Research Article

A Note on Jordan, Adamović-Mitrinović, and Cusa Inequalities

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We improve the Jordan, Adamović-Mitrinović, and Cusa inequalities. As applications, several new Shafer-Fink type inequalities for inverse sine function and bivariate means inequalities are established, and a new estimate for sine integral is given.

1. Introduction

The classical Jordan inequality [1] is given by

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1, \quad (1)$$

for $x \in (0, \pi/2)$.

Some new developments on refinements, generalizations, and applications for the Jordan inequality can be found in [2] and the references therein.

In the recent past, the following two-side inequality

$$(\cos x)^{1/3} < \frac{\sin x}{x} < \frac{2 + \cos x}{3} \quad \left(0 < x < \frac{\pi}{2}\right) \quad (2)$$

has attracted the attention of many researchers (see, e.g., [2–14]). The left inequality in (2) was obtained by Mitrinović (see [1, p. 238]), while the right one is due to Huygens (see, e.g., [15]) and it is called *Cusa inequality* [3, 5, 6, 8, 14].

In [16], the following open problem was proposed: for each $p > 0$, there are greatest value $q = q(p)$ and least value $r = r(p)$ such that the double inequality

$$\frac{q \sin x}{1 + p \cos x} < x < \frac{r \sin x}{1 + p \cos x} \quad (3)$$

holds for all $x \in (0, \pi/2)$. This was answered by Carver in [17]. In [1, p. 238, 3.4.15], it was listed that

$$\frac{(1+p) \sin x}{1+p \cos x} < x < \frac{(\pi/2) \sin x}{1+p \cos x} \quad (4)$$

for $p \in (0, 1/2]$ and $x \in [0, \pi/2]$. Wu [18] proved that the double inequality

$$\frac{(1+p) \cos x}{1+p \cos x} < \frac{\sin x}{x} < \frac{1+q}{1+q \cos x} \quad (5)$$

holds for $x \in (0, \pi/2)$, $p \in [-1, 2]$, and $q \in [-1/4, \infty)$. In particular, he obtained that for $x \in (0, \pi/2)$,

$$\frac{3 \cos x}{1+2 \cos x} < \frac{\sin x}{x} < \frac{3}{4 - \cos x}. \quad (6)$$

The first inequality in (6) is equivalent to the Huygen inequality:

$$2 \frac{\sin x}{x} + \frac{\tan x}{x} > 3. \quad (7)$$

Jiang [19] showed that for $x \in (0, \pi/2)$,

$$\frac{\sin x}{x} > \frac{1+2 \cos x}{2 + \cos x}. \quad (8)$$

Li and He [20] gave an improvement of (6) as follows:

$$\frac{7+5 \cos x}{11 + \cos x} < \frac{\sin x}{x} < \frac{9+6 \cos x}{14 + \cos x}. \quad (9)$$

The main purpose of this paper is to give sharp bounds for $(\sin x)/x$ in terms of the functions $H_1(\cos t, p)$ and $H_2(\cos t, p)$, where

$$H_1(x, p) = \frac{2p + (p+3)x}{3p+1+2x}, \quad x \in (0, 1), \quad (10)$$

$$p \in (-\infty, -1] \cup [0, \infty),$$

$$H_2(x, p) = \frac{3p+1}{\pi p} \frac{2p + (p+3)x}{(3p+1)+2x}, \quad x \in (0, 1), \quad (11)$$

$$p \in (-\infty, -1] \cup (0, \infty).$$

The rest of this paper is organized as follows. Several lemmas are given in Section 2. Main results and their proofs are given in Section 3, in which Theorem 7 unifies and generalizes Jordan and Cusa inequalities; Theorem 13 shows that Adamović-Mitrinović and Cusa inequalities (2) can be interpolated by $H_1(\cos x, p)$ for suitable p ; Theorem 18 gives a hyperbolic version of Theorem 7. In Section 4, some new Shafer-Fink type inequalities for inverse sine function and several inequalities for bivariate means are presented, and a simpler but more accurate estimate for sine integral is provided.

2. Lemmas

Lemma 1. Let H_1 and H_2 be defined by (10) and (11), respectively. Then H_1 and H_2 are, respectively, increasing and decreasing with respect to p on $(-\infty, -1] \cup (0, \infty)$ with the limits

$$\lim_{p \rightarrow -\infty} H_1(x, p) = \lim_{p \rightarrow \infty} H_1(x, p) = \frac{2+x}{3}, \quad (12)$$

$$\lim_{p \rightarrow -\infty} H_2(x, p) = \lim_{p \rightarrow \infty} H_2(x, p) = \frac{2+x}{\pi}.$$

Proof. From (10) and (11) we have

$$\begin{aligned} \frac{\partial H_1}{\partial p} &= \frac{2(x-1)^2}{(3p+1+2x)^2} > 0, \\ \frac{\partial H_2}{\partial p} &= -\frac{3x}{\pi p^2(3p+2x+1)^2} (p+1) \\ &\quad \times ((5-2x)p + (2x+1)). \end{aligned} \quad (13)$$

If $p \in (0, \infty)$, then we clearly see that $\partial H_2/\partial p < 0$. If $p \in (-\infty, -1)$, then $(5-2x)p + (2x+1) < 4(x-1) < 0$, and then $\partial H_2/\partial p < 0$.

Simple computations give (12). $\square$

Lemma 2. Let u_1, u_2 be defined on $(0, 1) \times (-\infty, -1] \cup [0, \infty)$ by

$$u_1(x, p) = (2p + (3+p)x)(3p+1+2x), \quad (14)$$

$$u_2(x, p) = 2(p+3)x^3 + 8px^2 + 2p(3p+1)x + 3(p+1)^2, \quad (15)$$

respectively. Then $u_1(x, p), u_2(x, p) > 0$.

Proof. It is not difficult to see that $u_1(x, p), u_2(x, p) > 0$ for $p \in [0, \infty)$. If $p \in (-\infty, -1]$, then

$$\begin{aligned} 2p + (3+p)x &< 2(x-1) < 0, \\ (3p+1+2x) &< 2(x-1) < 0, \end{aligned} \quad (16)$$

and then $u_1(x, p) > 0$. It remains to prove that $u_2(x, p) > 0$ for $p \in (-\infty, -1]$. Differentiation leads to

$$\frac{\partial u_2}{\partial p} = (12x+6)p + (2x^3 + 8x^2 + 2x + 6). \quad (17)$$

Hence, $\partial u_2/\partial p < -(12x+6) + (2x^3 + 8x^2 + 2x + 6) = 2x(x+5)(x-1) < 0$, which implies that u_2 is decreasing in p on $(-\infty, -1]$, and therefore,

$$u_2(x, p) > u_2(x, -1) = 4x(x-1)^2 > 0. \quad (18)$$

This completes the proof. $\square$

Lemma 3. Let u_3 be defined on $(0, 1) \times (-\infty, -1] \cup [0, \infty)$ by

$$\begin{aligned} u_3(x, p) &= (p+3)^2 x^2 + (p+3)(7p+3)x \\ &\quad + (-3p^3 + 13p^2 + 21p + 9). \end{aligned} \quad (19)$$

Then

- (i) $u_3(x, p) \geq 0$ for all $x \in (0, 1)$ if and only if $p \in (-\infty, p_3]$, where $p_3 \approx 5.6630$ is the unique solution of the equation $u_3(0, p) = -3p^3 + 13p^2 + 21p + 9 = 0$;
- (ii) $u_3(x, p) \leq 0$ if and only if $p \in [9, \infty)$;
- (iii) for every $p \in (p_3, 9)$, there exists a unique $x_1 \in (0, 1)$ such that $u_3(x, p) < 0$ for $x \in (0, x_1)$ and $u_3(x, p) > 0$ for $x \in (x_1, 1)$.

