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Research Article

Convertible Bonds with Higher Loan Rate: Model, Valuation, and Optimal Strategy

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We study the pricing problem for convertible bonds via backward stochastic differential equations (BSDEs). By virtue of reflected BSDEs and Malliavin derivatives, we establish the formulae for the fair price of convertible bonds and the hedging portfolio strategy explicitly. We also obtain the optimal conversion time when there is no dividends-paying for underlying common stocks. Furthermore, we consider the case that the loan rate is higher than riskless interest rate in a financial market, and conclude that it does not affect the price of convertible bonds actually. To illustrate our results, some numerical simulations are given and discussed at last.

1. Introduction

After it was first issued by American NEW YORK ERIE Company in 1843, the convertible bond is becoming one of the most important financial instruments for companies to raise capital fund nowadays. Generally speaking, a convertible bond is a kind of financial derivatives that gives holders the right to convert it to a specified number of shares of common stocks by forgoing future coupon and principal payments. Though a convertible bond is a hybrid security consisting of a straight bond and a call on the underlying stocks formally, various characteristics make it impossible to decouple the stock option from the riskless part. Therefore, how to price convertible bonds fairly attracts the interests of worldwide economists and mathematicians.

Theoretical study for the fair prices of convertible bonds first appeared in the 1960s. The main idea is that the price of convertible bonds should be equal to the present discounting of the maximum of its value as an ordinary bond or its value in common stocks (after conversion) at some time point in the future. This method or a slight modification thereby was employed by Poensgen [1, 2], Baumol et al. [3], Weil et al. [4], and so on. Later, the prices of convertible bonds are evaluated by the celebrated Black-Scholes formula as contingent claims on firm values, since the fundamental paper worked by Black and Scholes [5] for pricing financial derivatives was published in 1973. There is also rich literature along this line, for example, Ingersoll Jr. [6], Brennan and Schwartz [7, 8], in which authors took firm values as variables that determine the prices of convertible bonds, while in McConnell and Schwartz [9], Ho and Pfeffer [10] and Tsiveriotis and Fernandes [11], a convertible bond is viewed and valued as a derivative of the underlying equity, which is commonly the stocks of issuing firm.

However, all models mentioned above attempted to give convertible bonds fair prices by solving some partial differential equations (PDEs), which are originally developed by Black and Scholes [5]. As we know, nonlinear backward stochastic differential equations (BSDEs), which was introduced by Pardoux and Peng [12] and Duffie and Epstein [13] independently, is another powerful tool to price contingent claims. For a BSDE coupled with a forward SDE, Peng [14] gave a probabilistic interpretation for a large kind of the second order quasilinear partial differential equations. This result generalized the well-known Feynman-Kac formula to a nonlinear case. El Karoui et al. [15] gave some important properties of BSDEs and their applications to optimal controls and financial mathematics, such as

European option pricing problem in the constraint case. They also investigated Malliavin derivatives of solutions to BSDEs, which is a derivative defined in a weak sense. Since the price of convertible bonds should be always greater than the conversion value, it corresponds to the solution of a new type of backward equation called reflected BSDEs. An increasing process is introduced to keep the solution staying above a given stochastic process, called the obstacle. Bielecki et al. [16] employed doubly reflected BSDEs to price convertible bonds via decomposing them into bond components and option components. Throughout this paper, however, we attempt to evaluate convertible bonds by taking them as whole contingent claims and introduce a risk neutral measure under which the prices of convertible bonds are equal to the supreme discounted expected value of future payoff. In fact, the existence and uniqueness of such a measure is one of the most important reasons that we can adapt BSDEs method for pricing purpose. Moreover, inspired by El Karoui et al. [17], we also discuss the case that the loan rate is higher than riskless interest rate, which is never dealt with before. Moreover, to validate the theory proposed in this paper, we do some numerical simulations. The computation is closely dependent on probabilistic or analytic representation of solutions to BSDEs, whereas, in another paper [18], the authors focused on evaluating the corresponding PDEs via some numerical methods, in order to give out the prices of convertible bonds. Therefore, our approach is different from [18] and has distinctive features.

The rest of this paper is organized as follows. We introduce some key characteristics of a convertible bond and present some properties of BSDEs, as well as reflected BSDEs, and the Malliavin calculus in Section 2. In Section 3, we formulate the pricing model for convertible bonds and give formulae for the price and portfolio strategy. Moreover, we obtain an important fact of convertible bonds related to the optimal conversion time. In Section 4, we study the problem with higher loan rate by virtue of properties of convex reflected BSDEs. Some numerical simulations with constant coefficients are illustrated in Section 5. The last section is devoted to conclude the novelty of this paper and discuss the future research work in this field.

2. Preliminaries

In this section, let us first describe convertible bonds, and then recall the BSDEs, including its Malliavin calculus. Moreover, we will introduce the reflected BSDEs and their properties as preliminaries of solving our problem.

2.1. A Convertible Bond Indenture Agreement. Generally speaking, a convertible bond indenture agreement declares theexpiry date T before which a holder can convert the bond to a specified number of common stocks. The number of shares of common stocks that can be obtained upon the surrender of one share of convertible bond is specified by conversion ratio C. Otherwise, say, a holder never exercises the convertible bond; he can get the aggregate balloon payment which is equal toface price F according to the agreement at date T. Besides, it usually contains put term that allows an investor to choose holding the convertible bond or putting it to the issuer for a specified put value on each prefixed put date and call term that, on the prefixed call date, if the issuer wants to call the convertible bonds, an investor must elect to receive either the cash call price or the conversion value of convertible bonds.

