Observability Estimate for the Fractional Order Parabolic Equations on Measurable Sets

Guojie Zheng1,2 and M. Montaz Ali2,3

1 College of Mathematics and Information Science, Henan Normal University, Xinxiang 453007, China
2 School of Computational and Applied Mathematics, University of the Witwatersrand (Wits), Johannesburg 2050, South Africa
3 TCSE, Faculty of Engineering and Built Environment, University of the Witwatersrand (Wits), Johannesburg 2050, South Africa

Correspondence should be addressed to Guojie Zheng; guojiezeng@yeah.net

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Abstract and Applied Analysis

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We establish an observability estimate for the fractional order parabolic equations evolved in a bounded domain \( \Omega \) of \( \mathbb{R}^n \). The observation region is \( F \times \omega \), where \( \omega \) and \( F \) are measurable subsets of \( \Omega \) and \((0, T)\), respectively, with positive measure. This inequality is equivalent to the null controllable property for a linear controlled fractional order parabolic equation. The building of this estimate is based on the Lebeau-Robbiano strategy and a delicate result in measure theory provided in Phung and Wang (2013).

Moreover, the operator \( A^\alpha \) is a self-adjoint operator and \( -A^\alpha \) is an infinitesimal generator of a strong continuous semigroup \( \{ S_t(t) \}_{t \geq 0} \). Now, we consider the following linear controlled fractional order parabolic equation:

\[
\partial_t y (x, t) + A^\alpha y (x, t) = Bu (x, t), \quad (x, t) \in \Omega \times (0, T],
\]

where \( \alpha > 1/2 \), \( B \) is a linear bounded operator in \( L^2(\Omega) \) defined by \( Bu = \chi_F(t) \cdot \chi_\omega (x) u, \ y_0 \in L^2(\Omega) \), and \( u(\cdot, t) \) is a control function taken from the space \( L^2(0, T; L^2(\Omega)) \). We denote \( y(\cdot; y_0, u) \) to be the unique solution of (3) corresponding to the control \( u \) and the initial value \( y_0 \). We denote \( \| \cdot \| \) and \( \langle \cdot, \cdot \rangle \) to be the usual norm and the inner product in \( L^2(\Omega) \), respectively.

In recent years, extensive research has been devoted to the study of differential equations with fractional orders due to their importance for applications in various branches of applied sciences and engineering. Many important phenomena in signal processing, electromagnetics, crowded systems, and fluid mechanics are well described by fractional differential equation (see [1]). In this paper, we always discuss the fractional Laplacian. The fractional Laplacian \(-A^\alpha\) with
\( \alpha \in (0, 1] \), generates the rotationally invariant \( 2\alpha \) stable Lévy process. For \( \alpha = 1 \), this process is the normal Brownian motion \( B_t \) on \( \mathbb{R}^n \) (see [2]).

Now, we will focus on the issue of what the controllable property is for the controlled system (3). System (3) is said to be null controllable in time \( T \) if for any \( y_0 \in L^2(\Omega) \), there exists a control function \( u \in L^2(0, T; L^2(\Omega)) \), such that the solutions of (3) matches

\[
y(T; y_0, u) = 0.
\]

The problem of null controllability of parabolic equations has also been the object of numerous studies. Extensive related references can be found in [3–7] and the rich works cited therein. Especially, we refer to [5] for a null controllability result for the parabolic equations which plays a crucial role in establishing the main result in our paper. In the above works, the control region \( \omega \) is always assumed to contain an open ball. The reason is that the main technique used in the argument, Carleman inequality, is required to construct weight functions. The construction of such functions seems to be not possible, when \( \omega \) do not contain a ball. Recently, the null controllability for the parabolic equations with \( \omega \) that is a measurable subset of positive measure has been established in [8], where an inequality involving measurable sets for a class of real analytic functions was set up in a skillful way. On the other hand, the classical null controllability for some fractional order parabolic equation was studied in [9, 10].

In particular, in [9] the authors proved that one-dimensional problem is not controllable from the boundary for \( \alpha \in (0, 1/2) \). Indeed, the semigroup can be written as follows:

\[
S_\alpha(t) \psi = \sum_{i=1}^{\infty} e^{-\lambda_i^\alpha t} \langle \psi, \mathcal{E}_i \rangle \mathcal{E}_i, \quad \text{for } \psi \in L^2(\Omega). \quad (8)
\]

From this, it follows that

\[
\|S_\alpha(t)\psi\|_{L^2(\Omega)} \leq e^{-\lambda_1^\alpha t} \|\psi\|_{L^2(\Omega)},
\]

for any \( \alpha > 1/2 \) (see [12]).

