Research Article
On a New Criterion for Meromorphic Starlike Functions

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The main purpose of this paper is to derive a new criterion for meromorphic starlike functions of order \( \alpha \).

1. Introduction and Preliminaries
Let \( \Sigma_n \) denote the class of functions of the form
\[
f(z) = \frac{1}{z} + \sum_{k=n}^{\infty} a_{k-1} z^{k-1} \quad (n \in \mathbb{N} := \{1, 2, \ldots \}),
\]
which are analytic in the punctured open unit disk
\[
U^* := \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \} =: \mathbb{U} \setminus \{0\}.
\]
A function \( f \in \Sigma_n \) is said to be in the class \( \mathcal{M} \mathcal{S}_n^*(\alpha) \) of meromorphic starlike functions of order \( \alpha \) if it satisfies the condition
\[
\Re \left( \frac{z f'(z)}{f(z)} \right) < -\alpha \quad (z \in \mathbb{U}; 0 \leq \alpha < 1).
\]

For simplicity, we write \( \mathcal{M} \mathcal{S}_n^*(0) =: \mathcal{M} \mathcal{S}_n^* \).

For two functions \( f \) and \( g \), analytic in \( \mathbb{U} \), we say that the function \( f \) is subordinate to \( g \) in \( \mathbb{U} \) and write
\[
f(z) < g(z) \quad (z \in \mathbb{U}),
\]
if there exists a Schwarz function \( \omega \), which is analytic in \( \mathbb{U} \) with
\[
\omega(0) = 0, \quad |\omega(z)| < 1 \quad (z \in \mathbb{U}),
\]
such that
\[
f(z) = g(\omega(z)) \quad (z \in \mathbb{U}).
\]
Indeed, it is known that
\[
f(z) < g(z) \quad (z \in \mathbb{U})
\]
\[
\Rightarrow f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}).
\]

Furthermore, if the function \( g \) is univalent in \( \mathbb{U} \), then we have the following equivalence:
\[
f(z) < g(z) \quad (z \in \mathbb{U})
\]
\[
\iff f(0) = g(0), \quad f(\mathbb{U}) \subset g(\mathbb{U}).
\]

In a recent paper, Miller et al. [1] proved the following result.

**Theorem A.** Let \( n \in \mathbb{N}, 0 \leq \lambda \leq 1 \), and
\[
M_0(\lambda, n) = \frac{n + 1 - \lambda}{\sqrt{(n + 1 - \lambda)^2 + \lambda^2 + 1 - \lambda}}.
\]

If \( f \in \Sigma_n \) satisfies the condition
\[
|z^2 f'(z) + (1 - \lambda) z f(z) + \lambda| < M_0(\lambda, n) \quad (z \in \mathbb{U}),
\]
then \( f \in \mathcal{M} \mathcal{S}_n^* \).

More recently, Catas [2] improved Theorem A as follows.

**Theorem B.** Let \( n \in \mathbb{N}, 0 \leq \lambda < 1 \), and
\[
M(\lambda, n) = \max \{ M_0(\lambda, n), M_1(\lambda, n) \},
\]
where \( M_0(\lambda, n) \) is given by (9) and
\[
M_1(\lambda, n) = \frac{n + 1 - \lambda}{\sqrt{(n + 1 - \lambda)^2 + \lambda^2 + 1 - \lambda}}.
\]
If \( f \in \Sigma_n \) satisfies the condition

\[
|z^2 f'(z) + (1 - \lambda)zf(z) + \lambda| < M(\lambda, n) \quad (z \in U),
\]
then \( f \in \mathcal{M}^\star_n \).

In this paper, we aim at finding the conditions for starlikeness of the expression \( |z^2 f'(z) + \lambda zf(z) + 1 - \lambda| \) for \( \lambda > 1 \).

For some recent investigations of meromorphic functions, see, for example, the works of [3–12] and the references cited therein.

In order to prove our main results, we require the following subordination result due to Hallenbeck and Ruscheweyh [13].