2.2. Backward Stochastic Differential Equations. Let $(\Omega, \mathscr{F}, P)$ be a completed probability space endowed with filtration $\{\mathscr{F}_t; 0 \le t \le T\}$, which is generated by a d-dimensional standard Brownian motion $\{W_t, 0 \le t \le T\}$ defined on the space and satisfies the usual conditions. Denote $|\cdot|$ as the Euclidean norm on $\mathbb{R}^n$. For convenience, we use the following notations throughout this paper:

$$\begin{aligned}
&\mathscr{L}^2(\mathscr{F}_T, \mathbb{R}^m) \\
&\quad = \{\xi \text{ is a } \mathbb{R}^m\text{-valued } \mathscr{F}_T\text{-measurable random variable} \\
&\qquad \text{s.t. } \mathbb{E}\left(|\xi|^2\right) < +\infty\}, \\
&\mathscr{L}^2_{\mathscr{F}}(0, T; \mathbb{R}^m) \\
&\quad = \left\{ \{\varphi_t, 0 \le t \le T\} \text{ is a } \mathbb{R}^m\text{-valued adapted process} \right. \\
&\qquad \left. \text{s.t. } \mathbb{E}\left(\int_0^T |\varphi_t|^2 dt\right) < +\infty \right\}, \\
&\mathscr{S}^2_{\mathscr{F}}(0, T; \mathbb{R}^m) \\
&\quad = \left\{ \{\psi_t, 0 \le t \le T\} \text{ is a } \mathbb{R}^m\text{-valued adapted process} \right. \\
&\qquad \left. \text{s.t. } \mathbb{E}\left(\sup_{0 \le t \le T} |\psi_t|^2\right) < +\infty \right\}.
\end{aligned} \tag{1}$$

Consider a 1-dimensional BSDE:

$$\begin{aligned}
-dY_t &= f(t, Y_t, Z_t)\, ds - Z_t dW_t, \\
Y_T &= \xi,
\end{aligned} \tag{2}$$

where the data

$$\begin{aligned}
&\xi : \Omega \longrightarrow \mathbb{R}, \\
&f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \longrightarrow \mathbb{R}
\end{aligned} \tag{3}$$

satisfy the following standard conditions.

Assumption 1. We assume that

(1) $\xi \in \mathscr{L}^2(\mathscr{F}_T, \mathbb{R})$;

(2) $f(\cdot, y, z) \in \mathscr{L}^2_{\mathscr{F}}(0, T; \mathbb{R})$ for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, and there exists a constant $K > 0$ such that

$$\begin{aligned}
&\left|f(t, y, z) - f(t, y', z')\right| \le K\left(\left|y - y'\right| + \left|z - z'\right|\right), \quad \text{a.s.}, \\
&\qquad \forall (y, z), (y', z') \in \mathbb{R} \times \mathbb{R}^d.
\end{aligned} \tag{4}$$

Thus, (1) admits a unique solution pair $\{(Y_t, Z_t), 0 \le t \le T\} \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}) \times \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ from Pardoux and Peng [12]. Here let us recall the comparison theorem for BSDEs in El Karoui et al. [15], which will be used repeatedly in the sequel.

Theorem 2. *Let (ξ^1, f^1) and (ξ^2, f^2) be two standard data of BSDEs, and let (Y^1, Z^1) and (Y^2, Z^2) be the associated square-integrable solutions. We suppose that*

(i) $\xi^1 \ge \xi^2$, dP *a.s.;*

(ii) $f^1(t, Y^2_t, Z^2_t) \ge f^2(t, Y^2_t, Z^2_t)$, $dP \otimes dt$ *a.s., a.e.*

Then we have that a.s. for any time $t \in [0,T]$, $Y^1_t \ge Y^2_t$.

Moreover, [15] also present the Malliavin derivatives of solutions to BSDEs. Denote by $\mathcal{S}$ the space consisting of random variables in the form

$$\xi = \varphi\left(W\left(h^1\right), \ldots, W\left(h^k\right)\right), \tag{5}$$

where $\varphi \in C^\infty_b(\mathbb{R}^k, \mathbb{R})$, $h^1, \ldots, h^k \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^d)$ and $W(h^i) = \int_0^T h^i_s dW_s$. Define the Malliavin derivative on $\mathcal{S}$ as

$$D_\theta \xi = \sum_{j=1}^k \partial_{x_j} \varphi\left(W\left(h^1\right), \ldots, W\left(h^k\right)\right) h^j_\theta, \quad 0 \le \theta \le T. \tag{6}$$

Nualart [19] proved that the operator D on $\mathcal{S}$ has a closed extension to the space $\mathbb{D}_{1,2}$, the closure of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathbb{D}_{1,2}}$, which is defined by

$$\|\xi\|_{\mathbb{D}_{1,2}} = \left\{\mathbb{E}\left[|\xi|^2 + \int_0^T |D_\theta \xi|^2 d\theta\right]\right\}^{1/2}. \tag{7}$$

We denote by $D^i_\theta \xi$, $1 \le i \le d$ the i-th component of $D_\theta \xi$.

Let $\mathbb{L}^a_{1,2}(0,T;\mathbb{R}^m)$ be the set of $\mathbb{R}^m$-valued progressively measurable processes $\{u(t,\omega), 0 \le t \le T, \omega \in \Omega\}$ such that

(1) for all $t \in [0,T]$, $u(t,\cdot) \in (\mathbb{D}_{1,2})^m$;

(2) $(t,\omega) \to Du(t,\omega) \in (\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}))^{d\times m}$ admits a progressively measurable version;

(3) $\|u\|^2_{\mathbb{L}^a_{1,2}} = \mathbb{E}[\int_0^T |u(t)|^2 dt + \iint_0^T |D_\theta u(t)|^2 d\theta\, dt] < +\infty$.

Under these notions, for BSDE (2) with standard data, we make the following assumption in addition.

Assumption 3. Consider

(1) $\xi \in \mathbb{D}_{1,2}$ and $\mathbb{E}(\int_0^T |D_\theta \xi|^2 d\theta) < +\infty$.

