Research Article
Boundary Value Problems for First-Order Impulsive Functional $q$-Integrodifference Equations

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We discuss the existence and uniqueness of solutions for a first-order boundary value problem for impulsive functional $q_k$-integrodifference equations. The main results are obtained with the aid of some classical fixed point theorems. Illustrative examples are also presented.

1. Introduction

In this paper, we study the boundary value problem for impulsive functional $q_k$-integrodifference equation of the following form:

$$
\begin{align*}
&D_{q_k}x(t) = f(t, x(t), x(\theta(t)), (S_{q_k}x)(t)), \\
&\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m, \\
&\alpha x(0) = \beta x(T) + \sum_{i=0}^{m} \gamma_i \int_{t_i}^{t_{i+1}} x(s) \, dq_k s,
\end{align*}
$$

where $0 = t_0 < t_1 < t_2 < \cdots < t_k < \cdots < t_m < t_{m+1} = T$, $f : J \times \mathbb{R}^3 \to \mathbb{R}$, $\theta : J \to J$, $\phi : J \times \mathbb{R} \to [0, \infty)$ is a continuous function, $I_k \in C(\mathbb{R}, \mathbb{R})$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ for $k = 1, 2, \ldots, m$, $x(t_k^+)$ and $x(t_k^-)$ are left- and right-hand limits, respectively, $t_k = \lim_{s \to t_k^-} x(t_k + h)$, $a, b, \gamma_i$, $i = 0, 1, \ldots, m$ are real constants, and $0 < q_k < 1$ for $k = 0, 1, 2, \ldots, m$.

The notions of $q_k$-derivative and $q_k$-integral on finite intervals were introduced recently by the authors in [1]. Their basic properties were studied and applications existence and uniqueness results were proved for initial value problems for first- and second-order impulsive $q_k$-difference equations. In this paper, we continue the study on this new subject by considering the boundary value problem (1).

The book by Kac and Cheung [2] covers many of the fundamental aspects of the quantum calculus. In recent years, the topic of $q$-calculus has attracted the attention of several researchers and a variety of new results can be found in the papers [3–15] and the references cited therein. On the other hand, for some monographs on the impulsive differential equations we refer to [16–18].

The rest of this paper is organized as follows. In Section 2, we recall the notions of $q_k$-derivative and $q_k$-integral on finite intervals and present a preliminary result which will be used in this paper. In Section 3, we will consider the existence results for problem (1) while in Section 4, we will give examples to illustrate our main results.

2. Preliminaries

In this section, we recall the notions of $q_k$-derivative and $q_k$-integral on finite intervals. For a fixed $k \in \mathbb{N} \cup \{0\}$ let $J_k := [t_k, t_{k+1}] \subset \mathbb{R}$ be an interval and let $0 < q_k < 1$ be a constant. We define $q_k$-derivative of a function $f : J_k \to \mathbb{R}$ at a point $t \in J_k$ as follows.
Definition 1. Assume \( f : J_k \rightarrow \mathbb{R} \) is a continuous function and let \( t \in J_k \). Then the expression

\[
D_q t f(t) = \frac{f(t) - f(qt)}{(1 - q)t}, \quad t \neq t_k,
\]

\[
D_q t f(t_k) = \lim_{t \to t_k} D_q t f(t)
\]
is called the \( q_k \)-derivative of function \( f(t) \) at \( t \).

We say that \( f \) is \( q_k \)-differentiable on \( J_k \) provided that \( D_q t f(t) \) exists for all \( t \in J_k \). Note that if \( t_k = 0 \) and \( q_k = q \) in (3), then \( D_q t f = D_q f \), where \( D_q \) is the well-known \( q \)-derivative of the function \( f(t) \) defined by

\[
D_q t f(t) = \frac{f(t) - f(qt)}{(1 - q)t}.
\]

In addition, we should define the higher \( q_k \)-derivative of functions.

Definition 2. Let \( f : J_k \rightarrow \mathbb{R} \) be a continuous function; we call the second-order \( q_k \)-derivative \( D^{2}_q t f \) provided that \( D_q t f \) is \( q_k \)-differentiable on \( J_k \) with \( D^{2}_q t f = D_q t (D_q t f) : J_k \rightarrow \mathbb{R} \). Similarly, we define higher order \( q_k \)-derivative \( D^m_q t : J_k \rightarrow \mathbb{R} \).

