Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 356745, 8 pages
http://dx.doi.org/10.1155/2014/356745

Research Article

Multiple Solutions of Second-Order Damped Impulsive Differential Equations with Mixed Boundary Conditions

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Received 14 February 2014; Accepted 27 March 2014; Published 23 April 2014

Academic Editor: Fenghua Wen

We use variational methods to investigate the solutions of damped impulsive differential equations with mixed boundary conditions. The conditions for the multiplicity of solutions are established. The main results are also demonstrated with examples.

1. Introduction

Impulsive effect exists widely in many evolution processes in which their states are changed abruptly at certain moments of time. The theory of impulsive differential systems has been developed by numerous mathematicians [1–6]. Applications of impulsive differential equations with or without delays occur in biology, medicine, mechanics, engineering, chaos theory, and so on [7–11].

In this paper, we consider the following second-order damped impulsive differential equations with mixed boundary conditions:

$$\begin{gathered} -u''(t)+g(t)u'(t)-\lambda u(t)=f(t,u(t)), \\ t\neq t_j, \quad \text{a.e. } t\in[0,T], \\ -\Delta u'(t_j)=I_j(u(t_j)), \qquad j=1,2,\ldots,n, \\ u'(0)=0, \quad u(T)=0, \end{gathered} \tag{1}$$

where $0=t_0<t_1<t_2<\cdots<t_n<t_{n+1}=T$, $g\in C[0,T]$, $f:[0,T]\times R\to R$ is continuous, $I_j:R\to R$, $j=1,2,\ldots,n$ are continuous, and $\Delta u'(t_j)=u'(t_j^+)-u'(t_j^-)$ for $u'(t_j^\pm)=\lim_{t\to t_j^\pm}u'(t)$, $j=1,2,\ldots,n$.

The characteristic of (1) is the presence of the damped term $g(t)u'$. Most of the results concerning the existence of solutions of these equations are obtained using upper and lower solutions methods, coincidence degree theory, and fixed point theorems [12–15]. On the other hand, when there is no presence of the damped term, some researchers have used variational methods to study the existence of solutions for these problems [16–21]. However, to the best of our knowledge, there are few papers concerned with the existence of solutions for impulsive boundary value problems like problem (1) by using variational methods.

For this nonlinear damped mixed boundary problem (1), the variational structure due to the presence of the damped term $g(t)u'$ is not apparent. However, inspired by the work [22, 23], we will be able to transform it into a variational formulation. In this paper, our aim is to study the existence of n distinct pairs of nontrivial solutions of problem (1). Our main results extend the study made in [22, 23], in the sense that we deal with a class of problems that is not considered in those papers.

2. Preliminaries and Statements

Let $m=\min_{t\in[0,T]}e^{G(t)}$, $M=\max_{t\in[0,T]}e^{G(t)}$, $G(t)=-\int_0^t g(s)ds$, $t\in[0,T]$. We transform (1) into the following equivalent form:

$$\begin{gathered} -\left(e^{G(t)}u'(t)\right)'-\lambda e^{G(t)}u(t)=e^{G(t)}f(t,u(t)), \\ t\neq t_j, \quad \text{a.e. } t\in[0,T], \end{gathered}$$

$$-\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1, 2, \ldots, n,$$
$$u'(0) = 0, \; u(T) = 0. \tag{2}$$

Obviously, the solutions of (2) are solutions of (1).

Define the space $X = \{u(t) \mid u(t)$ is absolutely continuous on $[0,T], u'(\cdot) \in L^2[0,T], u(T) = 0\}$. It is easy to see that $H_0^1(0,T) \subset X \subset H^1(0,T)$ and X is a closed subset of $H^1(0,T)$. So X is a Hilbert space with the usual inner product in $H^1(0,T)$.

Consider the Hilbert spaces X with the inner product

$$(u,v) = \int_0^T e^{G(t)} u'(t) v'(t)\, dt, \tag{3}$$

inducing the norm

$$\|u\| = \left(\int_0^T e^{G(t)} |u'(t)|^2 dt\right)^{1/2}. \tag{4}$$

We also consider the inner product

$$(u,v) = \int_0^T u'(t) v'(t)\, dt, \tag{5}$$

inducing the norm

$$\|u\|_X = \left(\int_0^T |u'(t)|^2 dt\right)^{1/2}. \tag{6}$$

Consider the problem

$$-u''(t) = \lambda u(t), \quad t \in [0,T],$$
$$u'(0) = 0, \qquad u(T) = 0. \tag{7}$$

As is well known, (7) possesses a sequence of eigenvalues $(\lambda_i)(\lambda_i = [(2i-1)\pi/2T]^2)$ with

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_j < \cdots. \tag{8}$$

The corresponding eigenfunctions are normalized so that $\|\varphi_j\|_X = 1 = \lambda_j \int_0^T |\varphi_j(t)|^2 dt$; here

$$\varphi_j(t) = \sqrt{\frac{2}{T\lambda_j}} \cos\left(\sqrt{\lambda_j} t\right), \quad j = 1, 2, \ldots. \tag{9}$$