(2) f is continuously differentiable in (y,z) and its partial derivatives are bounded.

(3) For each (y,z), $f(\cdot, y, z)$ is in $\mathbb{L}^a_{1,2}(0,T;\mathbb{R})$ with the Malliavin derivative denoted by $D_\theta f(t,y,z)$; there exists a constant $L > 0$ such that

$$\left|D_\theta f\left(t, y^1, z^1\right) - D_\theta f\left(t, y^2, z^2\right)\right| \le L\left(\left|y^1 - y^2\right| + \left|z^1 - z^2\right|\right),$$
$$\forall \left(y^1, z^1\right), \left(y^2, z^2\right) \in \mathbb{R} \times \mathbb{R}^d, \quad t \in [0,T]. \tag{8}$$

(4) The solution (Y_t, Z_t) satisfies

$$\mathbb{E}\iint_0^T \left|D_\theta f\left(t, Y_t, Z_t\right)\right|^2 d\theta\, dt < +\infty. \tag{9}$$

Then we have a relationship between the first component $\{Y_t, 0 \le t \le T\}$ and the second one $\{Z_t, 0 \le t \le T\}$ of solution pair to BSDE (2), which is described by the following theorem.

Theorem 4. *Let Assumption 3 hold. For each $1 \le i \le d$, a version of $\{(D^i_\theta Y_t, D^i_\theta Z_t), 0 \le \theta, t \le T\}$ is given by*

$$D^i_\theta Y_t = 0, \quad D^i_\theta Z_t = 0, \quad 0 \le t < \theta \le T;$$
$$D^i_\theta Y_t = D^i_\theta \xi + \int_t^T \left[\partial_y f\left(s, Y_s, Z_s\right) D^i_\theta Y_s + \partial_z f\left(s, Y_s, Z_s\right) D^i_\theta Z_s + D^i_\theta f\left(s, Y_s, Z_s\right)\right] ds - \int_t^T D^i_\theta Z_s dW_s,$$
$$0 \le \theta \le t \le T. \tag{10}$$

Moreover, $\{D_t Y_t, 0 \le t \le T\}$ defined by (10) is a version of $\{Z_t, 0 \le t \le T\}$.

2.3. Reflected Backward Stochastic Differential Equations. A reflected BSDE is a special kind of BSDEs, where the solution is forced to stay above a given stochastic process, called the obstacle. Let us introduce a 1-dimensional reflected BSDE with the "obstacle" $\{S_t, 0 \le t \le T\}$:

$$Y_t = \xi + \int_t^T f\left(s, Y_s, Z_s\right) ds + K_T - K_t - \int_t^T Z_s dW_s, \tag{11}$$

where the standard data (ξ, f, S) satisfies the same conditions as that in Assumption 1, and $\{S_t, 0 \le t \le T\}$ is a $\mathbb{R}$-valued progressively measurable continuous process satisfying $\mathbb{E}(\sup_{0\le t\le T}(S^+_t)^2) < \infty$, and $S_T \le \xi$ a.s.

In El Karoui et al. [17], the authors proved the existence and uniqueness of solution triple $\{(Y_t, Z_t, K_t), 0 \le t \le T\}$ to reflected BSDE (11), and they also gave some properties of reflected BSDEs. Now we emphasize one of them which announces that the square-integrable solution $\{Y_t, 0 \le t \le T\}$ corresponds to the value function of an optimal stopping time problem.

Proposition 5. *Let $\{(Y_t, Z_t, K_t), 0 \le t \le T\}$ be the solution of (11). Then for each $t \in [0,T]$,*

$$Y_t = \operatorname{ess\,sup}_{\nu \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\nu f\left(s, Y_s, Z_s\right) ds + S_\nu \mathbf{1}_{\{\nu<T\}} + \xi \mathbf{1}_{\{\nu=T\}} \mathcal{F}_t\right], \tag{12}$$

where $\mathcal{T}$ is the set of all stopping times dominated by T, and

$$\mathcal{T}_t = \{\nu \in \mathcal{T}; t \le \nu \le T\}. \tag{13}$$

Moreover, the optimal stopping time is $D_t = \inf\{t \le u \le T : Y_u = S_u\}$ *with the convention that* $D_t = T$ *if* $Y_u > S_u$, $t \le u \le T$.

We give some further explanation of $\{K_t\}$, a particular component of solution triple to reflected BSDEs, which is a continuous and increasing process such that $K_0 = 0$ and

$$\int_0^T (Y_t - S_t)\, dK_t = 0. \tag{14}$$

Intuitively, dK_t/dt represents the amount of "push upward" that we add to $-dY_t/dt$, so that $\{Y_t\}$ keeps above the "obstacle" $\{S_t\}$. Equation (14) says that the push is minimal, in the sense that we push only when the constraint is saturated, that is, when $Y_t = S_t$.

3. Pricing Formula for Convertible Bonds

In this section, we formulate the pricing model for convertible bonds and give the optimal conversion time which plays a key role in the following discussion.

Consider two kinds of assets in our model: one is the bank account, whose process is

$$dB_t = r_t B_t dt, \tag{15}$$

and the other is the stock of a company with price process

$$dP_t = \mu_t P_t dt + \sigma_t P_t dW_t. \tag{16}$$

Here, r_t, μ_t, and σ_t are the riskless interest rate, the expected interest rate, and volatility rate of stocks, respectively, and $\{W_t, t \ge 0\}$ is a 1-dimensional standard Brownian motion under probability measure P. From now on, $\{\mathcal{F}_t\}$ stands for the natural filtration generated by this Brownian motion.

We also assume that the financial market as well as convertible bonds satisfies the following.