Proof. In order to prove the desired results, we need to rewrite $u_3(x, p)$ as

$$u_3(x, p) = \left((p+3)x + \frac{7p+3}{2} \right)^2 - 3 \left(p - \frac{9}{4} \right) (p+1)^2. \quad (20)$$

We clearly see that

$$\begin{aligned} u_3(0, p) &= -3p^3 + 13p^2 + 21p + 9, \\ u_3(1, p) &= -3(p+1)^2(p-9). \end{aligned} \quad (21)$$

We claim that there exists unique $p_3 \in (5, 6)$ such that $u_3(0, p) > 0$ for $p \in (-\infty, p_3)$ and $u_3(0, p) < 0$ for $p \in (p_3, \infty)$. Indeed, we have

$$\begin{aligned} u_3'(0, p) &= -9p^2 + 26p + 21 \\ &= -9 \left(p - \frac{13 - \sqrt{358}}{9} \right) \left(p - \frac{13 + \sqrt{358}}{9} \right), \end{aligned} \quad (22)$$

which implies that $u_3(0, p)$ is increasing on $((13 - \sqrt{358})/9, (13 + \sqrt{358})/9)$ and decreasing on $(-\infty, (13 - \sqrt{358})/9) \cup ((13 + \sqrt{358})/9, \infty)$. Since

$$u_3\left(0, \frac{13 - \sqrt{358}}{9}\right) = \frac{13952}{243} - \frac{716}{243} \sqrt{358} \approx 1.6652 > 0,$$

$$u_3\left(0, \frac{13 + \sqrt{358}}{9}\right) = \frac{716}{243} \sqrt{358} + \frac{13952}{243} > 0,$$

$$u_3(0, \infty) = -\infty, \quad (23)$$

there exists unique $p_3 \in ((13 + \sqrt{358})/9, \infty)$ such that $u_3(0, p_3) = 0$ and $u_3(0, p) > 0$ for $p \in (-\infty, p_3)$ and $u_3(0, p) < 0$ for $p \in (p_3, \infty)$. An easy calculation reveals that $p_3 \approx 5.6630$.

(i) Now we prove the necessary and sufficient condition for $u_3(x, p) \geq 0$ for all $x \in (0, 1)$. Since $u_3(x, -3) = 144 > 0$, we assume that $p \neq -3$. Denote the minimum point of $u_3(x, p)$ by x_0 . Then $x_0 = -(7p + 3)/(2(p + 3))$. And then, due to $\partial^2 u_3 / \partial x^2 > 0$, $u_3(x, p) \geq 0$ for all $x \in (0, 1)$ if and only if at least one of the following cases occur.

Case 1. Consider that $x_0 = -(7p + 3)/(2(p + 3)) \geq 1$, $u_3(1, p) \geq 0$. It is derived that $p \in (-3, -1]$.

Case 2. Consider that $x_0 = -(7p + 3)/(2(p + 3)) \leq 0$, $u_3(0, p) \geq 0$. It implies that $p \in (-\infty, -3) \cup [-3/7, p_3]$.

Case 3. Consider that $x_0 = -(7p + 3)/(2(p + 3)) \in (0, 1)$, $u_3(x_0, p) = -3(p - 9/4)(p + 1)^2 \geq 0$. It yields $p \in (-1, -3/7)$.

To sum up, $u_3(x, p) \geq 0$ for all $x \in (0, 1)$ if and only if $p \in (-\infty, p_3]$.

(ii) It is clear that $u_3(x, p) \leq 0$ if and only if $u_3(0, p) \leq 0$ and $u_3(1, p) \leq 0$. Solving the inequalities for p leads to $p \geq 9$.

(iii) In the case of $p \in (p_3, 9)$, we clearly see that $u_3(0, p) < 0$, $u_3(1, p) > 0$, and $x_0 = -(7p + 3)/(2(p + 3)) < 0$. This implies that there exists a unique $x_1 \in (0, 1)$ such that $u_3(x, p) < 0$ for $x \in (0, x_1)$ and $u_3(x, p) > 0$ for $x \in (x_1, 1)$.

This completes the proof. $\square$

Now let us consider the sign of function g defined on $(0, \pi/2) \times (-\infty, -1] \cup [0, \infty)$ by

$$\begin{aligned} g(t, p) &= t - \left((2p + (p + 3) \cos t)(3p + 1 + 2 \cos t) \right. \\ &\quad \times \left(2(p + 3) \cos^3 t + 8p \cos^2 t \right. \\ &\quad \left. \left. + 2p(3p + 1) \cos t + 3(p + 1)^2 \right)^{-1} \sin t \right) \\ &= t - \frac{u_1(\cos t, p)}{u_2(\cos t, p)} \sin t, \end{aligned} \quad (24)$$

where $u_1(x, p)$ and $u_2(x, p)$ are defined by (14) and (15), respectively. We have the following.

Lemma 4. Let g be defined on $(0, \pi/2) \times (-\infty, -1] \cup [0, \infty)$ by (24). Then

(i) $g(t, p) < 0$ for all $t \in (0, \pi/2)$ if and only if $p \in (-\infty, -1] \cup [9, \infty)$;

(ii) $g(t, p) > 0$ for all $t \in (0, \pi/2)$ if and only if $p \in [0, p_1]$, where

$$p_1 = \frac{2\sqrt{6\pi + 1} + 3\pi - 2}{12 - 3\pi} \approx 6.3433; \quad (25)$$

(iii) in the case of $p \in (p_1, 9)$, there exists a unique $t_0 \in (0, \pi/2)$ such that $g(t, p) > 0$ for $t \in (0, t_0)$ and $g(t, p) < 0$ for $t \in (t_0, \pi/2)$.

Proof. We first give two limit relations as follows:

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{g(t, p)}{t^5} &= -\frac{1}{45} \frac{p - 9}{p + 1} \quad \text{if } p \neq -1, \\ g\left(\frac{\pi^-}{2}, p\right) &= -\frac{12 - 3\pi}{6(p + 1)^2} (p - p_1)(p - p_2) \quad \text{if } p \neq -1, \end{aligned} \quad (26)$$

where

$$\begin{aligned} p_1 &= \frac{2\sqrt{6\pi + 1} + 3\pi - 2}{12 - 3\pi} \approx 6.3433, \\ p_2 &= -\frac{2\sqrt{6\pi + 1} - 3\pi + 2}{12 - 3\pi} < 0. \end{aligned} \quad (27)$$

In fact, if $p \neq -1$, then making use of power series we get

$$g(t, p) = -\frac{1}{45} \frac{p - 9}{p + 1} t^5 + o(t^5), \quad (28)$$

which implies the first relation. Direct computations yield the second one.

Differentiating $g(t, p)$ with respect to t leads to

$$\begin{aligned} \frac{\partial g}{\partial t} &= 1 - \frac{u_1(\cos t, p)}{u_2(\cos t, p)} \cos t + \left(\sin^2 t \right) \frac{d}{dx} \frac{u_1(x, p)}{u_2(x, p)} \Big|_{x=\cos t} \\ &= \frac{4(1 - x)(1 - x^2)}{u_2^2(x, p)} \times h(x, p), \end{aligned} \quad (29)$$

where $u_1(x, p)$ and $u_2(x, p)$ are defined by (14) and (15), respectively, and

$$h(x, p) = \left(x + \frac{3p + 1}{2} \right) \times u_3(x, p); \quad (30)$$

here $u_3(x, p)$ is defined by (19) and $x = \cos t \in (0, 1)$.

