Research Article

Refinements of Bounds for Neuman Means

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We present the sharp bounds for the Neuman means S_{HA} , S_{AH} , S_{CA} and S_{AC} in terms of the arithmetic, harmonic, and contraharmonic means. Our results are the refinements or improvements of the results given by Neuman.

1. Introduction

For $a, b > 0$ with $a \neq b$, the Schwab-Borchardt mean $SB(a, b)$ of a and b is given by

$$SB(a, b) = \begin{cases} \frac{\sqrt{b^2 - a^2}}{\cos^{-1}(a/b)}, & a < b, \\ \frac{\sqrt{a^2 - b^2}}{\cosh^{-1}(a/b)}, & a > b, \end{cases} \quad (1)$$

where $\cos^{-1}(x)$ and $\cosh^{-1}(x) = \log(x + \sqrt{x^2 - 1})$ are the inverse cosine and inverse hyperbolic cosine functions, respectively.

It is well-known that the mean $SB(a, b)$ is strictly increasing in both a and b , nonsymmetric and homogeneous of degree 1 with respect to a and b . Many symmetric bivariate means are special cases of the Schwab-Borchardt mean; for example,

$$P(a, b) = \frac{a - b}{2\sin^{-1}[(a - b)/(a + b)]} = SB(G, A)$$

is the first Seiffert mean,

$$T(a, b) = \frac{a - b}{2\tan^{-1}[(a - b)/(a + b)]} = SB(A, Q)$$

is the second Seiffert mean,

$$M(a, b) = \frac{a - b}{2\sinh^{-1}[(a - b)/(a + b)]} = SB(Q, A)$$

is the Neuman-Sándor mean,

$$L(a, b) = \frac{a - b}{2\tanh^{-1}[(a - b)/(a + b)]} = SB(A, G)$$

is the logarithmic mean,

(2)

where $G(a, b) = \sqrt{ab}$, $A(a, b) = (a + b)/2$, and $Q(a, b) = \sqrt{(a^2 + b^2)/2}$ denote the classical geometric mean, arithmetic mean, and quadratic mean of a and b , respectively. The Schwab-Borchardt mean $SB(a, b)$ was investigated in [1, 2].

Let $H(a, b) = 2ab/(a + b)$ and $C(a, b) = (a^2 + b^2)/(a + b)$ be the harmonic and contraharmonic means of two positive numbers a and b , respectively. Then, it is well-known that

$$\begin{aligned} H(a, b) &< G(a, b) < L(a, b) < P(a, b) < A(a, b) < M(a, b) \\ &< T(a, b) < Q(a, b) < C(a, b), \end{aligned} \quad (3)$$

for $a, b > 0$ with $a \neq b$.

Recently, the Schwab-Borchardt mean and its special cases have been the subject of intensive research. Neuman and Sándor [3, 4] proved that the inequalities

$$P(a, b) > \frac{2}{\pi} A(a, b),$$

$$\frac{A(a, b)}{\log(1 + \sqrt{2})} > M(a, b) > \frac{\pi}{4 \log(1 + \sqrt{2})} T(a, b),$$

$$T(A(a, b), G(a, b)) < P(a, b),$$

$$T(a, b) > T(A(a, b), Q(a, b)),$$

$$L(a, b) < L(A(a, b), G(a, b)),$$

$$M(a, b) < L(A(a, b), Q(a, b)),$$

$$L(a, b) > H(P(a, b), G(a, b)),$$

$$P(a, b) > H(L(a, b), A(a, b)),$$

$$M(a, b) > H(T(a, b), A(a, b)),$$

$$T(a, b) > H(M(a, b), Q(a, b)),$$

$$G(a, b) P(a, b) < L^2(a, b) < \frac{G^2(a, b) + P^2(a, b)}{2},$$

$$L(a, b) A(a, b) < P^2(a, b) < \frac{L^2(a, b) + A^2(a, b)}{2},$$

$$A(a, b) T(a, b) < M^2(a, b) < \frac{A^2(a, b) + T^2(a, b)}{2},$$

$$M(a, b) Q(a, b) < T^2(a, b) < \frac{M^2(a, b) + Q^2(a, b)}{2},$$

$$Q^{1/3}(a, b) A^{2/3}(a, b) < M(a, b) < \frac{1}{3} Q(a, b) + \frac{2}{3} A(a, b) \quad (4)$$

hold for all $a, b > 0$ with $a \neq b$. In [5], the author proved that the double inequalities

$$\begin{aligned} & \alpha Q(a, b) + (1 - \alpha) A(a, b) \\ & < M(a, b) < \beta Q(a, b) + (1 - \beta) A(a, b), \\ & \lambda C(a, b) + (1 - \lambda) A(a, b) \\ & < M(a, b) < \mu C(a, b) + (1 - \mu) A(a, b) \end{aligned} \quad (5)$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq [1 - \log(1 + \sqrt{2})]/[(\sqrt{2} - 1) \log(1 + \sqrt{2})] = 0.3249 \dots$, $\beta \geq 1/3$, $\lambda \leq [1 - \log(1 + \sqrt{2})]/\log(1 + \sqrt{2}) = 0.1345 \dots$, and $\mu \geq 1/6$. Chu and Long [6] found that the double inequality

$$M_p(a, b) < M(a, b) < qI(a, b) \quad (6)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $p \leq \log 2 / \log[2 \log(1 + \sqrt{2})] = 1.224 \dots$ and $q \geq e/[2 \log(1 + \sqrt{2})]$, where $M_p(a, b) = [(a^p + b^p)/2]^{1/p}$ ($p \neq 0$) and $M_0(a, b) = \sqrt{ab}$ is the p th power mean of a and b . Zhao et al. [7] presented the

