu)$ for all $\mu \in \mathcal{T}_\alpha$.

Proof. The proof is similar to Lemma 30 and thus it is omitted. $\square$

Theorem 39. Let L be a linearly order frame or let $0 \in L$ be prime. A topological levelwise stratified L -convergence space $(X, \bar{q})$ is k^* -regular if and only if it is p -regular for every ultrapre-topological levelwise stratified L -convergence structure $\bar{p} \geq \bar{q}$.

Proof. The sufficiency follows by Proposition 37. We prove only the necessity. Let $(X, \bar{q})$ be k^* -regular and let $\bar{p}$ be an arbitrary ultrapre-topological levelwise stratified L -convergence structure with $\bar{p} \geq \bar{q}$. Fix $\alpha \in L$; then, for each $y \in X$, there exists a $\mathcal{H}_y \in \mathcal{U}_L^s(X)$ such that $\mathcal{U}_p^\alpha(y) = \mathcal{H}_y \wedge [y]$. Obviously, $\mathcal{H}_y \xrightarrow{p_\alpha} y$ and then $\mathcal{H}_y \xrightarrow{q_\alpha} y$ by $\bar{p} \geq \bar{q}$.

Let $\phi \in \Sigma^*(X)$ be defined by $\phi(y) = \mathcal{H}_y$, for all $y \in X$. For each $\lambda \in \mathcal{T}_\alpha$, we check below $[\bar{\lambda}_p^\alpha, \hat{\phi}(\lambda)] = 1$. Here, $\mathcal{T}_\alpha$ ($\alpha \in L$) are the stratified L -topologies corresponding to $\bar{q}$.

Note that $[\bar{\lambda}_p^\alpha, \hat{\phi}(\lambda)] = \bigwedge_{y \in \iota(\bar{\lambda}_p^\alpha)} (\bar{\lambda}_p^\alpha(y) \rightarrow \phi(y)(\lambda))$. For each $y \in \iota(\bar{\lambda}_p^\alpha)$, it follows that $\bar{\lambda}_p^\alpha(y) = \bigvee_{\mathcal{F} \in c_p^\alpha(y)} \mathcal{F}(\lambda) > 0$, which means that there exists an $\mathcal{F}_y \xrightarrow{p_\alpha} y$ such that $\mathcal{F}_y(\lambda) > 0$. Thus, $\mathcal{F}_y(1_{\iota\lambda}) \geq \mathcal{F}_y(\lambda) > 0$. Fix $y \in \iota(\bar{\lambda}_p^\alpha)$; then $y \in \iota\lambda$ or $y \in X - \iota\lambda$.

Case 1. $y \in \iota\lambda$; that is, $\lambda(y) > 0$. Because $(X, \bar{q})$ is topological, thus $\lambda(y) = \mathcal{U}_q^\alpha(y)(\lambda) = \bigwedge \{\mathcal{F}(\lambda) \mid \mathcal{F} \xrightarrow{q_\alpha} y\} > 0$. From $\phi(y) \xrightarrow{q_\alpha} y$, we get $\phi(y)(\lambda) > 0$; indeed, $\phi(y)(\lambda) = 1$ since $\phi(y) \in \mathcal{U}_L^s(X)$ takes values in $\{0, 1\}$.

Case 2. $y \in X - \iota\lambda$; that is, $\lambda(y) = 0$. We assume that $\phi(y)(\lambda) \neq 1$; it follows by equality (25) that $\iota\lambda \notin \mathbb{F}_{\phi(y)}$. Because $\mathbb{F}_{\phi(y)}$ is an ultrafilter on X , then $X - \iota\lambda \in \mathbb{F}_{\phi(y)}$ and so $\phi(y)(1_{X-\iota\lambda}) = 1$. As we have known $\mathcal{F}_y \xrightarrow{p_\alpha} y$; hence, $\mathcal{F}_y \geq \mathcal{U}_p^\alpha(y) = \phi(y) \wedge [y]$; then $\mathcal{F}_y(1_{X-\iota\lambda}) \geq \phi(y)(1_{X-\iota\lambda}) \wedge 1_{X-\iota\lambda}(y) = 1$. Now,

$$\begin{aligned} 0 &= \mathcal{F}_y(1_{\iota\lambda} \wedge 1_{X-\iota\lambda}) \\ &\geq \mathcal{F}_y(1_{\iota\lambda}) \wedge \mathcal{F}_y(1_{X-\iota\lambda}) = \mathcal{F}_y(1_{\iota\lambda}) > 0. \end{aligned} \quad (31)$$

A contradiction! Thus, if $y \in X - \iota\lambda$, then $\phi(y)(\lambda) = 1$.

Combining of Cases 1 and 2 we get that if $y \in \iota(\bar{\lambda}_p^\alpha)$ then $\hat{\phi}(\lambda)(y) = 1$. It follows immediately that $[\bar{\lambda}_p^\alpha, \hat{\phi}(\lambda)] = 1$. Then similar to Lemma 30 we have $k_L\phi(\bar{\mathcal{U}}_q^\alpha(x)) \geq \mathcal{U}_q^\alpha(x)$. Let $\mathcal{F} \xrightarrow{q_\alpha} x$; then $\mathcal{F} \geq \mathcal{U}_q^\alpha(x)$ by the topologicalness of $\bar{q}$. Hence, $\bar{\mathcal{F}}_p^\alpha \geq \mathcal{U}_q^\alpha(x)_p^\alpha$ and then $k_L\phi(\bar{\mathcal{F}}_p^\alpha) \geq k_L\phi(\mathcal{U}_q^\alpha(x)_p^\alpha) \geq \mathcal{U}_q^\alpha(x)$, which means $k_L\phi(\bar{\mathcal{F}}_p^\alpha) \xrightarrow{q_\alpha} x$. Because $(X, \bar{q})$ is k^* -regular, then $\bar{\mathcal{F}}_p^\alpha \xrightarrow{q_\alpha} x$. It follows that $(X, \bar{q})$ is p -regular. $\square$

Remark 40. Similar to Remark 32, we guess that Theorem 39 holds for k -regularity only if $L = \{0, 1\}$.

5. Conclusions

In this paper, we introduce some weaker regularities for levelwise stratified L -convergence spaces and generalized stratified L -convergence spaces and study their characterizations and properties. For generalized stratified L -convergence spaces, we also investigate a notion of closures of stratified L -filters and then define by it a new p -regularity which is different from the p -regularity in [25] defined by the notion of α -level closures of stratified L -filters. At last, we discuss the relationships between weaker regularities and p -regularities. In addition, it seems that the p -regularity (for generalized stratified L -convergence spaces in [25]) has close relationships with k -regularity and k^* -regularity. But we fail to establish those relationships for it is difficult to find an appropriate definition for ultrapre-topological generalized stratified L -convergence spaces.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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