Now multiply (2) by $v \in X$ and integrate on the interval $[0,T]$:

$$\int_0^T e^{G(t)} u'(t) v'(t)\, dt - \lambda \int_0^T e^{G(t)} u(t) v(t)\, dt$$
$$= \sum_{j=1}^n e^{G(t_j)} I_j(u(t_j)) v(t_j) + \int_0^T e^{G(t)} f(t, u(t)) v(t)\, dt. \tag{10}$$

Then, a weak solution of (2) is a critical point of the following functional:

$$E(u) = \frac{1}{2}\int_0^T e^{G(t)} |u'(t)|^2 dt - \frac{\lambda}{2}\int_0^T e^{G(t)} |u(t)|^2 dt$$
$$- \sum_{j=1}^n e^{G(t_j)} \int_0^{u(t_j)} I_j(t)\, dt - \int_0^T e^{G(t)} F(t, u(t))\, dt, \tag{11}$$

where $F(t,u) = \int_0^u f(t,\xi) d\xi$.

We say that $u \in C[0,T]$ is a classical solution of IBVP (1) if it satisfies the following conditions: u satisfies the first equation of (1) a.e. on $[0,T]$; the limits $u'(t_j^+), u'(t_j^-)$, $j = 1, 2, \ldots, n$, exist and impulsive condition of (1) holds; u satisfies the boundary condition of (1).

Lemma 1. *If $u \in X$ is a weak solution of* (1)*, then u is a classical solution of* (1).

Proof. If $u \in X$ is a weak solution of (1), then u is a weak solution of (2), so $(E'(u), v) = 0$ holds for all $v \in X$; that is,

$$\int_0^T \left[e^{G(t)} u'(t) v'(t) + \lambda e^{G(t)} u(t) v(t)\right] dt$$
$$- \sum_{j=1}^n e^{G(t_j)} I_j(u(t_j)) v(t_j) \tag{12}$$
$$- \int_0^T e^{G(t)} f(t, u(t)) v(t)\, dt = 0.$$

By integrating by part, we have

$$\int_0^T \left[e^{G(t)} u'(t) v'(t) + \lambda e^{G(t)} u(t) v(t)\right] dt$$
$$- \sum_{j=1}^n e^{G(t_j)} I_j(u(t_j)) v(t_j)$$
$$- \int_0^T e^{G(t)} f(t, u(t)) v(t)\, dt$$
$$= \int_0^T \left[-\left(e^{G(t)} u'(t)\right)' + \lambda e^{G(t)} u(t) - e^{G(t)} f(t, u(t))\right] v(t)\, dt$$
$$- \sum_{j=1}^n e^{G(t_j)} \left[\Delta u'(t_j) + I_j(u(t_j))\right] v(t_j)$$
$$- e^{G(0)} u'(0) v(0) + e^{G(T)} u'(T) v(T)$$
$$= \int_0^T \left[-\left(e^{G(t)} u'(t)\right)' + \lambda e^{G(t)} u(t) - e^{G(t)} f(t, u(t))\right] v(t)\, dt$$
$$- \sum_{j=1}^n e^{G(t_j)} \left[\Delta u'(t_j) + I_j(u(t_j))\right] v(t_j) - u'(0) v(0). \tag{13}$$

Thus

$$\int_0^T \left[-\left(e^{G(t)}u'(t)\right)' + \lambda e^{G(t)}u(t) - e^{G(t)}f(t,u(t))\right]v(t)\,dt - \sum_{j=1}^{n} e^{G(t_j)}\left[\Delta u'(t_j) + I_j(u(t_j))\right]v(t_j) - u'(0)\,v(0) = 0 \tag{14}$$

holds for all $v \in X$. Without loss of generality, for any $j = \{1,2,\ldots,n\}$ and $v \in X$ with $v(t) \equiv 0$, for every $t \in [0,t_j] \cup [t_{j+1},T]$, then substituting v into (14), we get

$$-\left(e^{G(t)}u'(t)\right)' + \lambda e^{G(t)}u(t) - e^{G(t)}f(t,u(t)) = 0, \quad t \in (t_j, t_{j+1}). \tag{15}$$

Hence u satisfies the first equation of (2). Therefore, by (14) we have

$$-\sum_{j=1}^{n} e^{G(t_j)}\left[\Delta u'(t_j) + I_j(u(t_j))\right]v(t_j) - u'(0)\,v(0) = 0. \tag{16}$$