Assumption 6. In the financial market,

(1) it is perfect with no transactions costs, no taxes, and equal access to information for all investors;

(2) there are no dividend payments or other disbursements to common stockholders;

(3) the convertible bonds are not allowed to be called or putted, and the issuer will not default;

(4) the convertible bonds can be converted at any time before maturity date T, and the conversion ratio C is a constant;

(5) the riskless interest rate r_t, expected interest rate μ_t, and volatility rate σ_t of stocks are all deterministic bounded functions with respect to t; σ_t is invertible and the inverse σ_t^{-1} is also bounded.

Thus inspired by El Karoui et al. [20], for each time t, pricing a convertible bond is the choice of stopping time $\nu \in \mathcal{T}_t$ with payoff CP_ν on exercise if $\nu < T$ and $\xi := \max(CP_T, F)$ if $\nu = T$, under the constraint that the price at time t should be no less than CP_t. Denote

$$\widetilde{P}_s = CP_s \mathbf{1}_{\{s \le T\}} + \xi \mathbf{1}_{\{s = T\}}. \tag{17}$$

Then for any given $t \in [0, T]$ and stopping time $\nu \in \mathcal{T}_t$, there exists unique hedging portfolio strategy $(X_s(\nu, \widetilde{P}_\nu), \pi_s(\nu, \widetilde{P}_\nu)) \in \mathcal{S}^2_{\mathcal{F}} \times \mathcal{L}^2_{\mathcal{F}}$, denoted also by (X^ν_s, π^ν_s), that replicates $\widetilde{P}_\nu$. In fact, it corresponds to a classical BSDE associated with the terminal time ν and terminal value $\widetilde{P}_\nu$:

$$\begin{aligned} dX^\nu_s &= \left[r_s X^\nu_s + (\mu_s - r_s)\pi^\nu_s\right] ds + \sigma_s \pi^\nu_s dW_s, \\ X^\nu_\nu &= \widetilde{P}_\nu. \end{aligned} \tag{18}$$

Set

$$Z^\nu_s = \sigma_s \pi^\nu_s, \qquad \theta_s = \sigma_s^{-1}(\mu_s - r_s). \tag{19}$$

Thus (18) can be rewritten as

$$\begin{aligned} -dX^\nu_s &= -\left(r_s X^\nu_s + \theta_s Z^\nu_s\right) ds - Z^\nu_s dW_s, \\ X^\nu_\nu &= \widetilde{P}_\nu. \end{aligned} \tag{20}$$

Then, for rationality and fairness, the price of convertible bond at time t is given by a right-continuous adapted process $\{X_t, 0 \le t \le T\}$ satisfying

$$\begin{aligned} X_t &= \operatorname{ess\,sup}_{\nu \in \mathcal{T}_t} X_t\left(\nu, \widetilde{P}_\nu\right) \\ &= \operatorname{ess\,sup}_{\nu \in \mathcal{T}_t} \mathbb{E}\left[\int_t^\nu -\left(r_s X^\nu_s + \theta_s Z^\nu_s\right) ds + \widetilde{P}_\nu \mid \mathcal{F}_t\right]. \end{aligned} \tag{21}$$

By Proposition 5, it follows that the price process $\{X_t, 0 \le t \le T\}$ coincides with the solution of a reflected BSDE.

Theorem 7. *Let Assumption 6 hold. Then the price process* $\{X_t, 0 \le t \le T\}$ *of convertible bond satisfies the following reflected BSDE with "obstacle"* $\{CP_t, 0 \le t \le T\}$:

$$X_t = \xi - \int_t^T \left(r_s X_s + \theta_s Z_s\right) ds + K_T - K_t - \int_t^T Z_s dW_s. \tag{22}$$

Moreover, the stopping time $D_t = \inf\{t \le s \le T : X_s = CP_s\}$ is the execution time of convertible bond; that is,

$$X_t = \operatorname{ess\,sup}_{\nu \in \mathcal{T}_t} X_t\left(\nu, \widetilde{P}_\nu\right) = X_t\left(D_t, \widetilde{P}_{D_t}\right). \tag{23}$$

Generally, a triple $\{(x_t, z_t, k_t), 0 \le t \le T\}$ satisfying (22) with $x_t \ge CP_t$, $0 \le t \le T$ (but not necessarily $\int_0^T (x_t - CP_t) dk_t = 0$) is called a superhedging strategy for the convertible bond (CP_t, ξ). Consequently, the price X_t is equal to the so-called upper price defined as the smallest of the superhedging strategies for (CP_t, ξ). Moreover, the continuous and increasing process $\{K_t, 0 \le t \le T\}$ indicates the least amount of wealth that is needed extra in order to keep $X_t \ge CP_t$, $0 \le t \le T$. From the viewpoint of finance,

$\{K_t\}$ can be interpreted as cumulative consumption process during the hedging, and from (23), it may happen to consume only after conversion.

Recall that for any American call option without dividends-paying, the optimal execution time is always the maturity date. So its price is equal to that of a corresponding European call option. In fact, the convertible bonds possess a similar property.

Theorem 8. *Let Assumption 6 hold. Then the fair price of convertible bond is given by*

$$\begin{aligned} X_t &= \mathbb{E}\left[\int_t^T -\left(r_s X_s + \theta_s Z_s\right) ds + \xi \mid \mathcal{F}_t\right] \\ &= \mathbb{E}_{\overline{P}}\left[e^{-\int_t^T r_s ds}\xi \mid \mathcal{F}_t\right], \end{aligned} \tag{24}$$

where $\overline{P}$ is a risk neutral measure defined by

$$\frac{d\overline{P}}{dP} = \exp\left\{-\int_0^T \frac{\mu_t - r_t}{\sigma_t} dW_t - \frac{1}{2}\int_0^T \frac{\left(\mu_t - r_t\right)^2}{\sigma_t^2} dt\right\}. \tag{25}$$

Moreover, the optimal execution time for convertible bond is exactly the maturity date T; that is, the convertible bonds should not be converted in advance.