least values α_1 , α_2 , and α_3 and the greatest values β_1 , β_2 , and β_3 such that the double inequalities

$$\begin{aligned} & \alpha_1 H(a, b) + (1 - \alpha_1) Q(a, b) < M(a, b) \\ & < \beta_1 H(a, b) + (1 - \beta_1) Q(a, b), \\ & \alpha_2 G(a, b) + (1 - \alpha_2) Q(a, b) < M(a, b) \\ & < \beta_2 G(a, b) + (1 - \beta_2) Q(a, b), \\ & \alpha_3 H(a, b) + (1 - \alpha_3) C(a, b) < M(a, b) \\ & < \beta_3 H(a, b) + (1 - \beta_3) C(a, b) \end{aligned} \quad (7)$$

hold for all $a, b > 0$ with $a \neq b$.

Very recently, the bivariate means S_{AH} , S_{HA} , S_{CA} , and S_{AC} derived from the Schwab-Borchardt mean are defined by Neuman [8, 9] as follows:

$$\begin{aligned} S_{AH} &= SB(A, H), & S_{HA} &= SB(H, A), \\ S_{CA} &= SB(C, A), & S_{AC} &= SB(A, C). \end{aligned} \quad (8)$$

We call the means S_{AH} , S_{HA} , S_{CA} , and S_{AC} given in (8) the Neuman means. Moreover, let $v = (a - b)/(a + b) \in (-1, 1)$; then the following explicit formulas for S_{AH} , S_{HA} , S_{AC} , and S_{CA} are found by Neuman [8]:

$$S_{AH} = A \frac{\tanh(p)}{p}, \quad S_{HA} = A \frac{\sin(q)}{q}, \quad (9)$$

$$S_{CA} = A \frac{\sinh(r)}{r}, \quad S_{AC} = A \frac{\tan(s)}{s}, \quad (10)$$

where p , q , r , and s are defined implicitly as $\operatorname{sech}(p) = 1 - v^2$, $\cos(q) = 1 - v^2$, $\cosh(r) = 1 + v^2$, and $\sec(s) = 1 + v^2$, respectively. Clearly, $p \in (0, \infty)$, $q \in (0, \pi/2)$, $r \in (0, \log(2 + \sqrt{3}))$, and $s \in (0, \pi/3)$.

In [8, 9], Neuman proved that the inequalities

$$\begin{aligned} & H(a, b) < S_{AH}(a, b) < L(a, b) < S_{HA}(a, b) < P(a, b), \\ & T(a, b) < S_{CA}(a, b) < Q(a, b) < S_{AC}(a, b) < C(a, b), \end{aligned} \quad (11)$$

$$H^{1/3}(a, b) A^{2/3}(a, b) < S_{HA}(a, b) < \frac{1}{3} H(a, b) + \frac{2}{3} A(a, b),$$

$$C^{1/3}(a, b) A^{2/3}(a, b) < S_{CA}(a, b) < \frac{1}{3} C(a, b) + \frac{2}{3} A(a, b),$$

$$A^{1/3}(a, b) H^{2/3}(a, b) < S_{AH}(a, b) < \frac{1}{3} A(a, b) + \frac{2}{3} H(a, b),$$

$$A^{1/3}(a, b) C^{2/3}(a, b) < S_{AC}(a, b) < \frac{1}{3} A(a, b) + \frac{2}{3} C(a, b) \quad (12)$$

hold for $a, b > 0$ with $a \neq b$.

He et al. [10] found the greatest values $\alpha_1, \alpha_2 \in [0, 1/2]$, $\alpha_3, \alpha_4 \in [1/2, 1]$, and the least values $\beta_1, \beta_2 \in [0, 1/2]$, $\beta_3, \beta_4 \in [1/2, 1]$ such that the double inequalities

$$\begin{aligned} & H(\alpha_1 a + (1 - \alpha_1)b, \alpha_1 b + (1 - \alpha_1)a) < S_{AH}(a, b) \\ & < H(\beta_1 a + (1 - \beta_1)b, \beta_1 b + (1 - \beta_1)a), \\ & H(\alpha_2 a + (1 - \alpha_2)b, \alpha_2 b + (1 - \alpha_2)a) < S_{HA}(a, b) \\ & < H(\beta_2 a + (1 - \beta_2)b, \beta_2 b + (1 - \beta_2)a), \\ & C(\alpha_3 a + (1 - \alpha_3)b, \alpha_3 b + (1 - \alpha_3)a) < S_{CA}(a, b) \\ & < C(\beta_3 a + (1 - \beta_3)b, \beta_3 b + (1 - \beta_3)a), \\ & C(\alpha_4 a + (1 - \alpha_4)b, \alpha_4 b + (1 - \alpha_4)a) < S_{AC}(a, b) \\ & < C(\beta_4 a + (1 - \beta_4)b, \beta_4 b + (1 - \beta_4)a) \end{aligned} \quad (13)$$

hold for all $a, b > 0$ with $a \neq b$.