Next we will show that u satisfies the impulsive and the boundary condition in (2). If the impulsive condition in (2) does not hold, without loss of generality, we assume that there exists $j \in \{1,2,\ldots,n\}$ such that

$$\Delta u'(t_j) + I_j(u(t_j)) \neq 0. \tag{17}$$

Let $v(t) = \prod_{i=0, i\neq j}^{n+1}(t - t_i)$; then

$$-\sum_{j=1}^{n} e^{G(t_j)}\left[\Delta u'(t_j) + I_j(u(t_j))\right]v(t_j) - u'(0)\,v(0) = -e^{G(t_j)}\left[\Delta u'(t_j) + I_j(u(t_j))\right]v(t_j) \neq 0, \tag{18}$$

which contradicts (16). So u satisfies the impulsive condition in (2) and (16) implies

$$u'(0)\,v(0) = 0. \tag{19}$$

If $u'(0) \neq 0$, pick $v(t) = \prod_{i=1}^{n+1}(t - t_i)$; one has

$$u'(0)\prod_{i=1}^{n+1}(t_0 - t_i) \neq 0, \tag{20}$$

which contradicts (19), so u satisfies the boundary condition. Therefore, u is a solution of (1). □

Lemma 2. *Let $u \in X$. Then there exists a constant $\sigma > 0$, such that*

$$\|u\|_\infty \leq \sigma \|u\|, \tag{21}$$

where $\|u\|_\infty = \max_{t\in[0,T]}|u(t)|$.

Proof. By Hölder inequality, for $u \in X$,

$$\begin{aligned} |u(t)| &= \left|u(T) - \int_t^T u'(s)\,ds\right| \\ &\leq \left(\int_0^T \frac{1}{e^{G(s)}}ds\right)^{1/2}\left(\int_0^T e^{G(s)}\left|u'(s)\right|^2 ds\right)^{1/2} \\ &\leq \sqrt{\frac{T}{m}}\,\|u\| = \sigma\|u\|. \end{aligned} \tag{22}$$

□

Lemma 3 (see [24, Theorem 9.1]). *Let E be a real Banach space, $I \in C^1(E,R)$ with I even, bounded from below, and satisfying P.S. condition. Suppose $I(0) = 0$; there is a set $K \subset E$ such that K is homeomorphic to S^{j-1} by an odd map and $\sup_K I < 0$. Then I possesses at least j distinct pairs of critical points.*

3. Main Results

Theorem 4. *Suppose that the following conditions hold.*

(H1) *There exist $u_1 > 0$, $r > M\lambda_k/m$, λ_k which is the kth eigenvalue of (7) such that*

$$rMu_1 + e^{G(t)}f(t,u_1) = 0, \quad rMu_1 + e^{G(t)}f(t,u) > 0 \quad \text{for every } u \in (0,u_1). \tag{23}$$

(H2) *There exist $a_j, b_j > 0$ and $r_j \in [0,1)$ $(j = 1,2,\ldots,n)$ such that*

$$\left|I_j(u)\right| \leq a_j + b_j|u|^{r_j} \quad \text{for any } u \in R. \tag{24}$$

(H3) *$f(t,u)$ and $I_j(u)$ $(j = 1,2,\ldots,n)$ are odd about u.*

(H4) *$f(t,u) = o(|u|)$, $I_j(u) = o(|u|)$, as $|u| \to 0$, $j = 1,2,\ldots,n$.*

Then, for $\lambda \in (M\lambda_k/m, r]$, problem (1) has at least k distinct pairs of solutions.

Proof. Set

$$h_1(\lambda,t,u) = \begin{cases} \lambda e^{G(t)}u + e^{G(t)}f(t,u) & u \in [-u_1,u_1], \\ \lambda e^{G(t)}u_1 + e^{G(t)}f(t,u_1), & u \in [u_1,+\infty), \\ -\lambda e^{G(t)}u_1 - e^{G(t)}f(t,-u_1), & u \in (-\infty,-u_1]. \end{cases} \tag{25}$$

Consider

$$\begin{gathered} -\left(e^{G(t)}u'(t)\right)' = h_1(\lambda,t,u(t)), \\ t \neq t_j, \quad \text{a.e. } t \in [0,T], \\ -\Delta u'(t_j) = I_j(u(t_j)), \quad j = 1,2,\ldots,n, \\ u'(0) = 0, \quad u(T) = 0. \end{gathered} \tag{26}$$

Next, we will verify that the solutions of problem (26) are solutions of problem (1).