**Lemma 1.** Let \( \phi \) be a convex function with \( \phi(0) = 1 \), and let \( \gamma \neq 0 \) be a complex number with \( R(\gamma) \geq 0 \). If a function

\[
p(z) = 1 + p_n z^n + p_{n+1} z^{n+1} + \ldots
\]

satisfies the condition

\[
p(z) + \frac{1}{\gamma} z p'(z) < \phi(z),
\]
then

\[
p(z) < \chi(z) := \frac{\nu}{nz^{\nu/n}} \int_0^z \phi(t) t^{(\nu/n)-1} dt < \phi(z).
\]

**2. Main Results**

We begin by stating the following result.

**Theorem 2.** Let \( n \in \mathbb{N}, \lambda > 1, \) and \( 0 \leq \alpha < 1 \). If \( f \in \Sigma_n \) satisfies the inequality

\[
|z^2 f'(z) + \lambda zf(z) + 1 - \lambda| < M,
\]
where

\[
M := M(\lambda, \alpha, n) = \frac{(1 - \alpha)(\lambda + n - 1)}{\lambda - \alpha + \sqrt{(1 - \lambda)^2 + (\lambda + n - 1)^2}}.
\]

then \( f \in \mathcal{M}^\star_n (\alpha) \).

Proof. Suppose that

\[
q(z) := zf(z) \quad (z \in U).
\]

It follows from (19) that

\[
zq'(z) = zf(z) + z^2 f'(z).
\]

By combining (17), (19), and (20), we easily get

\[
|q(z) + \frac{1}{\lambda - 1} z q'(z) - 1| < \frac{M}{\lambda - 1},
\]
or equivalently

\[
q(z) + \frac{1}{\lambda - 1} z q'(z) < 1 + \frac{M}{\lambda - 1} z.
\]

An application of Lemma 1 yields

\[
q(z) < \frac{\lambda - 1}{nz^{(\lambda-1)/n}} \int_0^z \left(1 + \frac{M}{\lambda - 1} t\right)^{(\lambda-1)/n} dt
\]

\[
= 1 + \frac{M}{\lambda + n - 1} z.
\]

The subordination (23) is equivalent to

\[
|q(z) - 1| < \frac{M}{\lambda + n - 1} := N.
\]

From (18) and (24), we know that

\[
N < \frac{1 - \alpha}{\lambda - \alpha} < 1.
\]

We suppose that

\[
\frac{z f'(z)}{f(z)} := (1 - \alpha) p(z) + \alpha.
\]

By virtue of (19) and (26), we get

\[
z^2 f'(z) = -q(z) \left[(1 - \alpha) p(z) + \alpha\right],
\]
which implies that (17) can be written as

\[
|q(z) \left[(1 - \alpha) p(z) + \alpha - \lambda\right] + \lambda - 1| < M = (\lambda + n - 1) N.
\]

We now only need to show that (28) implies \( \Re(p(z)) > 0 \) in \( U \). Indeed, if this is false, since \( p(0) = 1 \), then there exists a point \( z_0 \in U \) such that \( p(z_0) = \beta i \), where \( \beta \) is a real number. Thus, in order to show that (28) implies \( \Re(p(z)) > 0 \) in \( U \), it suffices to obtain the contradiction from the inequality

\[
|q(z_0) \left[(1 - \alpha) \beta + \alpha - \lambda\right] + \lambda - 1| \geq (\lambda + n - 1) N \quad (\beta \in \mathbb{R}).
\]

By setting

\[
q(z_0) = u + iv \quad (u, v \in \mathbb{R}),
\]
we have
\[
E = |q(z_0) [(1 - \alpha) \beta i + \alpha - \lambda] + \lambda - 1|^2
\]
\[
= (u^2 + v^2) [(1 - \alpha)^2 \beta^2 + (\alpha - \lambda)^2]
\]
\[
- 2(1 - \lambda) \Re [(u + iv) [(1 - \alpha) \beta i + \alpha - \lambda]] + (1 - \lambda)^2
\]
\[
= (u^2 + v^2) (1 - \alpha)^2 \beta^2 + 2(1 - \lambda) (1 - \alpha) \beta v
\]
\[
+ [(u + iv) (\alpha - \lambda) - (1 - \lambda)|^2.
\]
(31)