Motivated by inequalities (12), it is natural to ask what the greatest values $\alpha_1, \alpha_2, \alpha_3$, and α_4 and the least values $\beta_1, \beta_2, \beta_3$, and β_4 are such that the double inequalities

$$\begin{aligned} & \alpha_1 \left[\frac{H(a, b)}{3} + \frac{2A(a, b)}{3} \right] + (1 - \alpha_1) H^{1/3}(a, b) A^{2/3}(a, b) \\ & < S_{HA}(a, b) < \beta_1 \left[\frac{H(a, b)}{3} + \frac{2A(a, b)}{3} \right] \\ & \quad + (1 - \beta_1) H^{1/3}(a, b) A^{2/3}(a, b), \\ & \alpha_2 \left[\frac{C(a, b)}{3} + \frac{2A(a, b)}{3} \right] + (1 - \alpha_2) C^{1/3}(a, b) A^{2/3}(a, b) \\ & < S_{CA}(a, b) < \beta_2 \left[\frac{C(a, b)}{3} + \frac{2A(a, b)}{3} \right] \\ & \quad + (1 - \beta_2) C^{1/3}(a, b) A^{2/3}(a, b), \\ & \alpha_3 \left[\frac{A(a, b)}{3} + \frac{2H(a, b)}{3} \right] + (1 - \alpha_3) A^{1/3}(a, b) H^{2/3}(a, b) \\ & < S_{AH}(a, b) < \beta_3 \left[\frac{A(a, b)}{3} + \frac{2H(a, b)}{3} \right] \\ & \quad + (1 - \beta_3) A^{1/3}(a, b) H^{2/3}(a, b), \\ & \alpha_4 \left[\frac{A(a, b)}{3} + \frac{2C(a, b)}{3} \right] + (1 - \alpha_4) A^{1/3}(a, b) C^{2/3}(a, b) \\ & < S_{AC}(a, b) < \beta_4 \left[\frac{A(a, b)}{3} + \frac{2C(a, b)}{3} \right] \\ & \quad + (1 - \beta_4) A^{1/3}(a, b) C^{2/3}(a, b) \end{aligned} \quad (14)$$

hold for all $a, b > 0$ with $a \neq b$.

The purpose of this paper is to answer these questions. All numerical computations are carried out using MATHEMATICA software. Our main results are the following Theorems 1–4.

Theorem 1. *The double inequality*

$$\begin{aligned} & \alpha_1 \left[\frac{H(a, b)}{3} + \frac{2A(a, b)}{3} \right] + (1 - \alpha_1) H^{1/3}(a, b) A^{2/3}(a, b) \\ & < S_{HA}(a, b) < \beta_1 \left[\frac{H(a, b)}{3} + \frac{2A(a, b)}{3} \right] \\ & \quad + (1 - \beta_1) H^{1/3}(a, b) A^{2/3}(a, b) \end{aligned} \quad (15)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_1 \leq 4/5$ and $\beta_1 \geq 3/\pi$.

Theorem 2. *The two-sided inequality*

$$\begin{aligned} & \alpha_2 \left[\frac{C(a, b)}{3} + \frac{2A(a, b)}{3} \right] + (1 - \alpha_2) C^{1/3}(a, b) A^{2/3}(a, b) \\ & < S_{CA}(a, b) < \beta_2 \left[\frac{C(a, b)}{3} + \frac{2A(a, b)}{3} \right] \\ & \quad + (1 - \beta_2) C^{1/3}(a, b) A^{2/3}(a, b) \end{aligned} \quad (16)$$

holds true for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_2 \leq 3[\sqrt[3]{2} \log(2 + \sqrt{3}) - \sqrt{3}]/[(3\sqrt[3]{2} - 4) \log(2 + \sqrt{3})] = 0.7528 \dots$ and $\beta_2 \geq 4/5$.

Theorem 3. *The double inequality*

$$\begin{aligned} & \alpha_3 \left[\frac{A(a, b)}{3} + \frac{2H(a, b)}{3} \right] + (1 - \alpha_3) A^{1/3}(a, b) H^{2/3}(a, b) \\ & < S_{AH}(a, b) < \beta_3 \left[\frac{A(a, b)}{3} + \frac{2H(a, b)}{3} \right] \\ & \quad + (1 - \beta_3) A^{1/3}(a, b) H^{2/3}(a, b) \end{aligned} \quad (17)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_3 \leq 0$ and $\beta_3 \geq 4/5$.

Theorem 4. *The two-sided inequality*

$$\begin{aligned} & \alpha_4 \left[\frac{A(a, b)}{3} + \frac{2C(a, b)}{3} \right] + (1 - \alpha_4) A^{1/3}(a, b) C^{2/3}(a, b) \\ & < S_{AC}(a, b) < \beta_4 \left[\frac{A(a, b)}{3} + \frac{2C(a, b)}{3} \right] \\ & \quad + (1 - \beta_4) A^{1/3}(a, b) C^{2/3}(a, b) \end{aligned} \quad (18)$$

holds true for all $a, b > 0$ with $a \neq b$ if and only if $\alpha_4 \leq 4/5$ and $\beta_2 \geq 3(3\sqrt{3} - \sqrt[3]{4}\pi)/[(5 - 3\sqrt[3]{4})\pi] = 0.8400 \dots$.

2. Two Lemmas

In order to prove our main results, we need two lemmas, which we present in this section.

Lemma 5. Let $p \in \mathbb{R}$ and

$$\begin{aligned} f(x) = & p^2 x^6 + 2p^2 x^5 + 3(-p^2 + 4p - 2)x^4 \\ & + 2(-2p^2 + 9p - 6)x^3 + (4p^2 + 6p - 9)x^2 \\ & + 6(p - 1)x + 3(p - 1). \end{aligned} \quad (19)$$

Then, the following statements are true.