Throughout the rest of the paper, the following notation will be used. For each measurable set \( A \subseteq \mathbb{R}, |A| \) stands for its Lebesgue measure in \( \mathbb{R} \). The following lemma is quoted from [13].

**Lemma 2.** Let \( E \subseteq [0, T] \) be a measurable set with a positive measure, and let \( l \) be a density point for \( E \). Then for each \( z > 1 \), there exists a \( l_i \in (l - 1, l) \) such that the sequence \( \{l_i\}_{i=1}^{\infty} \), given by

\[
l_{i+1} = l + \frac{1}{z} (l - l_i), \quad (10)
\]

satisfies

\[
|E \cap (l_{i+1}, l_i)| \geq \frac{1}{2} (l_i - l_{i+1}). \quad (11)
\]

Next, we recall the following results, which play a key role in this paper.

**Lemma 3.** Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \), \( n \geq 1 \). Suppose that \( \Omega \) have real analytic boundary. Then, for each subset \( \omega \subset \Omega \) with positive measure, there exist two positive constants \( C_1 > 1 \) and \( C_2 > 0 \), which only depend on \( \Omega, \omega \), such that

\[
\sum_{i \leq r} |a_i| \leq C_1 e^{C_2 r^\alpha} \int_\omega \left| \sum_{i \leq r} a_i \psi_i(x) \right|^2 dx \quad (12)
\]
for each finite $r > 0$ and each choice of the coefficients $\{a_i\}_{\lambda_i \leq r}$ with $a_i \in \mathbb{R}$.

This conclusion can be found in the literature [8].

Next, for each $r > 0$, we define $X_r = \text{span}\{e_i(x)\}_{\lambda_i \leq r}$ and $X_r^+ = \text{span}\{e_i(x)\}_{\lambda_i > r}$. Indeed, for each $r > 0$, $L^2(\Omega) = X_r \oplus X_r^+$.

**Lemma 4.** For any $\xi_0 \in X_r^+$, it always holds that

$$
\|S_0(t)\xi_0\|_{L^2(\Omega)} < e^{-rt} \|\xi_0\|_{L^2(\Omega)}.
$$

(13)

This lemma can be easily obtained by (8) and (9).

### 3. The Proof of Main Result

**Proof.** Let $E = \{t \mid t = T - s, \text{ where } s \in F\}$. Then, $|E| = |F| > 0$. Let $I$ be a density point for $E$. By Lemma 2, for $z > 1$, there exists a $i_0 \in (I, T)$ and a sequence $\{l_i\}_{i=1}^{\infty}$ satisfying (10) and (11).

We now define a subset of $(0, T)$ as follows:

$$
E_i = \left\{ t - \frac{l_i - l_{i+1}}{4} \mid t \in E \cap \left( l_{i+1} + \frac{l_i - l_{i+1}}{4}, l_{i+1} \right) \right\}.
$$

(14)

In fact, $E_i$ is a subset of $(l_{i+1}, l_{i+1} + (3/4)(l_i - l_{i+1}))$, and

$$
|E_i| = |E \cap \left( l_{i+1} + \frac{l_i - l_{i+1}}{4}, l_{i+1} \right)|

= |E \cap \left( (l_{i+1}, l_i) \setminus \left( l_{i+1}, l_{i+1} + \frac{l_i - l_{i+1}}{4} \right) \right)|.
$$

(15)

By (11), it follows that

$$
|E_i| \geq \frac{l_i - l_{i+1}}{4}.
$$

(16)

Let $i_0$ be the first natural number satisfying $i_0 > 1/(2\alpha - 1)$; namely, $i_0 = \lceil 1/(2\alpha - 1) \rceil + 1$. Let $b > 1$ be a positive number satisfying

$$
\frac{b^\alpha}{2} \left( 1 - \frac{1}{z} \right) (l_i - l_i) > 4C_2 b^{(1+1)/2} + 4 \ln (8C_1 z),
$$

for $i = i_0, i_0 + 1, i_0 + 2, \ldots$.