In fact, let $u_0(t)$ be the solution of problem (26). If $\max_{0\le t\le T}u_0(t) > u_1$, then there exists an interval $[a,b] \subset [0,T]$ such that

$$u_0(a) = u_0(b) = u_1, \quad u_0(t) > u_1 \quad \text{for any } t \in (a,b). \tag{27}$$

When $t \in [a,b]$, by (H1), we have

$$\begin{aligned} -\left(e^{G(t)}u_0'(t)\right)' &= h_1(\lambda,t,u) \\ &= \lambda e^{G(t)}u_1 + e^{G(t)}f(t,u_1) \le rMu_1 \\ &\quad + e^{G(t)}f(t,u_1) = 0. \end{aligned} \tag{28}$$

That is, $e^{G(t)}u_0'(t)$ is nondecreasing in $[a,b]$. By $u_0'(a) \ge 0$ and $u_0'(b) \le 0$, we have

$$0 \le e^{G(t)}u_0'(a) \le e^{G(t)}u_0'(t) \le e^{G(t)}u_0'(b) \le 0 \quad \text{for every } t \in [a,b]. \tag{29}$$

That is, $e^{G(t)}u_0'(t) \equiv 0$ for any $t \in [a,b]$. Since $e^{G(t)} \ne 0$, then $u_0'(t) \equiv 0$. So, there exists a constant ϵ such that $u_0(t) \equiv \epsilon$, which contradicts (27). Then $\max_{0\le t\le T}u_0(t) \le u_1$. Similarly, we can prove that $\min_{0\le t\le T}u_0(t) > -u_1$.

Therefore, any solution of (26) is a solution of (1). Hence to prove Theorem 4, it suffices to produce at least k distinct pairs of critical points of

$$\begin{aligned} E_1(u) &= \frac{1}{2}\int_0^T e^{G(t)}\left|u'(t)\right|^2 dt - \int_0^T H_1(\lambda,t,u(t))\,dt \\ &\quad - \sum_{j=1}^n e^{G(t_j)}\int_0^{u(t_j)} I_j(t)\,dt, \end{aligned} \tag{30}$$

where $H_1(\lambda,t,u(t)) = \int_0^u h_1(\lambda,t,s)ds$.

We will apply Lemma 3 to finish the proof.

By (30) and (H3), $E_1 \in C'(X,R)$ is even and $E_1(0) = 0$.

Next, we will show that E_1 is bounded from below.

Let $C_1 = \max\{a_1,a_2,\ldots,a_n\}$, $C_2 = \max\{b_1,b_2,\ldots,b_n\}$. By (H1) and (H3), we have $uh_1(\lambda,t,u(t)) \le 0$ for $|u| \ge u_1$; thus

$$\begin{aligned} \int_0^T H_1(\lambda,t,u(t))\,dt &= \int_0^T\int_0^{u(t)} h_1(\lambda,t,s)\,ds\,dt \\ &\le \int_0^T\int_0^{u_1} h_1(\lambda,t,s)\,ds\,dt \\ &\le \int_0^T\int_0^{u_1} [rMs + f(t,s)]\,ds\,dt = \rho > 0. \end{aligned} \tag{31}$$

So, we have

$$\begin{aligned} E_1(u) &= \frac{1}{2}\|u\|^2 - \int_0^T H_1(\lambda,t,u(t))\,dt \\ &\quad - \sum_{j=1}^n e^{G(t_j)}\int_0^{u(t_j)} I_j(t)\,dt \\ &\ge \frac{1}{2}\|u\|^2 - \rho - n\sigma C_1 M\|u\| - C_2 M\sum_{j=1}^n \sigma^{r_j+1}\|u\|^{r_j+1} \\ &> -\infty, \end{aligned} \tag{32}$$

for any $u \in X$. Therefore, E_1 is bounded from below.

In the following we will show that E_1 satisfies the P.S. condition. Let $\{u_k\} \subset X$ such that $\{E_1(u_k)\}$ is a bounded sequence and $\lim_{k\to\infty}E_1'(u_k) = 0$; then there exists $C_3 > 0$ such that

$$\left|E_1(u_k)\right| \le C_3. \tag{33}$$

By (32), we have

$$\begin{aligned} \frac{1}{2}\|u_k\|^2 &\le C_3 + \rho + n\sigma C_1 M\|u_k\| \\ &\quad + C_2 M\sum_{j=1}^n \sigma^{r_j+1}\|u\|^{r_j+1}. \end{aligned} \tag{34}$$

So $\{u_k\}$ is bounded in X. From the reflexivity of X, we may extract a weakly convergent subsequence that, for simplicity, we call $\{u_k\}$, $u_k \rightharpoonup u$ in X. In the following we will verify that $\{u_k\}$ strongly converges to u:

$$\begin{aligned} &\left(E_1'(u_k) - E_1'(u)\right)(u_k - u) \\ &\quad = \|u_k - u\|^2 - \int_0^T [h_1(\lambda,t,u_k(t)) - h_1(\lambda,t,u(t))] \\ &\qquad \times (u_k(t) - u(t))\,dt \\ &\qquad + \sum_{j=1}^n e^{G(t_j)}\left[I_j(u_k(t_j)) - I_j(u(t_j))\right] \\ &\qquad \times (u_k(t_j) - u(t_j)). \end{aligned} \tag{35}$$