- (1) If $p = 4/5$, then $f(x) < 0$ for all $x \in (0, 1)$ and $f(x) > 0$ for all $x \in (1, \sqrt[3]{2})$.
- (2) If $p = 3/\pi$, then there exists $\lambda_1 \in (0, 1)$ such that $f(x) < 0$ for $x \in (0, \lambda_1)$ and $f(x) > 0$ for $x \in (\lambda_1, 1)$.
- (3) If $p = 3[\sqrt[3]{2} \log(2 + \sqrt{3}) - \sqrt{3}]/[(3\sqrt[3]{2} - 4) \log(2 + \sqrt{3})] = 0.7528 \dots$, then there exists $\lambda_2 \in (1, \sqrt[3]{2})$ such that $f(x) < 0$ for $x \in (1, \lambda_2)$ and $f(x) > 0$ for $x \in (\lambda_2, \sqrt[3]{2})$.

Proof. For part (1), if $p = 4/5$, then (19) becomes

$$\begin{aligned} f(x) = & \frac{1}{25}(x - 1)(16x^5 + 48x^4 + 90x^3 + 86x^2 \\ & + 45x + 15). \end{aligned} \quad (20)$$

Therefore, part (1) follows easily from (20).

For part (2), if $p = 3/\pi$, then simple computations lead to

$$-p^2 + 4p - 2 = \frac{-2\pi^2 + 12\pi - 9}{\pi^2} > 0, \quad (21)$$

$$-2p^2 + 9p - 6 = \frac{-6\pi^2 + 27\pi - 18}{\pi^2} > 0, \quad (22)$$

$$4p^2 + 6p - 9 = \frac{-9\pi^2 + 18\pi + 36}{\pi^2} > 0, \quad (23)$$

$$f(0) = -\frac{3(\pi - 3)}{\pi} < 0, \quad (24)$$

$$f(1) = \frac{9(15 - 4\pi)}{\pi} > 0, \quad (25)$$

$$\begin{aligned} f'(x) = & 6p^2 x^5 + 10p^2 x^4 + 12(-p^2 + 4p - 2)x^3 \\ & + 6(-2p^2 + 9p - 6)x^2 + 2(4p^2 + 6p - 9)x \\ & + 6(p - 1), \end{aligned} \quad (26)$$

$$f'(0) = \frac{6(3 - \pi)}{\pi} < 0, \quad (27)$$

$$f'(1) = \frac{12(30 - 7\pi)}{\pi} > 0, \quad (28)$$

$$\begin{aligned} f''(x) = & 30p^2 x^4 + 40p^2 x^3 + 36(-p^2 + 4p - 2)x^2 \\ & + 12(-2p^2 + 9p - 6)x + 2(4p^2 + 6p - 9). \end{aligned} \quad (29)$$

It follows from (21)–(23) and (29) that $f'(x)$ is strictly increasing on $(0, 1)$. Then, (27) and (28) lead to the conclusion that there exists $x_0 \in (0, 1)$ such that $f(x)$ is strictly decreasing in $(0, x_0]$ and strictly increasing in $[x_0, 1)$.

Therefore, part (2) follows from (24) and (25) together with the piecewise monotonicity of $f(x)$.

For part (3), if $p = 3[\sqrt[3]{2} \log(2 + \sqrt{3}) - \sqrt{3}]/[(3\sqrt[3]{2} - 4) \log(2 + \sqrt{3})] = 0.7528 \dots$, then numerical computations lead to

$$-p^2 + 4p - 2 = 0.444 \dots > 0, \quad (30)$$

$$4p^2 + 6p - 9 = -2.215 \dots < 0, \quad (31)$$

$$6(p - 1) = -1.483 \dots < 0, \quad (32)$$

$$f(1) = 9(5p - 4) = -2.120 \dots < 0, \quad (33)$$

$$f(\sqrt[3]{2}) = 1.669 \dots > 0. \quad (34)$$

It follows from (26) and (30)–(32) that

$$\begin{aligned} f'(x) & > 6p^2 x^2 + 10p^2 x^2 + 12(-p^2 + 4p - 2)x^2 \\ & + 6(-2p^2 + 9p - 6)x^2 + 2(4p^2 + 6p - 9)x^2 \\ & + 6(p - 1)x^2 \\ & = 12(10p - 7)x^2 > 0 \end{aligned} \quad (35)$$

for $x \in (1, \sqrt[3]{2})$.

Therefore, part (3) follows easily from (33)–(35). $\square$

Lemma 6. Let $p \in \mathbb{R}$ and

$$\begin{aligned} g(x) = & 3(1 - p)x^6 + 6(1 - p)x^5 + (-4p^2 - 6p + 9)x^4 \\ & + 2(2p^2 - 9p + 6)x^3 + 3(p^2 - 4p + 2)x^2 \\ & - 2p^2 x - p^2. \end{aligned} \quad (36)$$

Then, the following statements are true.

- (1) If $p = 4/5$, then $g(x) < 0$ for all $x \in (0, 1)$ and $g(x) > 0$ for all $x \in (1, \sqrt[3]{2})$.
- (2) If $p = 3(3\sqrt{3} - \sqrt[3]{4}\pi)/[(5 - 3\sqrt[3]{4})\pi] = 0.8400 \dots$, then there exists $\lambda_3 \in (1, \sqrt[3]{2})$ such that $g(x) < 0$ for $x \in (1, \lambda_3)$ and $g(x) > 0$ for $x \in (\lambda_3, \sqrt[3]{2})$.