Taking $r_i = b_i$, by (8), it follows that, for any $\xi \in X_r$,

$$
\int_{l_{i+1}}^{l_{i+1}+(3/4)(l_i - l_{i+1})} X_E(t) \|S_0\left( l_{i+1} + \frac{3}{4} (l_i - l_{i+1}) \right) \xi \|_{L^2(\Omega)} dt

\leq \int_{l_{i+1}}^{l_{i+1}+(3/4)(l_i - l_{i+1})} X_E(t) \|S_0\xi \|_{L^2(\Omega)} dt.
$$

(17)

Combining with (16) and (12), this shows that

$$
\frac{l_i - l_{i+1}}{4} \|S_0\left( l_{i+1} + \frac{3}{4} (l_i - l_{i+1}) \right) \xi \|_{L^2(\Omega)}

\leq |E_i| \cdot \|S_0\left( l_{i+1} + \frac{3}{4} (l_i - l_{i+1}) \right) \xi \|_{L^2(\Omega)}

\leq \left[ \int_{l_{i+1}}^{l_{i+1}+(3/4)(l_i - l_{i+1})} X_E(t) \|S_0\xi \|_{L^2(\Omega)} dt \right] \|S_0(l_i)\phi_1\|_{L^2(\Omega)}.
$$

(18)

For each $\phi \in L^2(\Omega)$, we can write $\phi = \phi_1 + \phi_2$, where $\phi_1 \in X_r$ and $\phi_2 \in X_r^+$. Taking $\xi = S_0((l_i - l_{i+1})/4)\phi_1$ in (19), it follows that

$$
\frac{l_i - l_{i+1}}{4} \|S_0(l_i)\phi_1\|_{L^2(\Omega)}

\leq \frac{l_i - l_{i+1}}{4} \|S_0\left( l_{i+1} + \frac{3}{4} (l_i - l_{i+1}) \right) \phi_1 \|_{L^2(\Omega)}

\leq C_1 e^{C_2 \sqrt{T}} \int_{l_{i+1}}^{l_{i+1}+(3/4)(l_i - l_{i+1})} X_E(t) \|S_0\xi \|_{L^2(\Omega)} dt,
$$

(19)

By the definition of $E_i$, it is easily seen that

$$
X_E\left( t - \frac{l_i - l_{i+1}}{4} \right) = X_E(t),
$$

(20)

for any $t \in \left( l_{i+1} + \frac{l_i - l_{i+1}}{4}, l_{i+1} \right)$.

This, together with (20), deduces

$$
\frac{l_i - l_{i+1}}{4} \|S_0(l_i)\phi_1\|_{L^2(\Omega)}

\leq C_1 e^{C_2 \sqrt{T}} \int_{l_{i+1}+(l_i - l_{i+1})/4}^{l_{i+1}} X_E(t) \|S_0\|_{L^2(\omega)} + \|S_0(t)\phi_1\|_{L^2(\Omega)} dt,
$$

(21)

and

$$
\frac{l_i - l_{i+1}}{4} \|S_0\xi \|_{L^2(\Omega)}

\leq \int_{l_{i+1}+(l_i - l_{i+1})/4}^{l_{i+1}} X_E(t) \|S_0\xi \|_{L^2(\Omega)} dt

+ C_1 e^{C_2 \sqrt{T}} \left( l_i - l_{i+1} \right) \|S_0\left( l_{i+1} + \frac{l_i - l_{i+1}}{4} \right) \phi_2 \|_{L^2(\Omega)} dt.
$$

(22)

The last step is based on the energy decay property of $S_0(t)$. Along with Lemma 4, we derive that

$$
\frac{l_i - l_{i+1}}{4} \|S_0(l_i)\phi_1\|_{L^2(\Omega)}

\leq C_1 e^{C_2 \sqrt{T}} \int_{l_{i+1}+(l_i - l_{i+1})/4}^{l_{i+1}} X_E(t) \|S_0\xi \|_{L^2(\Omega)} dt