Proof. We begin with the first equality of (24). Noting (22) and $\{X_t, 0 \le t \le T\}$ given by (24) satisfying

$$\begin{aligned} -dX_t &= -\left(r_t X_t + \theta_t Z_t\right) dt - Z_t dW_t, \\ X_T &= \xi, \end{aligned} \tag{26}$$

it suffices to prove that

$$X_t \ge CP_t, \quad dP \otimes dt, \quad \text{a.s., a.e.} \tag{27}$$

Consider another BSDE:

$$\begin{aligned} -dX_t' &= -\left(r_t X_t' + \theta_t Z_t'\right) dt - Z_t' dW_t, \\ X_t' &= CP_T. \end{aligned} \tag{28}$$

It is obvious that $(CP_t, \sigma_t CP_t)$ is the unique solution of (28), where σ_t is the volatility rate of stock. Since $\xi \ge CP_T$, the claim (27) follows from Theorem 2. And the second equality comes from Girsanov's theorem directly.

The second assertion can be confirmed by Theorem 7 due to (24). This completes the proof. □

From Theorem 8, we can see that if there is no dividends-paying for underlying stocks, then, as rational investors, the best strategy is to keep the convertible bonds until maturity date T and decide whether to convert them or not by comparing the face value with conversion value at that time. Besides, there is no consumption throughout the whole process.

As mentioned in the Introduction, the risk neutral measure $\overline{P}$ plays a key role in the pricing of financial derivatives due to the fundamental theorem of asset pricing. From (21) and (25), it follows that

$$X_t = \operatorname{ess\,sup}_{\nu \in \mathcal{T}_t} \mathbb{E}_{\overline{P}}\left[e^{-\int_t^\tau r_s ds}\widetilde{P}_\tau \mid \mathcal{F}_t\right]. \tag{29}$$

Equation (29) implies that the discounted fair price process $\{e^{-\int_0^t r_s ds} X_t\}$ of convertible bond is exactly the Snell envelope of discounted payoff process $\{e^{-\int_0^t r_s ds}\widetilde{P}_t\}$. Then Theorem 8 tells us that it is equal to the conditional expectation of $e^{-\int_0^T r_s ds}\widetilde{P}_T$ in this case.

Remark 9. In fact, Theorem 8 is valid even when the coefficients r_t, μ_t and σ_t are random. The proof is similar as above. Theorem I in [6] also obtained the same conclusion about conversion time, whereas we adopt different methods.

4. Higher Loan Rate Case

In reality, the loan rate is usually higher than riskless interest rate in a financial market. Therefore, in this section, we discuss the pricing problem for convertible bonds under the following assumption.

Assumption 10. The loan rate R_t is higher than riskless interest rate r_t, which is also a deterministic bounded function with respect to t.

Similar to the procedure in the above section, denote

$$\begin{gathered} Z_t^* = \sigma_t \pi_t^*, \qquad \theta_t = \sigma_t^{-1}\left(\mu_t - r_t\right), \\ b\left(t, X_t^*, Z_t^*\right) = -\left[r_t X_t^* + \theta_t Z_t^* - \left(R_t - r_t\right)\left(X_t^* - \sigma_t^{-1} Z_t^*\right)^-\right]. \end{gathered} \tag{30}$$

Then the fair price X_t^* of convertible bond in this case satisfies the following reflected BSDE with "obstacle" $\{CP_t, 0 \le t \le T\}$:

$$X_t^* = \xi + \int_t^T b\left(s, X_s^*, Z_s^*\right) ds + K_T^* - K_t^* - \int_t^T Z_s^* dW_s. \tag{31}$$

Noting that the generator b of (31) is convex, we have the following result by virtue of the convex analysis method from El Karoui et al. [17].

Theorem 11. *Let Assumptions 6 and 10 hold. Then the fair price X_t^* of convertible bond is*

$$X_t^* = \operatorname{ess\,sup}\left\{X_t^\beta : r_t \le \beta_t \le R_t\right\}, \tag{32}$$

where X_t^β is the solution of the following reflected BSDE with "obstacle" $\{CP_t, 0 \le t \le T\}$:

$$\begin{aligned} X_t^\beta = \xi &- \int_t^T \left[\beta_s X_s^\beta + \sigma_s^{-1}\left(\mu_t - \beta_s\right) Z_s^\beta\right] ds \\ &+ K_T^\beta - K_t^\beta - \int_t^T Z_s^\beta dW_s. \end{aligned} \tag{33}$$

Proof. Since $b(t,x,z)$ is convex with respect to x and z, the polar process $B(t,\beta_t,\gamma_t)$ associated with b is given by

$$\begin{aligned} B(t,\beta_t,\gamma_t) &= \inf_{(x,z)} \{b(t,x,z) + \beta_t x + \gamma_t z\} \\ &= \inf_{(x,z)} \left\{(\beta_t - r_t)x + (R_t - r_t)\left(x - \sigma_t^{-1} z\right)^- \right. \\ &\qquad \left. + (\gamma_t - \theta_t) z\right\} \\ &= \inf_z \left\{\sigma_t^{-1}(\beta_t - r_t) z + (\gamma_t - \theta_t) z\right\} \\ &\quad \left(\text{the second part gets its infimum when } x = \sigma_t^{-1} z\right) \\ &= \begin{cases} 0, & \text{when } \sigma_t^{-1}(\beta_t - r_t) + \gamma_t - \theta_t = 0, \\ & \qquad r_t \le \beta_t \le R_t; \\ -\infty, & \text{otherwise.} \end{cases} \end{aligned} \tag{34}$$

Thus, the unique solution of (31) is given by

$$X_t^* = \text{ess sup}\left\{X_t^\beta; r_t \le \beta_t \le R_t\right\}, \tag{35}$$

where X_t^β is the solution of (33) corresponding to β. □

Usually, it is not easy to determine the value of β that makes X_t^β reach its essential supremum. However, comparing (31) and (33), we find that they have the same solution when $\beta_t = r_t$, $0 \le t \le T$, under the constraint that

$$\sigma_t^{-1} Z_t^* \le X_t^* \quad dP \otimes dt \quad \text{a.s., a.e.} \tag{36}$$

Furthermore, the solution coincides with that of (26) by Theorem 8. In fact, by means of Malliavin calculus technique, we have the following.