(i) We now prove that $g(t, p) \leq 0$ for all $t \in (0, \pi/2)$ if and only if $p \in (-\infty, -1] \cup [9, \infty)$. The necessity easily follows from the inequalities $\lim_{t \rightarrow 0^+} t^{-5} g(t, p) \leq 0$ and $g(\pi/2^-, p) \leq 0$ if $p \neq -1$ and $g(t, -1) = t - \tan t < 0$ together with the relation (26).

Next we prove the sufficiency. If $p \in [9, \infty)$, then by Lemma 3 $u_3(x, p) \leq 0$, and then $h(x, p) \leq 0$. This indicates that g is decreasing in t on $(0, \pi/2)$, and therefore, we get $g(t, p) < g(0^+, p) = 0$. If $p \in (-\infty, -1]$, then $u_3(x, p) \geq 0$ and $x + (3p + 1)/2 < x - 1 < 0$, which yields $h(x, p) \leq 0$. This also yields that g is decreasing in t on $(0, \pi/2)$, and so $g(t, p) < g(0^+, p) = 0$.

(ii) Similarly, we can prove that $g(t, p) > 0$ for all $t \in (0, \pi/2)$ if and only if $p \in [0, p_1]$. If $g(t, p) > 0$ for all $t \in (0, \pi/2)$, then we have $\lim_{t \rightarrow 0^+} t^{-5} g(t, p) \geq 0$ and $g(\pi/2^-, p) \geq 0$, which together with (26) and $p \in (-\infty, -1] \cup [0, \infty)$ lead to $p \in [0, p_1]$.

In order to prove the sufficiency, we distinguish two cases.

In the case of $p \in [0, p_3]$, by Lemma 3 we have $u_3(x, p) \geq 0$, which implies that g is increasing in t on $(0, \pi/2)$, and so, $g(t, p) > g(0^+, p) = 0$.

In the case of $p \in (p_3, p_1]$, from Lemma 3 there is a unique $x_1 \in (0, 1)$ such that $u_3(x, p) < 0$ for $x \in (0, x_1)$ and $u_3(x, p) > 0$ for $x \in (x_1, 1)$. This in conjunction with (30) and (29) shows that g is decreasing in t on $(\arccos x_1, \pi/2)$ and increasing on $(0, \arccos x_1)$, and consequently, we have

$$\begin{aligned} g(t, p) &> g(0^+, p) = 0 \quad \text{for } t \in (0, \arccos x_1), \\ g(t, p) &> g\left(\frac{\pi^+}{2}, p\right) = -\frac{12-3\pi}{6(p+1)^2} (p-p_1)(p-p_2) \geq 0 \\ &\quad \text{for } t \in \left(\arccos x_1, \frac{\pi}{2}\right), \end{aligned} \quad (31)$$

which proves the sufficiency.

(iii) In the case when $p \in (p_1, 9)$, we have seen that g is decreasing in t on $(\arccos x_1, \pi/2)$ and increasing on $(0, \arccos x_1)$ and $g(t, p) > 0$ for $t \in (0, \arccos x_1)$, but

$$g\left(\frac{\pi^-}{2}, p\right) = -\frac{12-3\pi}{6(p+1)^2} (p-p_1)(p-p_2) < 0. \quad (32)$$

Thus, there is a unique $t_0 \in (\arccos x_1, \pi/2)$ such that $g(t, p) > 0$ for $t \in (0, t_0)$ and $g(t, p) < 0$ for $t \in (t_0, \pi/2)$.

The whole proof is complete. $\square$

We next observe the function f defined on $(0, \pi/2) \times (-\infty, -1] \cup [0, \infty)$ by

$$\begin{aligned} f(t, p) &= \ln \frac{\sin t}{t} - \ln H_1(\cos t, p) \\ &= \ln \frac{\sin t}{t} - \ln \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t}. \end{aligned} \quad (33)$$

Differentiation yields that

$$\frac{\partial f}{\partial t} = \frac{\cos t}{\sin t} - \frac{1}{t} - \frac{2\sin t}{3p+1+2\cos t} + \frac{(p+3)\sin t}{2p+(p+3)\cos t}$$

$$\begin{aligned} &= \left((2(p+3)\cos^3 t + (3p^2 + 14p + 3)\cos^2 t \right. \\ &\quad \left. + 3(p+1)^2(\sin^2 t) + 2p(3p+1)\cos t \right) \\ &\quad \times ((\sin t)(2p+3\cos t + p\cos t)(3p+2\cos t+1))^{-1} - \frac{1}{t} \\ &= \frac{1}{\sin t} \frac{u_2(\cos t, p)}{u_1(\cos t, p)} - \frac{1}{t} = \frac{1}{t \sin t} \frac{u_2(\cos t, p)}{u_1(\cos t, p)} \times g(t, p), \end{aligned} \quad (34)$$

where $u_1(x, p)$, $u_2(x, p)$, and $g(t, p)$ are defined by (14), (15), and (24), respectively. From Lemmas 2 and 4 the following assertion is immediate.

Lemma 5. Let f be the function defined on $(0, \pi/2) \times (-\infty, -1] \cup [0, \infty)$ by (33). Then

- (i) f is decreasing in t on $(0, \pi/2)$ if and only if $p \in (-\infty, -1] \cup [9, \infty)$;
- (ii) f is increasing in t on $(0, \pi/2)$ if and only if $p \in [0, p_1]$, where $p_1 \approx 6.3433$ is given by (25);
- (iii) in the case when $p \in (p_1, 9)$, there is a unique $t_0 \in (0, \pi/2)$ such that f is increasing in t on $(0, t_0)$ and decreasing on $(t_0, \pi/2)$.

Lastly, for later use, we also give the following.

Lemma 6. Let H_1 be defined on $(0, 1) \times (-\infty, -1] \cup [0, \infty)$ by (10). Then $H_1(x^3, p) \geq x$ if and only if $p \in (-\infty, -1] \cup [1, \infty)$, and $H_1(x^3, p) \leq x$ if and only if $p = 0$.

Proof. For $p \in (-\infty, \infty)$, we define

$$\begin{aligned} u_4(x, p) &= 2x^2 + (1-p)x - 2p \\ &= 2\left(x - \frac{p-1}{4}\right)^2 - \frac{1}{7}(14p + p^2 + 1). \end{aligned} \quad (35)$$

Then $u_4(x, p) \geq 0$ holds for all $x \in (0, 1)$ if and only if $p \in (-\infty, 0]$.

In fact, $u_4(x, p) \geq 0$ if and only if at least one case of the following occurs.

Case 1. Consider that $(p-1)/4 \geq 1$, $u_4(1, p) = 3-3p \geq 0$. It is impossible.

Case 2. Consider that $(p-1)/4 \leq 0$, $u_4(0, p) = -2p \geq 0$. It indicates $p \in (-\infty, 0]$.

Case 3. Consider that $0 < (p-1)/4 < 1$, $u_4((p-1)/4, p) \geq 0$. It is impossible.

In the same way, we can prove that $u_4(x, p) \leq 0$ holds for all $x \in (0, 1)$ if and only if $p \in [1, \infty)$.