The \( q_k \)-integral is defined as follows.

Definition 3. Assume \( f : J_k \rightarrow \mathbb{R} \) is a continuous function. Then the \( q_k \)-integral is defined by

\[
\int_{t_k}^{t} f(s) d_q s = (1 - q_k)(t - t_k) \sum_{n=0}^{\infty} q^n_k f(q^n_k t + (1 - q^n_k) t_k)
\]

for \( t \in J_k \). Moreover, if \( a \in (t_k, t) \), then the definite \( q_k \)-integral is defined by

\[
\int_{a}^{t} f(s) d_q s = \int_{t_k}^{t} f(s) d_q s - \int_{a}^{t_k} f(s) d_q s.
\]

\[
= (1 - q_k)(t - t_k) \sum_{n=0}^{\infty} q^n_k f(q^n_k t + (1 - q^n_k) t_k)
- (1 - q_k)(a - t_k) \sum_{n=0}^{\infty} q^n_k f(q^n_k a + (1 - q^n_k) t_k).
\]

Note that if \( t_k = 0 \) and \( q_k = q \), then (5) reduces to \( q \)-integral of a function \( f(t) \), defined by

\[
\int_{0}^{t} f(s) d_q s = (1 - q) t \sum_{n=0}^{\infty} q^n f(q^n t) \quad \text{for} \ t \in [0, \infty).
\]

For the basic properties of \( q_k \)-derivative and \( q_k \)-integral we refer to [1].

Let \( J = [0, T] \), \( J_0 = [t_0, t_1] \), and \( J_k = [t_k, t_{k+1}] \) for \( k = 1, 2, \ldots, m \). Let \( PC(J, \mathbb{R}) = \{x : J \rightarrow \mathbb{R} : x(t) \) be continuous everywhere except for some \( t_k \) at which \( x(t'_k) \) and \( x(t_k) \) exist and \( x(t_k) = x(t_k), k = 1, 2, \ldots, m \). \( PC(J, \mathbb{R}) \) is a Banach space with the norm \( \|x\|_{PC} = \sup \{\|x(t)\| : t \in J \} \).

We now consider the following linear case:

\[
D_q t x(t) = h(t), \quad t \in [0, T], \ t \neq t_k,
\]

\[
\Delta x(t_k) = I_k(x(t_k)), \quad k = 1, 2, \ldots, m. \tag{8}
\]

\[
\alpha x(0) = \beta x(T) + \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} x(s) d_q s,
\]

where \( h : J \rightarrow \mathbb{R} \).

Lemma 4. Let \( \alpha \neq \beta = \beta \sum_{i=0}^{m} \gamma_i (t_{i+1} - t_i) \). The unique solution of problem (8) is given by

\[
x(t) = \frac{\beta}{\Omega} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} h(s) d_q s + \frac{\beta}{\Omega} \sum_{k=1}^{m} I_k(x(t_k))
+ \frac{1}{\Omega} \sum_{i=0}^{m} \int_{t_i}^{t_{i+1}} I_k(x(t_k)) + \int_{t_i}^{t_k} h(s) d_q s + \int_{t_i}^{t_{i+1}} I_k(x(t_k)) \sum_{0 \leq i < k} \left( \int_{t_i}^{t_k} h(s) d_q s + I_k(x(t_k)) \right)
+ \int_{t_i}^{t_k} h(s) d_q s,
\]

with \( \sum_{a}^{b} (\cdot) = 0 \) for \( a > b \), where

\[
\Omega = \frac{1}{\alpha - \beta - \sum_{i=0}^{m} \gamma_i (t_{i+1} - t_i)}. \tag{10}
\]

Proof. For \( t \in J_0 \), \( q_0 \)-integrating (8), it follows

\[
x(t) = x_0 + \int_{0}^{t} h(s) d_q s,
\]

which leads to

\[
x(t_1) = x_0 + \int_{0}^{t_1} h(s) d_q s.
\]

For \( t \in J_1 \), \( q_1 \)-integrating to (8), we have

\[
x(t) = x(t_1) + \int_{t_1}^{t} h(s) d_q s.
\]