By $u_k \rightharpoonup u$ in X, we see that $\{u_k\}$ uniformly converges to u in $C[0,T]$. So

$$\begin{aligned} &\int_0^T [h_1(\lambda,t,u_k(t)) - h_1(\lambda,t,u(t))] \\ &\quad \times (u_k(t) - u(t))\,dt \longrightarrow 0, \\ &\sum_{j=1}^n e^{G(t_j)}\left[I_j(u_k(t_j)) - I_j(u(t_j))\right] \\ &\quad \times (u_k(t_j) - u(t_j)) \longrightarrow 0, \\ &\left(E_1'(u_k) - E_1'(u)\right)(u_k - u) \longrightarrow 0, \quad \text{as } k \longrightarrow +\infty. \end{aligned} \tag{36}$$

So we obtain $\|u_k - u\| \to 0$, as $k \to +\infty$. That is, $\{u_k\}$ strongly converges to u in X, which means that E_1 satisfies the P.S. condition.

Now set $K = \{\sum_{i=1}^{k} c_i\varphi_i : \sum_{i=1}^{k} c_i^2 = c^2\}$, where φ_i is defined in (9). It is clear that K is homeomorphic to S^{k-1} by an odd map for any $c > 0$. In the following we verify that $E_1|_K < 0$ if c is sufficiently small.

For any $u \in K, u = \sum_{i=1}^{k} c_i\varphi_i$. By (H4) and (30), we have

$$\begin{aligned}
E_1(u) &= \frac{1}{2}\int_0^T e^{G(t)}\left[\left(\sum_{i=1}^{k} c_i\varphi_i(t)\right)'\right]^2 dt \\
&\quad - \int_0^T H_1(\lambda, t, u(t))\,dt \\
&\quad - \sum_{j=1}^{n} e^{G(t_j)}\int_0^{u(t_j)} I_j(t)\,dt \\
&= \frac{1}{2}\sum_{i=1}^{k} c_i^2\int_0^T e^{G(t)}\left[\varphi_i'(t)\right]^2 dt \\
&\quad - \frac{\lambda}{2}\sum_{i=1}^{k} c_i^2\int_0^T e^{G(t)}\left[\varphi_i(t)\right]^2 dt \\
&\quad - \int_0^T e^{G(t)}F(t, u(t))\,dt \\
&\quad - \sum_{j=1}^{n} e^{G(t_j)}\int_0^{u(t_j)} I_j(t)\,dt \\
&\le \frac{M}{2}\sum_{i=1}^{k} c_i^2\int_0^T \left[\varphi_i'(t)\right]^2 dt \\
&\quad - \frac{m\lambda}{2}\sum_{i=1}^{k} c_i^2\int_0^T \left[\varphi_i(t)\right]^2 dt \\
&\quad - \int_0^T e^{G(t)}F(t, u(t))\,dt - \sum_{j=1}^{n} e^{G(t_j)}\int_0^{u(t_j)} I_j(t)\,dt \\
&= \frac{1}{2}\sum_{i=1}^{k} c_i^2\left(M - \frac{m\lambda}{\lambda_i}\right) - \int_0^T e^{G(t)}F(t, u(t))\,dt \\
&\quad - \sum_{j=1}^{n} e^{G(t_j)}\int_0^{u(t_j)} I_j(t)\,dt \\
&\le \frac{1}{2}\left(M - \frac{m\lambda}{\lambda_k}\right)c^2 + o\left(c^2\right) + o\left(c^2\right),
\end{aligned} \tag{37}$$

for small $c > 0$. Since $\lambda \in (M\lambda_k/m, r]$, $E_1(u) < 0$ and the proof is complete. □

Theorem 5. *Suppose that the following conditions hold.*

(H1) *There exist $u_1 > 0$, $r > M\lambda_k/m, \lambda_k$ which is the kth eigenvalue of (7) such that*

$$rMu_1 + e^{G(t)}f(t, u_1) = 0, \quad rMu_1 + e^{G(t)}f(t, u) > 0 \quad \text{for every } u \in (0, u_1). \tag{38}$$

(H2) $\int_0^u I_j(s)ds \le 0$ *for any* $u \in R$ $(j = 1, 2, \ldots, n)$.

(H3) $f(t, u)$ *and* $I_j(u)$ $(j = 1, 2, \ldots, n)$ *are odd about* u.

(H4) $f(t, u) = o(|u|)$, $I_j(u) = o(|u|)$, *as* $|u| \to 0$, $j = 1, 2, \ldots, n$.

Then, for $\lambda \in (M\lambda_k/m, r]$, problem (1) has at least k distinct pairs of solutions.

Proof. The proof is similar to the proof of Theorem 4, and therefore we omit it. □

Theorem 6. *Suppose that the following conditions hold.*

(H1) *There exist $u_2 > 0$, $r > M\lambda_k/m, \lambda_k$ which is the kth eigenvalue of (7) such that*

$$rMu_2 + e^{G(t)}f(t, u_2) \le 0, \qquad I_j(u_2) \le 0, \quad j = 1, 2, \ldots, n. \tag{39}$$

(H2) $f(t, u)$ *and* $I_j(u)$ $(j = 1, 2, \ldots, n)$ *are odd about* u.