We now prove that $H_1(x^3, p) \geq x$ if and only if $p \in (-\infty, -1] \cup [1, \infty)$. Factoring yields

$$\begin{aligned} H_1(x^3, p) - x &= -2(x-1)^2 \frac{2x^2 + (1-p)x - 2p}{3p + 2x^3 + 1} \\ &= -(x-1)^2 \frac{u_4(x, p)}{3p + 2x^3 + 1}. \end{aligned} \quad (36)$$

If $p \in (-\infty, -1]$, then $3p + 2x^3 + 1 < 0$, and then, $H_1(x^3, p) \geq x$ if and only if $u_4(x, p) \geq 0$, which is equivalent to $p \in (-\infty, -1] \cap (-\infty, 0] = (-\infty, -1]$. If $p \in [0, \infty)$, then $3p + 2x^3 + 1 > 0$, and then, $H_1(x^3, p) \geq x$ if and only if $u_4(x, p) \leq 0$, which is equivalent to $p \in [0, \infty) \cap [1, \infty) = [1, \infty)$. Consequently, $H_1(x^3, p) \geq x$ if and only if $p \in (-\infty, -1] \cup [1, \infty)$.

Next we show that $H_1(x^3, p) \leq x$ if and only if $p = 0$. In fact, if $p \in (-\infty, -1]$, then $H_1(x^3, p) \leq x$ if and only if $u_4(x, p) \leq 0$, which yields $p \in [1, \infty)$. It is clearly a contradiction. If $p \in [0, \infty)$, then the statement in question if and only if $u_4(x, p) \geq 0$, which leads to $p \in [0, \infty) \cap (-\infty, 0] = \{0\}$. Thus the proof is complete. $\square$

3. Main Results

Theorem 7. Let $p \in (-\infty, -1] \cup [0, \infty)$. Then for $t \in (0, \pi/2)$,

$$\frac{\sin t}{t} < \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} \quad (37)$$

holds if and only if $p \in (-\infty, -1] \cup [9, \infty)$. Moreover, we have

$$\begin{aligned} H_2(\cos t, p) &= \lambda_p \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} < \frac{\sin t}{t} \\ &< \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} = H_1(\cos t, p) \end{aligned} \quad (38)$$

for $p \in (-\infty, -1] \cup [9, \infty)$, where $\lambda_p = (3p+1)/(\pi p)$ is the best possible. And the lower and upper bounds in (38) are decreasing and increasing in p on $(-\infty, -1] \cup (0, \infty)$, respectively.

Proof. Clearly, the desired result is equivalent to $f(t, p) < 0$ if and only if $p \in (-\infty, -1] \cup [9, \infty)$, where $f(t, p)$ is defined by (33). To this end, we give two limit relations. The first one follows by expanding $f(t, p)$ in power series for t . We have

$$f(t, p) = -\frac{1}{180} \frac{p-9}{p+1} t^4 + o(t^4) \quad \text{if } p \neq -1, \quad (39)$$

which yields

$$\lim_{t \rightarrow 0^+} \frac{f(t, p)}{t^4} = -\frac{1}{180} \frac{p-9}{p+1} \quad \text{if } p \neq -1. \quad (40)$$

The second one is derived by a simple computation; that is,

$$f\left(\frac{\pi^-}{2}, p\right) = \ln \frac{3p+1}{\pi p}. \quad (41)$$

Now we prove that $f(t, p) < 0$ for all $t \in (0, \pi/2)$ if and only if $p \in (-\infty, -1] \cup [9, \infty)$.

The necessity easily follows by solving the simultaneous inequalities:

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{f(t, p)}{t^4} &= -\frac{1}{180} \frac{p-9}{p+1} \leq 0 \quad \text{if } p \neq -1, \\ f(t, -1) &= \ln \frac{\sin t}{t} < 0, \\ f\left(\frac{\pi^-}{2}, p\right) &= \ln \frac{3p+1}{\pi p} \leq 0, \end{aligned} \quad (42)$$

which implies $p \in (-\infty, -1] \cup [9, \infty)$.

The sufficiency is due to Lemma 5. In fact, If $p \in (-\infty, -1] \cup [9, \infty)$, then by Lemma 5 we see that f is decreasing in t on $(0, \pi/2)$. Hence, $f(t, p) < f(0^+, p) = 0$.

Utilizing the monotonicity of f in t on $(0, \pi/2)$ gives (38). And from Lemma 1 it is seen that the lower and upper bounds in (38) are decreasing and increasing in p on $(-\infty, -1] \cup (0, \infty)$, respectively.

Thus the proof is finished. $\square$

By Theorem 7 and Lemma 1, we have the following interesting chain of inequalities.

Corollary 8. For $t \in (0, \pi/2)$, one has

$$\begin{aligned} \frac{2}{\pi} &= H_2(\cos t, -1) < \dots < H_2(\cos t, -\infty) \\ &= \frac{2 + \cos t}{\pi} = H_2(\cos t, \infty) < \dots < H_2(\cos t, 9) < \frac{\sin t}{t} \\ &< H_1(\cos t, 9) < \dots < H_1(\cos t, \infty) = \frac{2 + \cos t}{3} \\ &= H_1(\cos t, -\infty) \dots < H_1(\cos t, -1) = -1. \end{aligned} \quad (43)$$

Remark 9. It is clear that our results unify and refine Jordan and Cusa's inequalities and show that the first one in (9) is sharp. Also, Theorem 7 contains other known results, for example, taking $p = -3$ in (38) we get

$$\frac{8}{\pi} \frac{1}{4 - \cos t} < \frac{\sin t}{t} < 3 \frac{1}{4 - \cos t}, \quad (44)$$

which contain (6). After a simple transformation, (44) can be written as

$$\frac{8}{\pi} \frac{t}{\sin t} + \cos t < 4 < 3 \frac{t}{\sin t} + \cos t, \quad (45)$$

where the second inequality in (45) is due to Neuman and Sándor [6, (2.12)].

Theorem 10. Let $p \in (-\infty, -1] \cup [0, \infty)$. Then for $t \in (0, \pi/2)$

$$\frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} < \frac{\sin t}{t} \quad (46)$$

holds if and only if $p \in [0, p_0]$, where $p_0 = (\pi - 3)^{-1} \approx 7.0625$.

Moreover, for $p \in (0, p_1]$, one has

$$\begin{aligned} H_1(\cos t, p) &= \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} < \frac{\sin t}{t} \\ &< \lambda_p \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} = H_2(\cos t, p), \end{aligned} \quad (47)$$

where $p_1 \approx 6.3433$, $\lambda_p = (3p+1)/(\pi p)$ is the best possible. And $H_1(\cos t, p)$, $H_2(\cos t, p)$ are decreasing and increasing in p on $(-\infty, -1] \cup (0, \infty)$, respectively.

For $p \in (p_1, p_0]$ one has

$$\begin{aligned} H_1(\cos t, p) &= \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} < \frac{\sin t}{t} \\ &< \delta_p \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} = \delta_p H_1(\cos t, p), \end{aligned} \quad (48)$$

where $\delta_p = (\sin t_0/t_0)((3p+1)+2\cos t_0)/(2p+(p+3)\cos t_0)$ is the best possible and t_0 is the unique root of the equation

$$\begin{aligned} &\frac{(2p + (3+p)\cos t)(3p+1+2\cos t)}{2(p+3)\cos^3 t + 8p\cos^2 t + 2p(3p+1)\cos t + 3(p+1)^2} \\ &\times \sin t = t \end{aligned} \quad (49)$$

on $(0, \pi/2)$.

Proof. Since the inequality (46) is equivalent to $f(t, p) > 0$, it suffices to prove that $f(t, p) > 0$ holds for $t \in (0, \pi/2)$ if and only if $p \in [0, p_0]$.

Similarly, solving the simultaneous inequalities $\lim_{t \rightarrow 0^+} t^{-4} f(t, p) \geq 0$ and $f(\pi/2^-, p) \geq 0$ with $p \in (-\infty, -1] \cup (0, \infty)$ yields $p \in [0, p_0]$, which proves the necessity.