Proof. For part (1), if $p = 4/5$, then (36) becomes

$$\begin{aligned} g(x) = & \frac{1}{25}(x - 1)(15x^5 + 45x^4 + 86x^3 + 90x^2 \\ & + 48x + 16). \end{aligned} \quad (37)$$

Therefore, part (1) follows from (37).

For part (2), if $p = 3(3\sqrt{3} - \sqrt[3]{4\pi})/[(5 - 3\sqrt[3]{4})\pi] = 0.8400 \dots$, then numerical computations lead to

$$-4p^2 - 6p + 9 = 1.137 \dots > 0, \quad (38)$$

$$p^2 - 4p + 2 = -0.654 \dots < 0, \quad (39)$$

$$g(1) = 9(4 - 5p) = -1.801 \dots < 0, \quad (40)$$

$$g(\sqrt[3]{2}) = 1.635 \dots > 0, \quad (41)$$

$$\begin{aligned} g'(x) &= 18(1-p)x^5 + 30(1-p)x^4 \\ &\quad + 4(-4p^2 - 6p + 9)x^3 + 6(2p^2 - 9p + 6)x^2 \\ &\quad + 6(p^2 - 4p + 2)x - 2p^2. \end{aligned} \quad (42)$$

From (38) and (39) together with (42), we clearly see that

$$\begin{aligned} g'(x) &> 18(1-p)x^2 + 30(1-p)x^2 \\ &\quad + 4(-4p^2 - 6p + 9)x^2 + 6(2p^2 - 9p + 6)x^2 \\ &\quad + 6(p^2 - 4p + 2)x^2 - 2p^2x^2 \\ &= 6(22 - 25p)x^2 > 0 \end{aligned} \quad (43)$$

for $x \in (1, \sqrt[3]{2})$.

Therefore, part (2) follows from (40) and (41) together with (43). $\square$

3. Proofs of Theorems 1–4

Proof of Theorem 1. Without loss of generality, we assume that $a > b$. Let $v = (a-b)/(a+b)$, $\lambda = v\sqrt{2-v^2}$, $x = \sqrt[6]{1-\lambda^2}$, and $p \in \{4/5, 3/\pi\}$. Then, $v, \lambda, x \in (0, 1)$,

$$\begin{aligned} &\frac{S_{HA}(a, b) - H^{1/3}(a, b) A^{2/3}(a, b)}{H(a, b)/3 + 2A(a, b)/3 - H^{1/3}(a, b) A^{2/3}(a, b)} \\ &= \frac{\lambda/\sin^{-1}(\lambda) - (1-\lambda^2)^{1/6}}{2/3 + (1-\lambda^2)^{1/2}/3 - (1-\lambda^2)^{1/6}}, \\ S_{HA}(a, b) &= \left[p \left(\frac{1}{3} H(a, b) + \frac{2A(a, b)}{3} \right) \right. \\ &\quad \left. + (1-p) H^{1/3}(a, b) A^{2/3}(a, b) \right] \end{aligned} \quad (44)$$

$$\begin{aligned} &= A(a, b) \left[\frac{\lambda}{\sin^{-1}(\lambda)} - p \left(\frac{(1-\lambda^2)^{1/2}}{3} + \frac{2}{3} \right) \right. \\ &\quad \left. - (1-p)(1-\lambda^2)^{1/6} \right] \\ &= \left(A(a, b) \left[p \left((1-\lambda^2)^{1/2} + 2 \right) + 3(1-p)(1-\lambda^2)^{1/6} \right] \right) \\ &\quad \times (3\sin^{-1}(\lambda))^{-1} F(x), \end{aligned} \quad (45)$$

where

$$F(x) = \frac{3\sqrt{1-x^6}}{px^3 + 3(1-p)x + 2p} - \sin^{-1}(\sqrt{1-x^6}), \quad (46)$$

$$F(0) = \frac{3}{2p} - \frac{\pi}{2}, \quad (47)$$

$$F(1) = 0, \quad (48)$$

$$F'(x) = \frac{3(x-1)^2}{\sqrt{1-x^6}[px^3 + 3(1-p)x + 2p]^2} f(x), \quad (49)$$

where $f(x)$ is defined as in Lemma 5.

We divide the proof into two cases.

Case 1 ($p = 4/5$). Then, from Lemma 5(1) and (49), we clearly see that $F(x)$ is strictly decreasing in $(0, 1)$. Therefore,

$$\begin{aligned} S_{HA}(a, b) &> \frac{4}{5} \left[\frac{1}{3} H(a, b) + \frac{2}{3} A(a, b) \right] \\ &\quad + \frac{1}{5} H^{1/3}(a, b) A^{2/3}(a, b) \end{aligned} \quad (50)$$

for all $a, b > 0$ with $a \neq b$ follows from (45) and (48) together with the monotonicity of $F(x)$.

Case 2 ($p = 3/\pi$). Then, from (47) and (49) and Lemma 5(2), we know that

$$F(0) = 0 \quad (51)$$

and there exists $\lambda_1 \in (0, 1)$ such that $F(x)$ is strictly decreasing in $(0, \lambda_1]$ and strictly increasing in $[\lambda_1, 1)$. Therefore,

$$\begin{aligned} S_{HA}(a, b) &< \frac{3}{\pi} \left[\frac{1}{3} H(a, b) + \frac{2}{3} A(a, b) \right] \\ &\quad + \left(1 - \frac{3}{\pi} \right) H^{1/3}(a, b) A^{2/3}(a, b) \end{aligned} \quad (52)$$

for all $a, b > 0$ with $a \neq b$ follows from (45) and (48) together with (51) and the piecewise monotonicity of $F(x)$.