(H3) $f(t, u) = o(|u|)$, $I_j(u) = o(|u|)$, *as* $|u| \to 0$, $j = 1, 2, \ldots, n$.

Then, for $\lambda \in (M\lambda_k/m, r]$, problem (1) has at least k distinct pairs of solutions.

Proof. Set

$$\begin{aligned}
h_2(\lambda, t, u) &= \begin{cases} \lambda e^{G(t)}u + e^{G(t)}f(t, u), & u \in [-u_2, u_2], \\ \lambda e^{G(t)}u_2 + e^{G(t)}f(t, u_2), & u \in [u_2, +\infty), \\ -\lambda e^{G(t)}u_2 - e^{G(t)}f(t, -u_2), & u \in (-\infty, -u_2], \end{cases} \\
T_j(u) &= \begin{cases} I_j(u), & u \in [-u_2, u_2], \\ I_j(u_2), & u \in [u_2, +\infty), \\ I_j(-u_2), & u \in (-\infty, -u_2]. \end{cases}
\end{aligned} \tag{40}$$

Consider

$$\begin{aligned}
-\left(e^{G(t)}u'(t)\right)' &= h_2(\lambda, t, u(t)), \\
t \ne t_j, &\quad \text{a.e. } t \in [0, T], \\
-\Delta u'(t_j) &= T_j(u(t_j)), \quad j = 1, 2, \ldots, n, \\
u'(0) = 0, &\quad u(T) = 0.
\end{aligned} \tag{41}$$

Next, we will verify that the solutions of problem (41) are solutions of problem (1).

In fact, let $\omega_1 = \{t \in (a_1, b_1) \subseteq [0,T] : u(t) > u_2\}$. By the definitions of $h_2(\lambda, t, u)$ and $T_j(u)$, (41) is reduced to

$$\begin{aligned} -\left(e^{G(t)}u'(t)\right)' &= h_2(\lambda, t, u_2) = \lambda e^{G(t)}u_2 + e^{G(t)} f(t, u_2) \\ &\le rMu_2 + e^{G(t)} f(t, u_2) \le 0, \quad t \ne t_j, \quad \text{a.e. } t \in (a_1, b_1), \\ -\Delta u'(t_j) &= T_j(u(t_j)) = I_j(u_2) \le 0, \quad j = 1, 2, \ldots, n, \\ u(a_1) &= u(b_1) = u_2. \end{aligned} \tag{42}$$

The solution $u(t)$ of (42) satisfies $u(t) \le u_2, t \in (a_1, b_1)$. So $\omega_1 = \emptyset$ and $u(t) \le u_2$.

Let $\omega_2 = \{t \in (a_2, b_2) \subseteq [0,T] : u(t) < -u_2\}$. By the definitions of $h_2(\lambda, t, u)$ and $T_j(u)$, (41) is reduced to

$$\begin{aligned} -\left(e^{G(t)}u'(t)\right)' &= h_2(\lambda, t, -u_2) = -\lambda e^{G(t)}u_2 + e^{G(t)} f(t, -u_2) \\ &\ge -rMu_2 - e^{G(t)} f(t, u_2) \ge 0, \\ &\qquad t \ne t_j, \quad \text{a.e. } t \in (a_2, b_2), \\ -\Delta u'(t_j) &= T_j(u(t_j)) = -I_j(u_2) \ge 0, \quad j = 1, 2, \ldots, n, \\ u(a_2) &= u(b_2) = -u_2. \end{aligned} \tag{43}$$

The solution $u(t)$ of (43) satisfies $u(t) \ge -u_2, t \in (a_2, b_2)$. So $\omega_2 = \emptyset$ and $u(t) \ge -u_2$.

Therefore, the solutions of (41) are solutions of (1). Hence to prove Theorem 6, it suffices to produce at least k distinct pairs of critical points of

$$\begin{aligned} E_2(u) = \frac{1}{2}\int_0^T e^{G(t)}\left|u'(t)\right|^2 dt - \int_0^T H_2(\lambda, t, u(t))\, dt \\ - \sum_{j=1}^n e^{G(t_j)} \int_0^{u(t_j)} T_j(t)\, dt, \end{aligned} \tag{44}$$

where $H_2(\lambda, t, u(t)) = \int_0^u h_2(\lambda, t, s) ds$.

We will apply Lemma 3 to finish the proof.

By (44) and (H2), $E_2 \in C'(X, R)$ is even and $E_2(0) = 0$.

Next, we will show that E_2 is bounded from below.