Conversely, the condition $p \in [0, p_0]$ is also sufficient for $f(t, p) > 0$ to be valid. For this end, we divide the proof into two cases.

Case 1. Consider that $p \in [0, p_1]$. By Lemma 5 it is seen that f is increasing in t on $(0, \pi/2)$, which indicates that $f(t, p) > f(0^+, p) = 0$.

Case 2. Consider that $p \in (p_1, p_0]$. By Lemma 5 we see that there is a unique $t_0 \in (0, \pi/2)$ such that f is increasing in t on $(0, t_0)$ and decreasing on $(t_0, \pi/2)$. It is acquired that

$$\begin{aligned} f(t_0, p) &> f(t, p) > f(0^+, p) = 0 \quad \text{for } t \in (0, t_0), \\ f(t_0, p) &> f(t, p) > f(\pi/2^-, p) = \ln \frac{3p+1}{\pi p} \geq 0 \\ &\quad \text{for } t \in \left(t_0, \frac{\pi}{2}\right); \end{aligned} \quad (50)$$

that is,

$$f(t_0, p) \geq f(t, p) > 0 \quad \text{for } t \in (0, \pi/2), \quad (51)$$

which proves the sufficiency.

In the first case, application of the monotonicity of f in t on $(0, \pi/2)$ leads to (47), and $\lambda_p = (3p+1)/(\pi p)$. In the second case, (51) also yields (47), and

$$\delta_p = \exp f(t_0, p) = \frac{\sin t_0}{t_0} \frac{(3p+1) + 2\cos t_0}{2p + (p+3)\cos t_0}. \quad (52)$$

Thus we complete the proof. $\square$

Remark 11. Taking $p = 7$ in (46), we get the first inequality in (9).

Letting $p = p_0 = (\pi - 3)^{-1}$ and solving (49) by mathematical computation software, we find that $t_0 \approx 1.3055$ and $\delta_{p_0} \approx 1.0015$. Letting $p = p_1$ be defined by (25) yields $\lambda_{p_1} = (3p_1+1)/(\pi p_1) \approx 1.0051$. By Theorem 10 we get the following.

Corollary 12. For $t \in (0, \pi/2)$, one has

$$\begin{aligned} \frac{2p_0 + (p_0+3)\cos t}{(3p_0+1) + 2\cos t} &< \frac{\sin t}{t} < \delta_{p_0} \frac{2p_0 + (p_0+3)\cos t}{(3p_0+1) + 2\cos t}, \\ \frac{2p_1 + (p_1+3)\cos t}{(3p_1+1) + 2\cos t} &< \frac{\sin t}{t} < \lambda_{p_1} \frac{2p_1 + (p_1+3)\cos t}{(3p_1+1) + 2\cos t}, \end{aligned} \quad (53)$$

where $\delta_{p_0} \approx 1.0015$ and $\lambda_{p_1} \approx 1.0051$ are the best possible constants.

Letting $x = \cos^{1/3} t$ in Lemma 6 and using Theorems 7 and 10, we obtain a chain of inequalities that interpolates Adamović-Mitrinović and Cusa's inequalities (2) by $H_1(\cos x, p)$.

Theorem 13. For $t \in (0, \pi/2)$, the inequalities

$$\begin{aligned} \frac{2p + (p+3)\cos t}{(3p+1) + 2\cos t} &< \cos^{1/3} t < \frac{2q + (q+3)\cos t}{(3q+1) + 2\cos t} \\ &< \frac{\sin t}{t} < \frac{2r + (r+3)\cos t}{(3r+1) + 2\cos t} < \frac{2 + \cos t}{3} \\ &< \frac{2s + (s+3)\cos t}{(3s+1) + 2\cos t} \end{aligned} \quad (54)$$

hold if and only if $p = 0$, $q \in [0, p_0]$, $r \in [9, \infty)$, and $s \in (-\infty, -1]$, where $p_0 = (\pi - 3)^{-1}$.

Using the monotonicity of $f(t, p)$ in t on $(0, \pi/4)$ given by parts one and two of Lemma 5, we see that

$$\begin{aligned} &\ln \left(\frac{4}{\pi} \frac{3p + \sqrt{2} + 1}{(2\sqrt{2} + 1)p + 3} \right) \\ &= f\left(\frac{\pi}{4}, p\right) \leq f\left(\frac{t}{2}, p\right) = \ln \frac{2\sin(t/2)}{t} \\ &\quad - \ln H_1\left(\cos \frac{t}{2}, p\right) \leq f(0, p) = 0 \end{aligned} \quad (55)$$

hold for $p \in (-\infty, -1] \cup [9, \infty)$. And then we have

$$\frac{4}{\pi} \frac{3p + \sqrt{2} + 1}{(2\sqrt{2} + 1)p + 3} H_1\left(\cos \frac{t}{2}, p\right) \cos \frac{t}{2} < \frac{\sin t}{t} = H_1\left(\cos \frac{t}{2}, p\right) \cos \frac{t}{2}. \quad (56)$$

It is clear that the right-hand in (56) is increasing in p on $(-\infty, -1] \cup [0, \infty)$, but the monotonicity of left-hand is to be checked. We define

$$H_3(x, p) = \frac{4}{\pi} \frac{3p + \sqrt{2} + 1}{(2\sqrt{2} + 1)p + 3} H_1(x, p), \quad (57)$$

where $x = \cos(t/2) \in [1/\sqrt{2}, 1]$. Logarithmic differentiation leads to

$$\begin{aligned} \frac{\partial \ln H_3}{\partial p} &= \frac{3}{(3p + \sqrt{2} + 1)} - \frac{2\sqrt{2} + 1}{(p(2\sqrt{2} + 1) + 3)} \\ &\quad - \frac{3}{(3p + 2x + 1)} + \frac{x + 2}{2p + x(p + 3)} \\ &= -\left(\left(6(2\sqrt{2} + 1)\left(x - \frac{\sqrt{2}}{2}\right)\left(\frac{22 - 9\sqrt{2}}{7} - x\right)\right)\right. \\ &\quad \times \left((3p + \sqrt{2} + 1)(p(2\sqrt{2} + 1) + 3)(3p + 2x + 1)\right. \\ &\quad \times \left.(2p + x(p + 3))\right)^{-1}\bigg)(p + 1)(p - u_5(x)), \end{aligned} \quad (58)$$

where

$$u_5(x) = \frac{(5 - 2\sqrt{2})x - (\sqrt{2} + 2)}{(5\sqrt{2} - 2) - (2\sqrt{2} + 1)x}. \quad (59)$$

Since

$$u'_5(x) = -\frac{12(3 - 2\sqrt{2})}{(5\sqrt{2} - 2 - (2\sqrt{2} + 1)x)^2} < 0, \quad (60)$$

we have $-1 = u_5(1) < u_5(x) < u_5(1/\sqrt{2}) = -(24\sqrt{2} + 5)/49 \approx -0.7947$. Consequently, $\partial(\ln H_3)/\partial p < 0$ for $p \in (-\infty, -1] \cup [0, \infty)$.

The result can be stated as a theorem.

Theorem 14. Let $p \in (-\infty, -1] \cup [0, \infty)$. Then for $t \in (0, \pi/2)$ the inequalities

$$\begin{aligned} \sigma_p \frac{2p \cos(t/2) + (p + 3) \cos^2(t/2)}{(3p + 1) + 2 \cos(t/2)} &< \frac{\sin t}{t} < \frac{2p \cos(t/2) + (p + 3) \cos^2(t/2)}{(3p + 1) + 2 \cos(t/2)}. \end{aligned} \quad (61)$$

hold if and only if $p \in (-\infty, -1] \cup [9, \infty)$, where $\sigma_p = (4/\pi)((3p + \sqrt{2} + 1)/((2\sqrt{2} + 1)p + 3))$ is the best constant. And the right-hand and left-hand in (61) are increasing and decreasing in p , respectively. Inequality (61) is reversed if and only if $p \in [0, p_1]$, where $p_1 \approx 6.3433$ is defined by (25).