Lemma 12. *Let Assumptions 6 and 10 hold. Then the solution* $\{(X_t, Z_t), 0 \le t \le T\}$ *of* (26) *satisfies*

$$\sigma_t^{-1} Z_t \le X_t, \quad dP \otimes dt \quad \textit{a.s., a.e.} \tag{37}$$

Proof. By Theorem 4, the version of $\{(D_u X_t, D_u Z_t), 0 \le u \le t \le T\}$, where $\{(X_t, Z_t), 0 \le t \le T\}$ is the solution of (26), is given by

$$\begin{aligned} -dD_u X_t &= -\left(r_t D_u X_t + \theta_t D_u Z_t\right) dt - D_u Z_t dW_t, \\ D_u X_T &= D_u \xi. \end{aligned} \tag{38}$$

And $\{D_t X_t, 0 \le t \le T\}$ is a version of $\{Z_t, 0 \le t \le T\}$. So (37) is equivalent to

$$\sigma_t^{-1} D_t X_t \le X_t \quad dP \otimes dt \quad \text{a.s., a.e.} \tag{39}$$

Denote $X_t^u = \sigma_u^{-1} D_u X_t$ and $Z_t^u = \sigma_u^{-1} D_u Z_t$ for $0 \le u \le t \le T$. Then $\{(X_t^u, Z_t^u), 0 \le u \le t \le T\}$ is the solution of following BSDE:

$$\begin{aligned} -dX_t^u &= -\left(r_t X_t^u + \theta_t Z_t^u\right) dt - Z_t^u dW_t, \\ X_T^u &= \sigma_u^{-1} D_u \xi. \end{aligned} \tag{40}$$

Thus, by Theorem 2 for (40) and (26), if the terminal condition satisfies

$$\sigma_t^{-1} D_t \xi \le \xi \quad dP \otimes dt \quad \text{a.s., a.e.,} \tag{41}$$

then (39) and (37) hold obviously. In fact, if we set $\xi = f(CP_T)$, where $f(x) = F + (x - F)^+$, then f can be approximated by C^1 functions f_n as

$$\begin{aligned} f_n(x) &= f(x), \quad \text{for } |x - F| \ge \frac{1}{n}, \\ 0 &\le f_n'(x) \le 1, \quad \forall x. \end{aligned} \tag{42}$$

Thus we have

$$\begin{aligned} D_t \xi &= \lim_{n\to\infty} D_t f_n(\xi) \\ &= \chi_{[F,\infty)}(CP_T) D_t(CP_T) \\ &= \chi_{[F,\infty)}(CP_T) \cdot CP_T \cdot \sigma_t. \end{aligned} \tag{43}$$

Since

$$CP_T = CP_0 \exp\left\{\int_0^T \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt + \int_0^T \sigma_t dW_t\right\}, \tag{44}$$

it follows that

$$\begin{aligned} D_t(CP_T) &= D_t\left(CP_0 \exp\left\{\int_0^T \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt + \int_0^T \sigma_t dW_t\right\}\right) \\ &= CP_T \cdot \sigma_t + CP_T D_t\left(\int_0^T \left(\mu_t - \frac{1}{2}\sigma_t^2\right) dt\right) \\ &= CP_T \cdot \sigma_t. \end{aligned} \tag{45}$$

Consequently, we get

$$\sigma_t^{-1} D_t \xi = \chi_{[F,\infty)}(CP_T) CP_T \le F + (CP_T - F)^+ = \xi. \tag{46}$$

This completes the proof. □

Remark 13. Notice that (38) holds under the assumption that r_t, μ_t, and σ_t are deterministic. Therefore (5) in Assumption 6 is necessary.

Concluding all discussions above, we have the main result of this paper.

Theorem 14. *Let Assumptions 6 and 10 hold. Then the pricing formula of convertible bond is not affected by higher loan rate* R_t. *The fair price* X_t^* *is given by* (24). *Moreover, the portfolio strategy* π_t^* *is given by*

$$\pi_t^* = \pi_t = \sigma_t^{-1} D_t X_t = \mathbb{E}_{\overline{P}}\left[e^{-\int_t^T r_s ds} \chi_{[F,\infty)}(CP_T) CP_T \mid \mathcal{F}_t\right], \tag{47}$$

where $\overline{P}$ *is defined as* (25).

Proof. Similar to the proof of Theorem 8, the solution of (26) coincides with that of reflected BSDE (33) with $\beta_t = r_t$. Thus, by Lemma 12, it follows that $\sigma_t^{-1} Z_t \leq X_t$, a.s., a.e. for (33). Further it has the same solution with (31) by Theorem 11.

In a word, the price X_t^* of convertible bond is the solution of (26) actually. Hence the first claim is true. At last, (47) can be easily obtained by the proof of Lemma 12. □

In fact, if all parameters in (24) and (47) are constants, we can get the following explicit representation of fair price X_t^* and portfolio strategy π_t^*.