Note that

$$\lim_{\lambda \rightarrow 0^+} \frac{\lambda/\sin^{-1}(\lambda) - (1-\lambda^2)^{1/6}}{2/3 + (1-\lambda^2)^{1/2}/3 - (1-\lambda^2)^{1/6}} = \frac{4}{5}, \quad (53)$$

$$\lim_{\lambda \rightarrow 1^-} \frac{\lambda/\sin^{-1}(\lambda) - (1-\lambda^2)^{1/6}}{2/3 + (1-\lambda^2)^{1/2}/3 - (1-\lambda^2)^{1/6}} = \frac{3}{\pi}. \quad (54)$$

Therefore, Theorem 1 follows from (50) and (52)–(54) together with the following statements.

- (i) If $\alpha > 4/5$, then (44) and (53) imply that there exists small enough $\delta > 0$ such that $S_{HA}(a, b) < \alpha(H(a, b)/3 + 2A(a, b)/3) + (1 - \alpha)H^{1/3}(a, b)A^{2/3}(a, b)$ for all $a > b > 0$ with $b/a \in (0, \delta)$.
- (ii) If $\beta < 3/\pi$, then (44) and (54) imply that there exists large enough $M > 1$ such that $S_{HA}(a, b) > \beta(H(a, b)/3 + 2A(a, b)/3) + (1 - \beta)H^{1/3}(a, b)A^{2/3}(a, b)$ for all $a > b > 0$ with $a/b \in (M, +\infty)$. $\square$

Proof of Theorem 2. Without loss of generality, we assume that $a > b$. Let $v = (a - b)/(a + b)$, $\mu = v\sqrt{2 + v^2}$, $x = \sqrt[6]{1 + \mu^2}$, and $p \in \{3[\sqrt[3]{2}\log(2 + \sqrt{3}) - \sqrt{3}]/[(3\sqrt[3]{2} - 4)\log(2 + \sqrt{3})], 4/5\}$. Then, $v \in (0, 1)$, $\mu \in (0, \sqrt{3})$, $x \in (1, \sqrt[3]{2})$,

$$\begin{aligned} & \frac{S_{CA}(a, b) - C^{1/3}(a, b)A^{2/3}(a, b)}{C(a, b)/3 + 2A(a, b)/3 - C^{1/3}(a, b)A^{2/3}(a, b)} \\ &= \frac{\mu/\sinh^{-1}(\mu) - (1 + \mu^2)^{1/6}}{2/3 + (1 + \mu^2)^{1/2}/3 - (1 + \mu^2)^{1/6}}, \\ & S_{CA}(a, b) - \left[p \left(\frac{1}{3}C(a, b) + \frac{2A(a, b)}{3} \right) \right. \\ & \quad \left. + (1 - p)C^{1/3}(a, b)A^{2/3}(a, b) \right] \\ &= A(a, b) \left[\frac{\mu}{\sinh^{-1}(\mu)} - p \left(\frac{(1 + \mu^2)^{1/2}}{3} + \frac{2}{3} \right) \right. \\ & \quad \left. - (1 - p)(1 + \mu^2)^{1/6} \right] \\ &= \left(A(a, b) \left[p \left((1 + \mu^2)^{1/2} + 2 \right) + 3(1 - p)(1 + \mu^2)^{1/6} \right] \right) \\ & \quad \times (3\sinh^{-1}(\mu))^{-1} G(x), \end{aligned} \quad (56)$$

where

$$G(x) = \frac{3\sqrt{x^6 - 1}}{px^3 + 3(1 - p)x + 2p} - \sinh^{-1}(\sqrt{x^6 - 1}), \quad (57)$$

$$G(1) = 0, \quad (58)$$

$$G(\sqrt[3]{2}) = \frac{3\sqrt{3}}{(4 - 3\sqrt[3]{2})p + 3\sqrt[3]{2}} - \log(1 + \sqrt{3}), \quad (59)$$

$$G'(x) = -\frac{3(x - 1)^2}{\sqrt{x^6 - 1}[px^3 + 3(1 - p)x + 2p]^2} f(x), \quad (60)$$

where $f(x)$ is defined as in Lemma 5.

We divide the proof into two cases.

Case 1 ($p = 3[\sqrt[3]{2}\log(2 + \sqrt{3}) - \sqrt{3}]/[(3\sqrt[3]{2} - 4)\log(2 + \sqrt{3})] = 0.7528 \dots$). Then, from (59) and (60) together with Lemma 5(3), we clearly see that there exists $\lambda_2 \in (1, \sqrt[3]{2})$ such that $G(x)$ is strictly increasing in $(1, \lambda_2]$ and strictly decreasing in $[\lambda_2, \sqrt[3]{2})$, and

$$G(\sqrt[3]{2}) = 0. \quad (61)$$

Therefore,

$$\begin{aligned} & S_{CA}(a, b) \\ & > \frac{3(\sqrt[3]{2}\log(2 + \sqrt{3}) - \sqrt{3})}{(3\sqrt[3]{2} - 4)\log(2 + \sqrt{3})} \left[\frac{1}{3}C(a, b) + \frac{2}{3}A(a, b) \right] \\ & \quad + \left(1 - \frac{3(\sqrt[3]{2}\log(2 + \sqrt{3}) - \sqrt{3})}{(3\sqrt[3]{2} - 4)\log(2 + \sqrt{3})} \right) \\ & \quad \times C^{1/3}(a, b)A^{2/3}(a, b) \end{aligned} \quad (62)$$

for all $a, b > 0$ with $a \neq b$ follows easily from (56) and (58) together with (61) and the piecewise monotonicity of $G(x)$.