Putting $p = 9, \infty, 0, 1$ in Theorem 14 we have the following.

Corollary 15. For $t \in (0, \pi/2)$ the following inequalities hold:

$$\frac{2(41\sqrt{2} - 25)}{7\pi} \frac{2\cos^2(t/2) + 3\cos(t/2)}{\cos(t/2) + 14} < \frac{\sin t}{t} < 3 \frac{2\cos^2(t/2) + 3\cos(t/2)}{\cos(t/2) + 14}, \quad (62)$$

$$\frac{4(2\sqrt{2} - 1)}{7} \frac{\cos^2(t/2) + 2\cos(t/2)}{\pi} < \frac{\sin t}{t} < \frac{\cos^2(t/2) + 2\cos(t/2)}{3}, \quad (63)$$

$$3 \frac{\cos^2(t/2)}{2\cos(t/2) + 1} < \frac{\sin t}{t} < \frac{4(\sqrt{2} + 1)}{\pi} \frac{\cos^2(t/2)}{2\cos(t/2) + 1}, \quad (64)$$

$$\frac{2\cos^2(t/2) + 1}{\cos(t/2) + 2} < \frac{\sin t}{t} < \frac{2(3 - \sqrt{2})}{\pi} \frac{2\cos^2(t/2) + 1}{\cos(t/2) + 2}. \quad (65)$$

Further, let H_4 be defined on $[1/\sqrt{2}, 1] \times (-\infty, -1] \cup [0, \infty)$ by

$$H_4(x, p) = \frac{H_1(2x^2 - 1, p)}{xH_1(x, p)}, \quad (66)$$

where H_1 is defined by (10). We can show that the monotonicity of H_4 in x for certain fixed p . Differentiation again yields

$$\begin{aligned} \frac{\partial \ln H_4(x, p)}{\partial x} &= \frac{2}{1 + 3p + 2x} - \frac{1}{x} - \frac{p + 3}{2p + (p + 3)x} \\ &\quad + \frac{4(p + 3)x}{(p - 3) + 2(p + 3)x^2} - \frac{8x}{4x^2 + 3p - 1}. \end{aligned} \quad (67)$$

It is easy to verify that

$$\begin{aligned} \frac{\partial \ln H_4(x, 9)}{\partial x} &= -2 \frac{(x - 1)^2 (594x^2 + 240x^3 + 8x^4 + 910x + 273)}{x(2x + 3)(x + 14)(2x^2 + 13)(4x^2 + 1)} \\ &< 0, \\ \frac{\partial \ln H_4(x, \infty)}{\partial x} &= -2 \frac{(1 - x)(2x + 1)}{x(x + 2)(2x^2 + 1)} < 0, \\ \frac{\partial \ln H_4(x, 1)}{\partial x} &= 2 \frac{(1 - x)(2x^3 + 8x^2 + x + 1)}{x(2x - 1)(x + 2)(2x^2 + 1)} > 0. \end{aligned} \quad (68)$$

Consequently, we have

$$1 = \frac{H_1(1, p)}{H_1(1, p)} < \frac{H_1(2x^2 - 1, p)}{xH_1(x, p)} \\ < \frac{H_1(0, p)}{(1/\sqrt{2})H_1(1/\sqrt{2}, p)} = \frac{4p}{3p+1} \frac{3p+1+\sqrt{2}}{(2\sqrt{2}+1)p+3} \quad (69) \\ \text{for } p = 9, \infty.$$

It is reversed for $p = 1$. From these we can obtain the following.

Theorem 16. For $t \in (0, \pi/2)$ the following inequalities hold:

$$\begin{aligned} \frac{28}{9\pi} \frac{6 \cos t + 9}{\cos t + 14} &< \frac{41(2\sqrt{2} - 25)}{7\pi} \frac{2\cos^2(t/2) + 3 \cos(t/2)}{\cos(t/2) + 14} \\ &< \frac{\sin t}{t} < \frac{6\cos^2(t/2) + 9 \cos(t/2)}{\cos(t/2) + 14} \\ &< \frac{6 \cos t + 9}{\cos t + 14}, \\ \frac{2 + \cos t}{\pi} &< \frac{12(2\sqrt{2} - 1)}{7\pi} \frac{\cos^2(t/2) + 2 \cos(t/2)}{3} \\ &< \frac{\sin t}{t} < \frac{\cos^2(t/2) + 2 \cos(t/2)}{3} \\ &< \frac{2 + \cos t}{3}, \\ \frac{2 \cos t + 1}{\cos t + 2} &< \frac{2\cos^2(t/2) + 1}{\cos(t/2) + 2} < \frac{\sin t}{t} \\ &< \frac{2(3 - \sqrt{2})}{\pi} \frac{2\cos^2(t/2) + 1}{\cos(t/2) + 2} \\ &< \frac{4}{\pi} \frac{2 \cos t + 1}{\cos t + 2}. \end{aligned} \quad (70)$$

Additionally, Lemma 4 implies an optimal two-side inequality.

Theorem 17. Let $p \in (-\infty, -1] \cup [0, \infty)$ and let $u_1(x, p)$ and $u_2(x, p)$ be defined by (14) and (15), respectively. Then for $t \in (0, \pi/2)$ the two-side inequality

$$\frac{u_2(\cos t, p)}{u_1(\cos t, p)} < \frac{\sin t}{t} < \frac{u_2(\cos t, q)}{u_1(\cos t, q)} \quad (71)$$

holds if and only if $p \in (-\infty, -1] \cup [9, \infty)$ and $q \in [0, p_1]$, where $p_1 \approx 6.3433$. And, for $x \in (0, 1)$, the function $p \mapsto u_2(x, p)/u_1(x, p)$ is decreasing on $(-\infty, -1] \cup [0, \infty)$.

Proof. Since $u_1(x, p), u_2(x, p) > 0$ for $p \in (-\infty, -1] \cup [0, \infty)$ and $x \in (0, 1)$ by Lemma 2 and $g(t, p)$ defined by (24) can be written as

$$g(t, p) = -t \frac{u_1(\cos t, p)}{u_2(\cos t, p)} \left(\frac{\sin t}{t} - \frac{u_2(\cos t, p)}{u_1(\cos t, p)} \right), \quad (72)$$

it follows from Lemma 4 that (71) holds if and only if $p \in (-\infty, -1] \cup [9, \infty)$ and $q \in [0, p_1]$. It remains to check the monotonicity of $u_2(\cos t, p)/u_1(\cos t, p)$ in p . Differentiation yields

$$\begin{aligned} \frac{d}{dp} \frac{u_2(x, p)}{u_1(x, p)} &= -6(x+1)(x-1)^2 \\ &\times \frac{(p+1)((5+x)p+5x+1)}{(2p+3x+px)^2(3p+2x+1)^2}, \end{aligned} \quad (73)$$

where $x \in (0, 1)$. If $p \in [0, \infty)$, then the numerator of the fraction in right-hand above is clearly positive. Consider that $(p+1)((5+x)p+5x+1) > 0$. If $p \in (-\infty, -1]$, then $(p+1) \leq 0$ and $((5+x)p+5x+1) \leq 5(x-1) < 0$, which yields that the numerator is nonnegative.