+ C_1 e^{C_2 \sqrt{T}} \left( l_i - l_{i+1} \right) e^{-C_2(l_i - l_{i+1})/4} \|S_0(l_i)\phi_2\|_{L^2(\Omega)}.$$
\[
\leq C_1 e^{C_2 \sqrt{\tau}} \int_{l_i}^{l_i+1} \chi_E(t) \|s_i(t)\|_{L^2(\omega)} dt + C_4 e^{C_2 \sqrt{\tau}} \left( l_i - l_{i+1} \right) e^{-\tau_0(l_i-l_{i+1})/4} \|s_i(l_{i+1})\|_{L^2(\Omega)}.
\]
(23)

Therefore,
\[
\frac{l_i - l_{i+1}}{4} \|s_i(l_i)\|_{L^2(\Omega)} \leq C_1 e^{C_2 \sqrt{\tau}} \int_{l_i}^{l_i+1} \chi_E(t) \|s_i(t)\|_{L^2(\omega)} dt + C_4 e^{C_2 \sqrt{\tau}} \left( l_i - l_{i+1} \right) e^{-\tau_0(l_i-l_{i+1})/4} \|s_i(l_{i+1})\|_{L^2(\Omega)}
\]
\[
+ \frac{l_i - l_{i+1}}{4} \|s_i(l_{i+1})\|_{L^2(\Omega)}.
\]
(24)

Thus,
\[
\frac{l_i - l_{i+1}}{4} \|s_i(l_i)\|_{L^2(\Omega)} \leq C_1 e^{C_2 \sqrt{\tau}} \int_{l_i}^{l_i+1} \chi_E(t) \|s_i(t)\|_{L^2(\omega)} dt + \left( l_i - l_{i+1} \right) e^{-\tau_0(l_i-l_{i+1})/4} \left( C_4 e^{C_2 \sqrt{\tau}} + 1 \right) \|s_i(l_{i+1})\|_{L^2(\Omega)}.
\]
(25)

This leads to
\[
\frac{l_i - l_{i+1}}{4C_1 e^{C_2 \sqrt{\tau}}} \|s_i(l_i)\|_{L^2(\Omega)} \leq \frac{C_1 e^{C_2 \sqrt{\tau}} + 1}{C_4 e^{C_2 \sqrt{\tau}}} \left( l_i - l_{i+1} \right) e^{-\tau_0(l_i-l_{i+1})/4} \|s_i(l_{i+1})\|_{L^2(\Omega)}
\]
\[
\leq \int_{l_i}^{l_{i+1}} \chi_E(t) \|s_i(t)\|_{L^2(\omega)} dt.
\]
(26)

Summing (26) from \( i_0 \) to \( \infty \), it follows that
\[
\frac{l_i - l_{i+1}}{4C_1 e^{C_2 \sqrt{\tau}}} \|s_i(l_i)\|_{L^2(\Omega)} + \sum_{i_{i+1}}^{\infty} k_i \|s_i(l_{i+1})\|_{L^2(\Omega)}
\]
\[
\leq \int_0^T \chi_E(t) \|s_i(t)\|_{L^2(\omega)} dt,
\]
(27)

where
\[
k_i = \frac{l_{i+1} - l_{i+2}}{4C_1 e^{C_2 \sqrt{\tau}}} = \frac{C_1 e^{C_2 \sqrt{\tau}} + 1}{C_4 e^{C_2 \sqrt{\tau}}} \left( l_i - l_{i+1} \right) e^{-\tau_0(l_i-l_{i+1})/4},
\]
\[
i = i_0, i_0 + 1, i_0 + 2, \ldots
\]
(28)

Actually, by (17), we can derive that
\[
k_i > 0, \quad \text{for any } i = i_0, i_0 + 1, i_0 + 2, \ldots
\]
(29)

This, together with (27), shows that
\[
\|s_i(T)\|_{L^2(\omega)} \leq \|s_i(l_i)\|_{L^2(\Omega)} \leq \frac{4C_1 e^{C_2 \sqrt{\tau}}}{l_i - l_{i+1}} \int_E \|s_i(t)\|_{L^2(\omega)} dt.
\]
(30)

Now, we are in the position to prove (7). The solution of (5) can be written as follows:
\[
\varphi(t) = s_i(T - t) \varphi_0.
\]
(31)

Along with (30), we can deduce that
\[
\|\varphi(x, 0)\|_{L^2(\omega)} \leq \frac{4C_1 e^{C_2 \sqrt{\tau}}}{l_i - l_{i+1}} \int_E \|\varphi(t)\|_{L^2(\omega)} dt.
\]
(32)

This completes the proof of the main result.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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