Case 2 ($p = 4/5$). Then, Lemma 5(1) and (60) lead to the conclusion that $G(x)$ is strictly decreasing in $(1, \sqrt[3]{2})$. Therefore,

$$\begin{aligned} S_{CA}(a, b) &< \frac{4}{5} \left[\frac{1}{3}C(a, b) + \frac{2}{3}A(a, b) \right] \\ & \quad + \frac{1}{5}C^{1/3}(a, b)A^{2/3}(a, b) \end{aligned} \quad (63)$$

for all $a, b > 0$ with $a \neq b$ follows from (56) and (58) together with the monotonicity of $G(x)$.

Note that

$$\lim_{\mu \rightarrow 0^+} \frac{\mu/\sinh^{-1}(\mu) - (1 + \mu^2)^{1/6}}{2/3 + (1 + \mu^2)^{1/2}/3 - (1 + \mu^2)^{1/6}} = \frac{4}{5}, \quad (64)$$

$$\begin{aligned} & \lim_{\mu \rightarrow \sqrt{3}^-} \frac{\mu/\sinh^{-1}(\mu) - (1 + \mu^2)^{1/6}}{2/3 + (1 + \mu^2)^{1/2}/3 - (1 + \mu^2)^{1/6}} \\ &= \frac{3(\sqrt[3]{2}\log(2 + \sqrt{3}) - \sqrt{3})}{(3\sqrt[3]{2} - 4)\log(2 + \sqrt{3})}. \end{aligned} \quad (65)$$

Therefore, Theorem 2 follows from (55) and (62)–(65). $\square$

Proof of Theorem 3. Without loss of generality, we assume that $a > b$. Let $v = (a - b)/(a + b)$, $\lambda = v\sqrt{2 - v^2}$, $x = \sqrt[6]{1 - \lambda^2}$, and $p \in \{4/5, 0\}$. Then, $v, \lambda, x \in (0, 1)$ and (9) leads to

$$S_{AH}(a, b) = A(a, b) \frac{\lambda}{\tanh^{-1}(\lambda)}. \quad (66)$$

It follows from (66) that

$$\begin{aligned}
 & \frac{S_{AH}(a, b) - A^{1/3}(a, b) H^{2/3}(a, b)}{A(a, b)/3 + 2H(a, b)/3 - A^{1/3}(a, b) H^{2/3}(a, b)} \\
 &= \frac{\lambda / \tanh^{-1}(\lambda) - (1 - \lambda^2)^{1/3}}{1/3 + 2(1 - \lambda^2)^{1/2}/3 - (1 - \lambda^2)^{1/3}}, \\
 & S_{AH}(a, b) - \left[p \left(\frac{1}{3} A(a, b) + \frac{2H(a, b)}{3} \right) \right. \\
 & \quad \left. + (1 - p) A^{1/3}(a, b) H^{2/3}(a, b) \right] \\
 &= A(a, b) \left[\frac{\lambda}{\tanh^{-1}(\lambda)} - p \left(\frac{2(1 - \lambda^2)^{1/2}}{3} + \frac{1}{3} \right) \right. \\
 & \quad \left. - (1 - p) (1 - \lambda^2)^{1/3} \right] \\
 &= \frac{A(a, b) \left[p \left(2(1 - \lambda^2)^{1/2} + 1 \right) + 3(1 - p) (1 - \lambda^2)^{1/3} \right]}{3 \tanh^{-1}(\lambda)} \\
 & \quad \times H(x),
 \end{aligned} \tag{67}$$

where

$$H(x) = \frac{3\sqrt{1-x^6}}{2px^3 + 3(1-p)x^2 + p} - \tanh^{-1}(\sqrt{1-x^6}) \tag{69}$$

$$H(1) = 0, \tag{70}$$

$$H'(x) = -\frac{3(1-x)^2}{x\sqrt{1-x^6}[2px^3 + 3(1-p)x^2 + p]^2} g(x), \tag{71}$$

where $g(x)$ is defined as in Lemma 6.

If $p = 4/5$, then Lemma 6(1) and (71) lead to the conclusion that $H(x)$ is strictly increasing in $(0, 1)$. Therefore,

$$\begin{aligned}
 S_{AH}(a, b) &< \frac{4}{5} \left(\frac{1}{3} A(a, b) + \frac{2H(a, b)}{3} \right) \\
 &+ \frac{1}{5} A^{1/3}(a, b) H^{2/3}(a, b)
 \end{aligned} \tag{72}$$

for all $a, b > 0$ with $a \neq b$ follows from (68) and (70) together with the monotonicity of $H(x)$.