This proves the assertion. $\square$

Similarly, we can obtain a hyperbolic version of Theorems 7 and 10

Theorem 18. Let $p \in (-\infty, -1] \cup [0, \infty)$. Then for $t \in (0, \infty)$

$$\frac{2 + (1+3p) \cosh t}{3 + p + 2p \cosh t} < \frac{\sinh t}{t} \quad (74)$$

holds if and only if $p \in (-\infty, -1] \cup [1/9, \infty)$. It is reversed if and only if $p = 0$.

Proof. Let F be the function defined on $(0, \infty) \times (-\infty, -1] \cup [0, \infty)$ by

$$F(t, p) = \frac{3 + p + 2p \cosh t}{2 + (1+3p) \cosh t} \sinh t - t. \quad (75)$$

Then the inequalities (74) are equivalent to $F(t, p) > 0$. Expanding in power series yields

$$F(t, p) = \frac{t^5}{180} \frac{9p-1}{p+1} + o(t^5), \quad (76)$$

which implies

$$\lim_{t \rightarrow 0} \frac{F(t, p)}{t^5} = \frac{1}{20} \frac{p-1/9}{p+1} \quad \text{if } p \neq -1, \quad (77)$$

$$F(t, -1) = \sinh t - t > 0.$$

On the other hand, we have

$$\lim_{t \rightarrow \infty} \frac{F(t, p)}{\sinh t} = \frac{2p}{1+3p}. \quad (78)$$

Now we prove desired results.

(i) We first prove that $F(t, p) > 0$ holds if and only if $p \in (-\infty, -1] \cup [1/9, \infty)$.

If $F(t, p) > 0$ for all $t > 0$, then we have

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{F(t, p)}{t^5} &= \frac{1}{20} \frac{p-1/9}{p+1} \geq 0, \\ F(t, -1) &= \sinh t - t > 0, \end{aligned} \quad (79)$$

$$\lim_{t \rightarrow \infty} \frac{F(t, p)}{\sinh t} = \frac{2p}{1+3p} \geq 0.$$

Solving the inequalities yields $p \in (-\infty, -1] \cup [1/9, \infty)$.

We prove the condition $p \in (-\infty, -1] \cup [1/9, \infty)$ is sufficient for $F(t, p) > 0$ to hold for $t \in (0, \infty)$. Differentiation gives

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{(3p+1+2\cosh t)}{2p+(p+3)\cosh t} \cosh t \\ &\quad - \frac{3(p+1)^2 \sinh^2 t}{(2p+(p+3)\cosh t)^2} - 1 \\ &= (x-1)^2 \frac{2p(3p+1)x + (3p^2+6p-1)}{(x+3px+2)^2}, \end{aligned} \quad (80)$$

where $x = \cosh t \in (1, \infty)$.

Due to $p \in (-\infty, -1] \cup [1/9, \infty)$, we see that $2p(3p+1) > 0$, which yields

$$\begin{aligned} 2p(3p+1)x + (3p^2+6p-1) &> 2p(3p+1) + (3p^2+6p-1) \\ &= (p+1)(9p-1) \geq 0. \end{aligned} \quad (81)$$

Then $\partial F/\partial t > 0$; that is, F is increasing in t on $(0, \infty)$. It is obtained that $F(t, p) > F(0, p) = 0$, which proves the sufficiency.

(ii) Next we prove that the reverse inequality of (74) holds if and only if $p = 0$. The necessity follows from

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{F(t, p)}{t^5} &= \frac{1}{20} \frac{p-1/9}{p+1} \leq 0, \\ \lim_{t \rightarrow \infty} \frac{F(t, p)}{\sinh t} &= \frac{2p}{1+3p} \leq 0, \end{aligned} \quad (82)$$

and the assumption $p \in (-\infty, -1] \cup [0, \infty)$. We get $p = 0$.

Now we prove $F(t, p) < 0$ when $p = 0$. We have

$$\frac{\partial F}{\partial t} = -\frac{(x-1)^2}{(x+3px+2)^2} < 0, \quad (83)$$

where $x = \cosh t \in (1, \infty)$, then $F(t, 0) < F(0, 0) = 0$.

Thus the proof of Theorem 18 is complete. $\square$

Denote

$$H_5(x, p) = \frac{2 + (1+3p)x}{3 + p + 2px}. \quad (84)$$

It is easy to verify that $H_5(x, p) = H_1(x, p^{-1})$ for $p \neq 0$. By Lemma 1, we see that H_5 is decreasing in p on $(-\infty, -1] \cup [0, \infty)$. Thus, as a consequence of Theorem 14, we have the following.

Corollary 19. *One has*

$$\begin{aligned} \frac{2 + \cosh t}{3} &> \frac{\sinh t}{t} > H_5\left(\cosh t, \frac{1}{9}\right) \\ &> \cdots > H_5(\cosh t, \infty) = \frac{3 \cosh t}{2 \cosh t + 1} \\ &= H_5(\cosh t, -\infty) > \cdots > H_5(\cosh t, -1) = 1. \end{aligned} \quad (85)$$

Furthermore, note that $H_5(x, p) = H_1^{-1}(x^{-1}, p)$ and by Lemma 6 we have the following.

Corollary 20. *One has*

$$\begin{aligned} \frac{2 + \cosh t}{3} &> \frac{\sinh t}{t} > \cosh^{1/3} t > \frac{1 + 2 \cosh t}{2 + \cosh t} \\ &> H_5(\cosh t, p), \end{aligned} \quad (86)$$

where $p \in (-\infty, -1] \cup (1, \infty)$.

4. Applications

In this section, we give some applications of our results.

4.1. Shafer-Fink Type Inequalities. In [1, p. 247, 3.4.31], it was listed that the inequality

$$\arcsin x > \frac{6(\sqrt{x+1} - \sqrt{1-x})}{4 + \sqrt{x+1} + \sqrt{1-x}} > \frac{3x}{2 + \sqrt{1-x^2}} \quad (87)$$

holds for $x \in (0, 1)$, which is due to Shafer [21]. Fink [22] proved that the double inequality

$$\frac{3x}{2 + \sqrt{1-x^2}} \leq \arcsin x \leq \frac{\pi x}{2 + \sqrt{1-x^2}} \quad (88)$$

is true for $x \in [0, 1]$. There has been some improvements and generalizations of Shafer-Fink inequality (see [23]). Letting $\sin t = x$ in Theorems 7, 10, 13, 14, 16 and 17 we can obtain corresponding Shafer-Fink type inequalities, which clearly contain many known results. For example, Theorems 7 and 10 can be changed into the following.

Proposition 21. *For $x \in (0, 1)$, the two-side inequality*

$$\begin{aligned} &\frac{x}{H_1(\sqrt{1-x^2}, p)} \\ &= x \frac{(3p+1) + 2\sqrt{1-x^2}}{2p + (p+3)\sqrt{1-x^2}} < \arcsin x \\ &< \frac{\pi p}{3p+1} x \frac{(3p+1) + 2\sqrt{1-x^2}}{2p + (p+3)\sqrt{1-x^2}} \\ &= \frac{x}{H_2(\sqrt{1-x^2}, p)} \end{aligned} \quad (89)$$

holds if and only if $p \in (-\infty, -1] \cup [9, \infty)$, where $\pi p/(3p+1)$ is the best possible. And, the lower and upper bounds in (89) are decreasing and increasing in p on $(-\infty, -1] \cup (0, \infty)$, respectively.

Inequality (89) is reversed if $p \in [0, p_1]$, where $p_1 \approx 6.3433$ is defined by (25).