By (H1) and (H2), we have $uh_2(\lambda, t, u(t)) \le 0$ and $uT_j(u) \le 0$ for $|u| \ge u_2$; thus

$$\begin{aligned} \int_0^T H_2(\lambda, t, u(t))\, dt &= \int_0^T \int_0^{u(t)} h_2(\lambda, t, s)\, ds\, dt \\ &\le \int_0^T \int_0^{u_2} h_2(\lambda, t, s)\, ds\, dt \\ &\le \int_0^T \int_0^{u_2} [rMs + f(t, s)]\, ds\, dt = \rho > 0, \\ \int_0^{u(t_j)} T_j(t)\, dt &\le \int_0^{u_2} T_j(t)\, dt = \delta > 0. \end{aligned} \tag{45}$$

So, we have

$$\begin{aligned} E_2(u) &= \frac{1}{2}\|u\|^2 - \int_0^T H_2(\lambda, t, u(t))\, dt \\ &\quad - \sum_{j=1}^n e^{G(t_j)} \int_0^{u(t_j)} T_j(t)\, dt \\ &\ge \frac{1}{2}\|u\|^2 - \rho - nM\delta \\ &> -\infty, \end{aligned} \tag{46}$$

for any $u \in X$. Therefore, E_2 is bounded from below.

In the following we will show that E_2 satisfies the P.S. condition. Let $\{u_k\} \subset X$ such that $\{E_2(u_k)\}$ is a bounded sequence and $\lim_{k\to\infty} E_2'(u_k) = 0$; then there exists $C_4 > 0$ such that

$$\left|E_2(u_k)\right| \le C_4. \tag{47}$$

By (46), we have

$$\frac{1}{2}\|u_k\|^2 \le C_4 + \rho + nM\delta. \tag{48}$$

So $\{u_k\}$ is bounded in X. From the reflexivity of X, we may extract a weakly convergent subsequence that, for simplicity, we call $\{u_k\}$, $u_k \rightharpoonup u$ in X. In the following we will verify that $\{u_k\}$ strongly converges to u:

$$\begin{aligned} &\left(E_2'(u_k) - E_2'(u)\right)(u_k - u) \\ &\quad = \|u_k - u\|^2 - \int_0^T [h_2(\lambda, t, u_k(t)) - h_2(\lambda, t, u(t))] \\ &\qquad\qquad \times (u_k(t) - u(t))\, dt \\ &\quad + \sum_{j=1}^n e^{G(t_j)} \left[T_j(u_k(t_j)) - T_j(u(t_j))\right] \\ &\qquad \times \left(u_k(t_j) - u(t_j)\right). \end{aligned} \tag{49}$$

By $u_k \rightharpoonup u$ in X, we see that $\{u_k\}$ uniformly converges to u in $C[0,T]$. So

$$\begin{aligned} &\int_0^T [h_2(\lambda, t, u_k(t)) - h_2(\lambda, t, u(t))](u_k(t) - u(t))\, dt \longrightarrow 0, \\ &\sum_{j=1}^n e^{G(t_j)} \left[T_j(u_k(t_j)) - T_j(u(t_j))\right]\left(u_k(t_j) - u(t_j)\right) \longrightarrow 0, \\ &\left(E_2'(u_k) - E_2'(u)\right)(u_k - u) \to 0, \quad \text{as } k \longrightarrow +\infty. \end{aligned} \tag{50}$$

So we obtain $\|u_k - u\| \to 0$, as $k \to +\infty$. That is, $\{u_k\}$ strongly converges to u in X, which means E_2 satisfies the P.S. condition.

Now set $K = \{\sum_{i=1}^k c_i\varphi_i : \sum_{i=1}^k c_i^2 = c^2\}$, where φ_i is defined in (9). It is clear that K is homeomorphic to S^{k-1} by an odd

map for any $c > 0$. In the following we verify that $E_2|_K < 0$ if c is sufficiently small.

For any $u \in K$, $u = \sum_{i=1}^{k} c_i\varphi_i$. By (H3) and (44), we have

$$\begin{aligned}
E_2(u) &= \frac{1}{2}\int_0^T e^{G(t)}\left[\left(\sum_{i=1}^{k} c_i\varphi_i(t)\right)'\right]^2 dt \\
&\quad - \int_0^T H_2(\lambda, t, u(t))\,dt \\
&\quad - \sum_{j=1}^{n} e^{G(t_j)}\int_0^{u(t_j)} T_j(t)\,dt \\
&= \frac{1}{2}\sum_{i=1}^{k} c_i^2\int_0^T e^{G(t)}\left[\varphi_i'(t)\right]^2 dt \\
&\quad - \frac{\lambda}{2}\sum_{i=1}^{k} c_i^2\int_0^T e^{G(t)}\left[\varphi_i(t)\right]^2 dt \\
&\quad - \int_0^T e^{G(t)}F(t, u(t))\,dt \\
&\quad - \sum_{j=1}^{n} e^{G(t_j)}\int_0^{u(t_j)} T_j(t)\,dt \\
&\le \frac{M}{2}\sum_{i=1}^{k} c_i^2\int_0^T \left[\varphi_i'(t)\right]^2 dt - \frac{m\lambda}{2}\sum_{i=1}^{k} c_i^2\int_0^T \left[\varphi_i(t)\right]^2 dt \\
&\quad - \int_0^T e^{G(t)}F(t, u(t))\,dt - \sum_{j=1}^{n} e^{G(t_j)}\int_0^{u(t_j)} T_j(t)\,dt \\
&= \frac{1}{2}\sum_{i=1}^{k} c_i^2\left(M - \frac{m\lambda}{\lambda_i}\right) - \int_0^T e^{G(t)}F(t, u(t))\,dt \\
&\quad - \sum_{j=1}^{n} e^{G(t_j)}\int_0^{u(t_j)} T_j(t)\,dt \\
&\le \frac{1}{2}\left(M - \frac{m\lambda}{\lambda_k}\right)c^2 + o\left(c^2\right) + o\left(c^2\right),
\end{aligned} \tag{51}$$