Note that

$$\lim_{\lambda \rightarrow 0^+} \frac{\lambda / \tanh^{-1}(\lambda) - (1 - \lambda^2)^{1/3}}{1/3 + 2(1 - \lambda^2)^{1/2}/3 - (1 - \lambda^2)^{1/3}} = \frac{4}{5}, \tag{73}$$

$$\lim_{\lambda \rightarrow 1^-} \frac{\lambda / \tanh^{-1}(\lambda) - (1 - \lambda^2)^{1/3}}{1/3 + 2(1 - \lambda^2)^{1/2}/3 - (1 - \lambda^2)^{1/3}} = 0. \tag{74}$$

Therefore, Theorem 3 follows from (12) and (67) together with (72)–(74). $\square$

Proof of Theorem 4. Without loss of generality, we assume that $a > b$. Let $v = (a - b)/(a + b)$, $\mu = v\sqrt{2 + v^2}$, $x = \sqrt[3]{1 + \mu^2}$, and $p \in \{3(3\sqrt{3} - \sqrt[3]{4}\pi)/[(5 - 3\sqrt[3]{4})\pi], 4/5\}$. Then, $v \in (0, 1)$, $\mu \in (0, \sqrt{3})$, and $x \in (1, \sqrt[3]{2})$ and (10) leads to

$$S_{AC}(a, b) = A(a, b) \frac{\mu}{\tan^{-1}(\mu)}. \tag{75}$$

It follows from (75) that

$$\begin{aligned}
 & \frac{S_{AC}(a, b) - A^{1/3}(a, b) C^{2/3}(a, b)}{A(a, b)/3 + 2C(a, b)/3 - A^{1/3}(a, b) C^{2/3}(a, b)} \\
 &= \frac{\mu / \tan^{-1}(\mu) - (1 + \mu^2)^{1/3}}{1/3 + 2(1 + \mu^2)^{1/2}/3 - (1 + \mu^2)^{1/3}}, \\
 & S_{AC}(a, b) - \left[p \left(\frac{1}{3} A(a, b) + \frac{2C(a, b)}{3} \right) \right. \\
 & \quad \left. + (1 - p) A^{1/3}(a, b) C^{2/3}(a, b) \right] \\
 &= A(a, b) \left[\frac{\mu}{\tan^{-1}(\mu)} - p \left(\frac{2(1 + \mu^2)^{1/2}}{3} + \frac{1}{3} \right) \right. \\
 & \quad \left. - (1 - p) (1 + \mu^2)^{1/3} \right] \\
 &= \frac{A(a, b) \left[p \left(2(1 + \mu^2)^{1/2} + 1 \right) + 3(1 - p) (1 + \mu^2)^{1/3} \right]}{3 \tan^{-1}(\mu)} \\
 & \quad \times J(x),
 \end{aligned} \tag{76}$$

where

$$J(x) = \frac{3\sqrt{x^6 - 1}}{2px^3 + 3(1-p)x^2 + p} - \tan^{-1}(\sqrt{x^6 - 1}), \tag{78}$$

$$J(1) = 0, \tag{79}$$

$$J(\sqrt[3]{2}) = \frac{3\sqrt{3}}{(5 - 3\sqrt[3]{4})p + 3\sqrt[3]{4}} - \frac{\pi}{3}, \tag{80}$$

$$J'(x) = \frac{3(x-1)^2}{\sqrt{x^6 - 1}[2px^3 + 3(1-p)x^2 + p]^2} g(x), \tag{81}$$

where $g(x)$ is defined as in Lemma 6.

We divide the proof into two cases.

Case 1 ($p = 4/5$). Then, (81) and Lemma 6(1) lead to the conclusion that $J(x)$ is strictly increasing in $(1, \sqrt[3]{2})$. Therefore,

$$\begin{aligned}
 S_{AC}(a, b) &> \frac{4}{5} \left(\frac{1}{3} A(a, b) + \frac{2C(a, b)}{3} \right) \\
 &+ \frac{1}{5} A^{1/3}(a, b) C^{2/3}(a, b)
 \end{aligned} \tag{82}$$

for all $a, b > 0$ with $a \neq b$ follows easily from (77) and (79) together with the monotonicity of $J(x)$.

Case 2 ($p = 3(3\sqrt{3} - \sqrt[3]{4}\pi)/(5 - 3\sqrt[3]{4}\pi)$). Then, (80) and (81) together with Lemma 6(2) lead to the conclusion that there exists $\lambda_3 \in (1, \sqrt[3]{2})$ such that $J(x)$ is strictly decreasing in $(1, \lambda_3]$ and strictly increasing in $[\lambda_3, \sqrt[3]{2})$, and

$$J(\sqrt[3]{2}) = 0. \quad (83)$$

Therefore,

$$\begin{aligned} S_{AC}(a, b) &< \frac{3(3\sqrt{3} - \sqrt[3]{4}\pi)}{(5 - 3\sqrt[3]{4})\pi} \left(\frac{1}{3} A(a, b) + \frac{2C(a, b)}{3} \right) \\ &+ \left(1 - \frac{3(3\sqrt{3} - \sqrt[3]{4}\pi)}{(5 - 3\sqrt[3]{4})\pi} \right) A^{1/3}(a, b) C^{2/3}(a, b) \end{aligned} \quad (84)$$

for all $a, b > 0$ with $a \neq b$ follows easily from (77) and (79) together with (83) and the piecewise monotonicity of $J(x)$.

Note that

$$\lim_{\mu \rightarrow 0^+} \frac{\mu/\tan^{-1}(\mu) - (1 + \mu^2)^{1/3}}{1/3 + 2(1 + \mu^2)^{1/2}/3 - (1 + \mu^2)^{1/3}} = \frac{4}{5}, \quad (85)$$

$$\lim_{\mu \rightarrow \sqrt{3}} \frac{\mu/\tan^{-1}(\mu) - (1 + \mu^2)^{1/3}}{1/3 + 2(1 + \mu^2)^{1/2}/3 - (1 + \mu^2)^{1/3}} = \frac{3(3\sqrt{3} - \sqrt[3]{4}\pi)}{(5 - 3\sqrt[3]{4})\pi}. \quad (86)$$

Therefore, Theorem 4 follows from (76) and (82) together with (84)–(86). $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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