By means of (24), we obtain
\[
\begin{align*}
|u + iv| (\alpha - \lambda) - (1 - \lambda)|
\end{align*}
\]
\[
= |[(u + iv) (\alpha - \lambda) + \alpha - \lambda - 1 + \lambda|
\]
\[
= |(\alpha - \lambda) (u + iv - 1) - (1 - \alpha)|
\]
\[
\geq 1 - \alpha - (\lambda - \alpha)|u + iv - 1|
\]
\[
\geq 1 - \alpha - (\lambda - \alpha) N
\]

It follows from (31) and (32) that
\[
E \geq (u^2 + v^2) (1 - \alpha)^2 \beta^2 + 2(1 - \lambda) (1 - \alpha) \beta v
\]
\[
+ [1 - \alpha - (\lambda - \alpha) N|^2
\]
\[
\tag{33}
\]

We now set
\[
F(\beta) = E - M^2
\]
\[
\geq (u^2 + v^2) (1 - \alpha)^2 \beta^2 + 2(1 - \lambda) (1 - \alpha) \beta v
\]
\[
+ [1 - \alpha - (\lambda - \alpha) N|^2 - (\lambda + n - 1)^2 N^2.
\]
(34)

If \(F(\beta) \geq 0\), then (29) holds true. Since \((u^2 + v^2)(1 - \alpha)^2 > 0\), the inequality \(F(\beta) \geq 0\) holds if the discriminant \(\Delta \leq 0\); that is,
\[
\Delta = (1 - \alpha)^2
\]
\[
\times \{(1 - \lambda)^2 v^2 - (u^2 + v^2)
\]
\[
\times \{(1 - \alpha - (\lambda - \alpha) N)^2 - (\lambda + n - 1)^2 N^2\} \leq 0,
\]
(35)

and the last inequality is equivalent to
\[
v^2 \left[ (1 - \lambda)^2 - (1 - \alpha - (\lambda - \alpha) N)^2 + (\lambda + n - 1)^2 N^2 \right]
\]
\[
\leq u^2 \left[ (1 - \alpha - (\lambda - \alpha) N)^2 - (\lambda + n - 1)^2 N^2 \right].
\]
(36)

Furthermore, in view of (24) and (36), after a geometric argument, we deduce that
\[
\frac{v^2}{u^2} \leq \frac{N^2}{1 - N^2}
\]
\[
\leq \frac{(1 - \alpha - (\lambda - \alpha) N)^2 - (\lambda + n - 1)^2 N^2}{(1 - \lambda)^2 - (1 - \alpha - (\lambda - \alpha) N)^2 + (\lambda + n - 1)^2 N^2}.
\]
(37)

It follows from (37) that \(\Delta \leq 0\), which implies that \(F(\beta) \geq 0\). But this contradicts (28). Therefore, we know that \(\Re(p(z)) > 0\) in \(U\). By virtue of (26), we conclude that
\[
\Re \left( \frac{zf^{(n)}(z)}{f(z)} \right) < -\Re \left( (1 - \alpha)p(z) + \alpha \right) < -\alpha.
\]
(38)

This evidently completes the proof of Theorem 2.

\[\Box\]

Taking \(\alpha = 0\) in Theorem 2, we obtain the following result.

**Corollary 3.** Let \(n \in \mathbb{N}\) and \(\lambda > 1\). If \(f \in \Sigma_n\) satisfies the inequality
\[
|z^2 f''(z) + \lambda zf(z) + 1 - \lambda| < \frac{(\lambda + n - 1)}{\lambda + \sqrt{(1 - \lambda)^2 + (\lambda + n - 1)^2}}\]
\[
then f \in \mathcal{M} \Delta_n^*.
\]

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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