Letting $\sin t = x$, then $\cos(t/2) = (1/2)(\sqrt{1+x} + \sqrt{1-x})$. Theorem 14 can be restated as follows.

Proposition 22. For $x \in (0, 1)$, the two-side inequality

$$\begin{aligned} & \frac{(3p+1)(\sqrt{1+x}-\sqrt{1-x})+2x}{4p+(p+3)(\sqrt{1+x}+\sqrt{1-x})} \\ & < \arcsin x < \frac{2}{\sigma_p} \frac{(3p+1)(\sqrt{1+x}-\sqrt{1-x})+2x}{4p+(p+3)(\sqrt{1+x}+\sqrt{1-x})} \end{aligned} \quad (90)$$

holds if and only if $p \in (-\infty, -1] \cup [9, \infty)$, where $\sigma_p = (4/\pi)((3p + \sqrt{2} + 1)/((2\sqrt{2} + 1)p + 3))$ is the best constant. And, the lower and upper bounds in (90) are decreasing and increasing in p on $(-\infty, -1] \cup (0, \infty)$, respectively.

Inequality (90) is reversed if $p \in [0, p_1]$, where $p_1 \approx 6.3433$ is defined by (25).

As another example, Theorem 16 can be rewritten as follows.

Proposition 23. For $x \in (0, 1)$, all the following chains of inequalities hold:

$$\begin{aligned} \frac{x\sqrt{1-x^2}+14}{3\sqrt{1-x^2}+3} & < \frac{1}{3} \frac{x+14(\sqrt{x+1}-\sqrt{1-x})}{3+\sqrt{x+1}+\sqrt{1-x}} < \arcsin x \\ & < \frac{(41\sqrt{2}+25)\pi}{782} \frac{x+14(\sqrt{x+1}-\sqrt{1-x})}{3+\sqrt{x+1}+\sqrt{1-x}} \\ & < \frac{3\pi x}{28} \frac{\sqrt{1-x^2}+14}{2\sqrt{1-x^2}+3}, \end{aligned} \quad (91)$$

$$\begin{aligned} \frac{3x}{2+\sqrt{1-x^2}} & < \frac{6(\sqrt{x+1}-\sqrt{1-x})}{4+\sqrt{x+1}+\sqrt{1-x}} < \arcsin x \\ & < \frac{(1+2\sqrt{2})\pi}{12} \frac{6(\sqrt{x+1}-\sqrt{1-x})}{4+\sqrt{x+1}+\sqrt{1-x}} \\ & < \frac{\pi x}{2+\sqrt{1-x^2}}, \end{aligned} \quad (92)$$

$$\begin{aligned} \frac{\pi x}{4} \frac{\sqrt{1-x^2}+2}{2\sqrt{1-x^2}+1} & < \frac{(\sqrt{2}+3)\pi}{14} \frac{x+2(\sqrt{x+1}-\sqrt{1-x})}{1+\sqrt{x+1}+\sqrt{1-x}} \\ & < \arcsin x < \frac{x+2(\sqrt{x+1}-\sqrt{1-x})}{1+\sqrt{x+1}+\sqrt{1-x}} \\ & < x \frac{\sqrt{1-x^2}+2}{2\sqrt{1-x^2}+1}. \end{aligned} \quad (93)$$

Remark 24. Inequalities (92) are due to Zhu [23].

4.2. Inequalities for Certain Means. For $a, b > 0$ with $a \neq b$, the first and second Seiffert means [24, 25]; Nueman-Sándor means [26] are defined by

$$\begin{aligned} P &= P(a, b) = \frac{a-b}{2 \arcsin((a-b)/(a+b))}, \\ T &= T(a, b) = \frac{a-b}{2 \arctan((a-b)/(a+b))}, \\ \text{NS} &= \text{NS}(a, b) = \frac{a-b}{2 \operatorname{arcsinh}((a-b)/(a+b))}, \end{aligned} \quad (94)$$

respectively. More new means can be found in [27]. We also denote the logarithmic mean, arithmetic mean, geometric mean, and quadratic mean of a and b by L , A , G , and Q . There has been some inequalities for these means; we quote [7, 26–36]. Now we establish some new ones involving these means.

Let $x = \arcsin((b-a)/(a+b))$, $\arctan((b-a)/(a+b))$. Then $(\sin x)/x = P/A$, $\cos x = G/A$; $(\sin x)/x = T/Q$, $\cos x = A/Q$. And then Theorems 7, 10, 13, 14, 16 and 17 can be stated as equivalent ones involving means P , A , G , and T , Q . For example, from Theorems 7 and 17 we have the following.

Proposition 25. For $a, b > 0$ with $a \neq b$, both the two-side inequalities

$$\begin{aligned} \frac{2pA+(p+3)G}{(3p+1)A+2G} A & < P < A \frac{2qA+(q+3)G}{(3q+1)A+2G}, \\ \frac{2pQ+(p+3)A}{(3p+1)Q+2A} Q & < T < Q \frac{2qQ+(q+3)A}{(3q+1)Q+2A} \end{aligned} \quad (95)$$

hold if and only if $p \in [0, p_0]$ and $q \in (-\infty, -1] \cup [9, \infty)$, where $p_0 = (\pi - 3)^{-1} \approx 7.0625$.

Making changes of variables $x = \operatorname{arctanh}((b-a)/(a+b))$, $\operatorname{arcsinh}((b-a)/(a+b))$ yield $(\sinh x)/x = L/G$, $\cosh x = A/G$; $(\sinh x)/x = \text{NS}/A$, $\cosh x = Q/A$, respectively. And then, Theorem 18 can be equivalently written as follows.

Proposition 26. For $a, b > 0$ with $a \neq b$, both the inequalities

$$\begin{aligned} \frac{2G+(1+3p)A}{(3+p)G+2pA} G & < L, \\ \frac{2A+(1+3p)Q}{(3+p)A+2pQ} A & < \text{NS} \end{aligned} \quad (96)$$

hold if and only if $p \in (-\infty, -1] \cup [1/9, \infty)$. They are reversed if and only if $p = 0$.

4.3. The Estimate for the Sine Integral. For the estimations for the sine integral defined by

$$\text{Si}(x) = \int_0^x \frac{\sin t}{t} dt, \quad (97)$$

there has been some results (see [37–39]). By our results we can obtain many estimates for $\text{Si}(x)$. Here we give a simpler but more accurate one.

Proposition 27. For $x \in (0, \pi/2]$, we have

$$\begin{aligned} & \frac{4\sqrt{2}-2}{7\pi} \left(x + \sin x + 8 \sin \frac{x}{2} \right) \\ & < \text{Si}(x) < \frac{1}{6} \left(x + \sin x + 8 \sin \frac{x}{2} \right). \end{aligned} \quad (98)$$

Proof. By (63) we see that the inequalities

$$\begin{aligned} & \frac{4(2\sqrt{2}-1)}{7} \frac{\cos^2(t/2) + 2 \cos(t/2)}{\pi} \\ & < \frac{\sin t}{t} < \frac{\cos^2(t/2) + 2 \cos(t/2)}{3} \end{aligned} \quad (99)$$

hold for $t \in [0, \pi/2]$. Integrating both sides over $[0, x]$ and simple calculation yield (98). $\square$

Remark 28. By (98) we have

$$\begin{aligned} 1.3682 & \approx \frac{2\sqrt{2}-1}{7\pi} (\pi + 8\sqrt{2} + 2) < \int_0^{\pi/2} \frac{\sin t}{t} dt \\ & < \frac{1}{12} (\pi + 8\sqrt{2} + 2) \approx 1.3713. \end{aligned} \quad (100)$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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