Corollary 15. *Let the loan rate R_t, the riskless interest rate r_t, and the volatility rate σ_t be all constants. Then the fair price X_t^* and portfolio strategy π_t^* are given by*

$$\begin{aligned} X_t^* &= CP_t N(d_1) - Fe^{-r(T-t)} N(d_2) + Fe^{-r(T-t)}, \\ \pi_t^* &= CP_t N(d_1), \end{aligned} \tag{48}$$

where d_1 and d_2 are defined as

$$\begin{aligned} d_1 &= \frac{\ln(CP_t/F) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\ln(CP_t/F) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}, \end{aligned} \tag{49}$$

and $N(x)$ stands for the cumulation distribution function of standard normal distribution.

Proof. By (24) and (47), the conclusion follows easily from usual calculational techniques for classical probability. So we just omit the details. □

5. Numerical Simulation

In this section, we calculate hedging strategies at the initial time $t = 0$ explicitly in the case that all coefficients are constants according to Corollary 15 and discuss the influence of the following parameters, which can further illustrate our results obtained in this paper. Here we fix $C = 4.5$, $\mu = 0.11$, $\sigma = 0.30$, and $T = 10$ years throughout this section.

5.1. The Influence of Initial Stock Price P_0. We set $r = 0.05$ and $F = 1000$ yuan, then Figure 1 depicts the influence of P_0 to X_0^* and π_0^*. We find that the higher the initial stock price, the higher the price of convertible bond, and one should put more money into the stock market for investment, correspondingly. For example, when $P_0 = 39.2$ yuan, the price of convertible bond is 619.6 yuan, and to hedge the risk of convertible bond, one should invest 35.98 yuan into the stock market. When $P_0 = 60$ yuan, the price increases to 645.8 yuan and the amount invested into stocks is 95.16 yuan. This coincides with our intuition because the latter case seems to bring more profit for investors.

5.2. The Influence of Riskless Interest Rate r. We set $P_0 = 50$ and $F = 1000$ yuan and let riskless interest rate r vary from

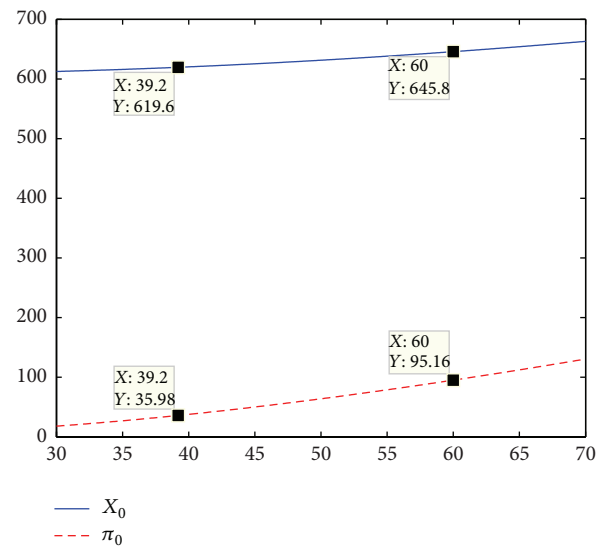

Figure 1: The influence of initial stock price.

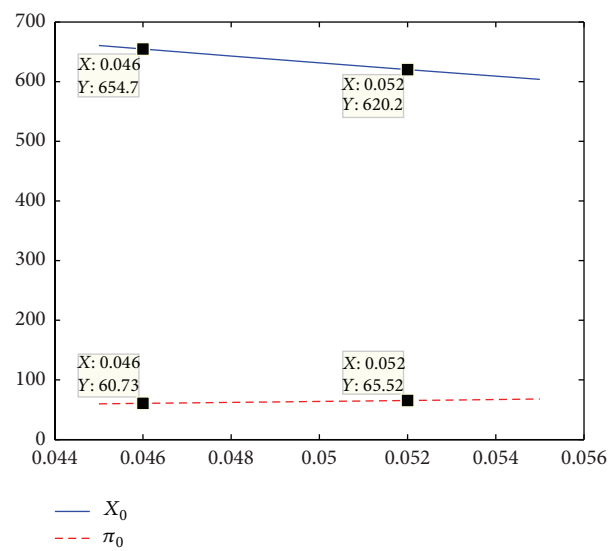

Figure 2: The influence of riskless interest rate.

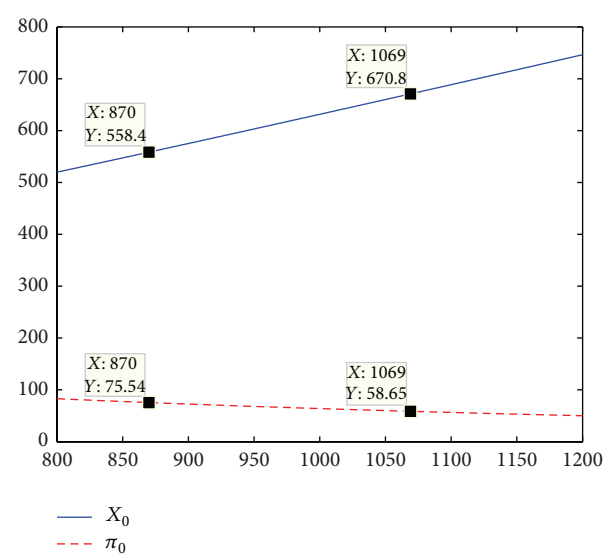

Figure 3: The influence of face price.