Since \( x(t'_1) = x(t_1 + I_1(x(t_1))) \), then we have

\[
x(t) = x_0 + \int_{0}^{t_1} h(s) d_q s + \int_{t_1}^{t} h(s) d_q s + I_1(x(t_1)).
\]
Again \( q_2 \)-integrating (8) from \( t_2 \) to \( t \), where \( t \in I_2 \), then
\[
x(t) = x(t_2) + \int_{t_2}^{t} h(s) \, dq_2 s = x_0 + \int_{t_0}^{t} h(s) \, dq_2 s + \int_{t_1}^{t_2} h(s) \, dq_2 s + \int_{t_2}^{t} h(s) \, dq_2 s + I_1 (x(t_1)) + I_2 (x(t_2)) .
\]
(15)

Repeating the above process, for \( t \in J \), we obtain
\[
x(t) = x_0 + \sum_{k=0}^{m} \left( \sum_{i=0}^{k} \int_{t_{i+1}}^{t_{i}} h(s) \, dq_{i+1} s + I_k (x(t_k)) \right)
+ \int_{t}^{t_{i+1}} h(s) \, dq_i s .
\]
(16)

Further, \( q_i \)-integrating (16) from \( t_i \) to \( t_{i+1} \), it follows
\[
\int_{t_i}^{t_{i+1}} x(u) \, dq_{i+1} u = x_0 (t_{i+1} - t_i)
+ \sum_{k=0}^{m} \left( \sum_{i=0}^{k} \int_{t_{i+1}}^{t_{i}} h(s) \, dq_{i+1} s + I_k (x(t_k)) \right) (t_{i+1} - t_i)
+ \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s) \, dq_i s \, dq_{i+1} u .
\]
(18)

Applying the boundary condition of (8), one has
\[
\alpha x_0 = \beta x_0 + \beta \sum_{k=0}^{m} \left( \int_{t_k}^{t_{k+1}} h(s) \, dq_{k+1} s + I_k (x(t_k)) \right)
+ \beta \int_{t_m}^{T} h(s) \, dq_m s + x_0 \sum_{i=0}^{m} y_i (t_{i+1} - t_i)
+ \sum_{i=0}^{m} y_i \sum_{k=1}^{i} \left( \int_{t_k}^{t_{k+1}} h(s) \, dq_{k+1} s + I_k (x(t_k)) \right) (t_{i+1} - t_i)
+ \sum_{i=0}^{m} y_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s) \, dq_i s \, dq_{i+1} u .
\]

Since \( T = t_{m+1} \) and \( \sum_{i=a}^{b} () = 0 \) for \( a > b \), we have
\[
x_0 \left( \alpha - \beta \sum_{i=0}^{m} y_i (t_{i+1} - t_i) \right)
= \beta \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} h(s) \, dq_{k+1} s + I_k (x(t_k))
+ \sum_{i=0}^{m} y_i (t_{i+1} - t_i) \int_{t_k}^{t_{k+1}} h(s) \, dq_{k+1} s
+ \sum_{i=0}^{m} y_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s) \, dq_i s \, dq_{i+1} u .
\]
(20)

Therefore,
\[
x_0 = \frac{\beta}{\alpha - \beta \sum_{i=0}^{m} y_i (t_{i+1} - t_i)} \sum_{k=0}^{m} \int_{t_k}^{t_{k+1}} h(s) \, dq_{k+1} s + I_k (x(t_k))
+ \frac{1}{\alpha - \beta \sum_{i=0}^{m} y_i (t_{i+1} - t_i)} \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s) \, dq_i s \, dq_{i+1} u
+ \sum_{i=0}^{m} y_i \int_{t_i}^{t_{i+1}} \int_{t_i}^{u} h(s) \, dq_i s \, dq_{i+1} u .
\]
(21)

Substituting the constant \( x_0 \) into (16), we obtain (9) as requested.
\[\square\]
3. Main Results