for small $c > 0$. Since $\lambda \in (M\lambda_k/m, r]$, $E_2(u) < 0$ and the proof is complete. □

4. Example

To illustrate how our main results can be used in practice we present the following example.

Example 1. Let $T = \pi/4, g(t) = -2t$, and consider the following problem:

$$\begin{aligned}
&-u''(t) - 2tu'(t) - \lambda u(t) \\
&= (1+t)\left(u - u^2\right) - 1000e^{(\pi^2/16)-t^2}, \quad t \in \left[0, \frac{\pi}{4}\right],\ t \ne t_j, \\
&-\Delta u'\left(t_j\right) = 2 - \sqrt[3]{u\left(t_j\right)}, \quad j = 1, 2, \ldots, n, \\
&u'(0) = 0, \quad u\left(\frac{\pi}{4}\right) = 0.
\end{aligned} \tag{52}$$

Compared with (1), $f(t,u) = (1+t)(u-u^2) - 1000e^{(\pi^2/16)-t^2}$, $I_j(u) = 2 - \sqrt[3]{u(t)}$. Obviously (H2), (H3), and (H4) are satisfied. Let $u_1 = 1$, $r = 1000$; then (H1) is satisfied. By Theorem 4, for $(M\lambda_k/m, 1000] = (4e^{\pi^2/16}(2k-1)^2\pi^2, 1000]$, $k = 1, 2$, problem (1) has at least k distinct pairs of solutions.

Example 2. Let $T = \pi/2$, $g(t) = -t/4$, and consider the following problem:

$$\begin{aligned}
&-u''(t) - \frac{t}{4}u'(t) - \lambda u(t) \\
&= \left(1+t^2\right)\left(2u - u^2\right) - 1000e^{(\pi^2/32)-t^2}, \\
&\qquad t \in \left[0, \frac{\pi}{2}\right],\ t \ne t_j, \\
&-\Delta u'\left(t_j\right) = -u\left(t_j\right), \quad j = 1, 2, \ldots, n, \\
&u'(0) = 0, \quad u\left(\frac{\pi}{2}\right) = 0.
\end{aligned} \tag{53}$$

Compared with (1), $f(t,u) = (1+t^2)(2u-u^2) - 1000e^{(\pi^2/32)-t^2}$, $I_j(u) = -u(t_j)$. Obviously (H2), (H3), and (H4) are satisfied. Let $u_1 = 2$, $r = 500$; then (H1) is satisfied. By Theorem 5, for $(M\lambda_k/m, 500] = (e^{\pi^2/32}(2k-1)^2\pi^2, 500]$, $k = 1, 2, 3, 4$, problem (53) has at least k distinct pairs of solutions.

Example 3. Let $T = \pi/2$, $g(t) = -t/2$, and consider the following problem:

$$\begin{aligned}
&-u''(t) - \frac{t}{2}u'(t) - \lambda u(t) \\
&= -e^{\pi^2/16}\left(1+t^2\right)u^3(t), \quad t \in \left[0, \frac{\pi}{2}\right],\ t \ne t_j, \\
&-\Delta u'\left(t_j\right) = -3u^3\left(t_j\right), \quad j = 1, 2, \ldots, n, \\
&u'(0) = 0, \quad u\left(\frac{\pi}{2}\right) = 0.
\end{aligned} \tag{54}$$

Compared with (1), $f(t,u) = -e^{\pi^2/16}(1+t^2)u^3(t)$, $I_j(u) = -3u^3(t_j)$. Obviously (H2) and (H3) are satisfied. Let $u_2 = 25, r = 625$; then (H1) is satisfied. By Theorem 6, for $(M\lambda_k/m, 625] = (e^{\pi^2/16}(2k-1)^2\pi^2, 625]$, $k = 1, 2, 3$, problem (54) has at least k distinct pairs of solutions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

This work is partially supported by the National Natural Science Foundation of China (no. 71201013) and the Innovation Platform Open Funds for Universities in Hunan Province (no. 13K059).

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