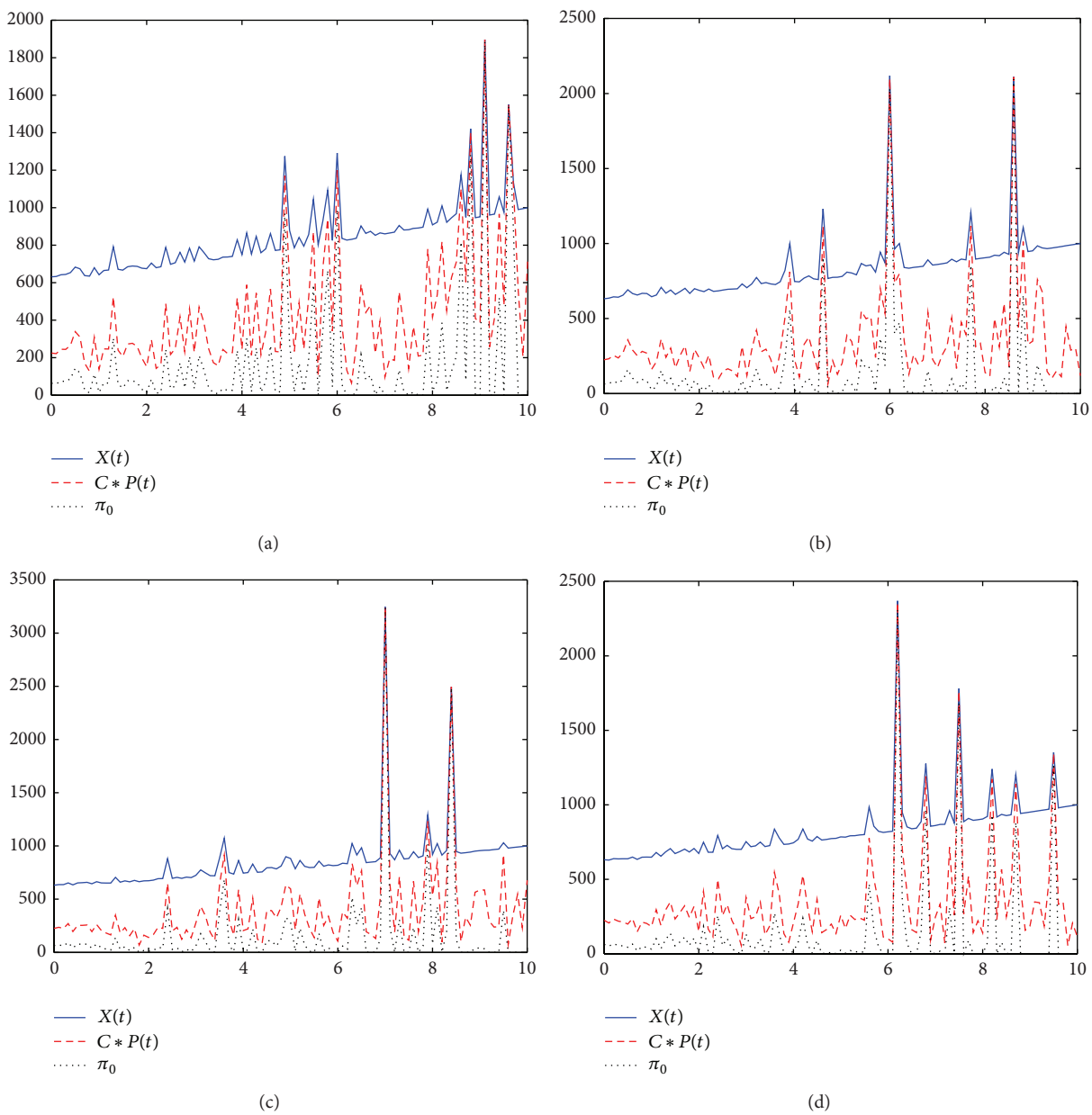

Figure 4: The simulation for the price.

0.045 to 0.055. Figure 2 indicates that high riskless interest rate depresses the price of convertible bond but provokes the investment into the stock market. In fact, under this situation, the price of convertible bond decreases because of the high discounting rate; simultaneously investors tend to put more money into the stock market.

5.3. The Influence of Face Price F. Figure 3 illustrates that another factor, the face price, also makes positive effect on the price of convertible bond. It is reasonable since the higher the face price is, the more profits an investor gains at the maturity date from the bond characteristic of convertible bond. Therefore, the desire of investors to put money into the stock market will be weakened at the same time.

From Figures 1, 2, and 3, we can see that the amount invested into the stock market is always less than the total wealth of an investor in the hedging portfolio, which confirms our results obtained in Section 4.

5.4. The Simulation for the Price of Convertible Bond X_t^*. Fixing $P_0 = 50$ yuan, $r = 0.05$, and $F = 1000$ yuan, Figure 4 gives out 4 times of simulations of the stock price process and the corresponding price process of convertible bond, according to (16) and (48), respectively. It shows that the price of convertible bond is higher than the conversion value of it throughout the expiry time and again the amount invested into stocks is less than the total wealth of an investor in the hedging portfolio. Therefore, for a rational investor, he just

needs to hold the convertible bond and consider executing the conversion right only at the maturity date, which has been confirmed theoretically by Theorems 8 and 14.

6. Conclusion and Extension

It is the first attempt to take the difference of riskless interest rate and loan rate into consideration when formulating the pricing model for convertible bonds, to the authors' knowledge. There are three distinctive features of our paper. (1) We establish the pricing formula for convertible bonds through reflected BSDEs, which introduce an increasing process to push the price upwards of conversion value. (2) Thanks to comparison theorem, we obtain the fair price and hedging portfolio explicitly by degenerating reflected BSDEs to BSDEs. We also conclude that the optimal conversion time for convertible bonds is the maturity date as a byproduct. (3) We prove an important fact for convertible bonds that the fair price is not affected by the higher loan rate in a financial market by Malliavin calculus. Besides, some numerical simulations are given and discussed to illustrate our conclusions.

In this paper, there are two main assumptions: one is that the underlying stocks are without dividend, and the other is that the convertible bonds are not allowed to be called or putted. In order to apply our results into practice better, we will try to relax the constraints to obtain more available models in our future work.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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