In view of Lemma 4, we define an operator $\mathcal{K} : \text{PC}(J, \mathbb{R}) \rightarrow \text{PC}(J, \mathbb{R})$ by

$$(\mathcal{K}x)(t) = \frac{\beta}{\Omega} \sum_{k=1}^{m+1} \int_{\tau_{k-1}}^{\tau_k} f\left(s, x(s), x(\theta(s)), \left(S_{q_{k-1}} x\right)(s)\right) ds\, d_{q_{k-1}}s$$

where

$$\Lambda_1 = \frac{[\beta] + |\Omega|}{|\Omega|} \sum_{k=1}^{m+1} \left[ (L_1 + L_2) (t_k - t_{k-1}) \right.$$

$$\left. + \frac{\phi_0 L_3(t_k - t_{k-1})^2}{1 + q_{k-1}} \right]$$

$$+ \frac{L_4}{|\Omega|} \sum_{i=1}^{m} \left| y_i \right| (t_{i+1} - t_i) + \frac{m |\beta| + |\Omega| L_4}{|\Omega|},$$

$$(23)$$

and

$$(24)$$

Theorem 5. Assume the following.

$(H_1)$ The function $\phi : J^2 \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous and there exists a constant $\phi_0 > 0$ such that

$$|\phi(t,s,y) - \phi(t,s,z)| \leq \phi_0 |y - z|,$$

for each $t, s \in J$ and $y, z \in \mathbb{R}.$

$(H_2)$ The function $f : J \times \mathbb{R}^3 \rightarrow \mathbb{R}$ is continuous and there exist constants $L_1, L_2, L_3 > 0$ such that

$$|f(t,y_1,y_2,y_3) - f(t,z_1,z_2,z_3)| \leq L_1 |y_1 - z_1| + L_2 |y_2 - z_2| + L_3 |y_3 - z_3|,$$

for each $t \in J$ and $y, z, i \in \mathbb{R}, i = 1, 2, 3.$

$(H_3)$ The functions $I_k : \mathbb{R} \rightarrow \mathbb{R}$ are continuous and there exists a constant $L_4 > 0$ such that

$$|I_k(y) - I_k(z)| \leq L_4 |y - z|,$$

for each $y, z \in \mathbb{R}, k = 1, 2, \ldots, m.$

If

$$\Lambda_1 \leq \delta < 1,$$

where $\Lambda_1$ is defined by $(23),$ then the boundary value problem (1) has a unique solution on $J.$

Proof. Firstly, we transform the boundary value problem (1) into a fixed point problem, $x = \mathcal{K}x,$ where the operator $\mathcal{K}$ is defined by $(22).$ Using the Banach contraction principle, we will show that $\mathcal{K}$ has a fixed point which is the unique solution of the boundary value problem (1).

Let $M_1$ and $M_2$ be nonnegative constants such that

$$\sup_{t \in J} |f(t,0,0,0)| = M_1$$

and

$$\sup \{|I_k(0)| : k = 1, 2, \ldots, m\} = M_2.$$
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where $\delta \leq \epsilon < 1$ and $\Lambda_2$ defined by (24), we will show that $\mathcal{K}B_r \subset B_r$, where a suitable ball $B_r$ is defined by $B_r = \{x \in PC(J, \mathbb{R}) : \|x\| \leq r\}$. For $x \in B_r$, we have

$$|\mathcal{K}x(t)| \leq \frac{[\beta] + |\Omega|}{|\Omega|} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \left| f(s, x(s), x(\theta(s)), (S_{q_k}x)(s)) - f(s, 0, 0, 0) \right| ds + \frac{|\beta| + |\Omega|}{|\Omega|} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \left| f(s, x(s), x(\theta(s)), (S_{q_k}x)(s)) \right| ds \frac{d_q}{s}$$

$$+ \frac{1}{|\Omega|} \sum_{i=0}^{m} \left| \gamma_i \right| |t_{i+1} - t_i| \left| I_k(x(t_k)) \right| + \frac{1}{|\Omega|} \sum_{i=0}^{m} \left| \psi_i \right| |t_{i+1} - t_i| \left| I_k(x(t_k)) \right|$$

$$+ \frac{1}{|\Omega|} \sum_{i=0}^{m} \left| \psi_i \right| |t_{i+1} - t_i| \left| I_k(x(t_k)) \right|$$

which yields $\mathcal{K}B_r \subset B_r$. 

(30)
For any $x, y \in \text{PC}(J, \mathbb{R})$ and for each $t \in J$, we have

$$|\mathcal{X}x(t) - \mathcal{X}y(t)| \leq \frac{|\beta| + |\Omega|}{|\Omega|} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} \left| f \left( s, x(s), x(\theta(s)), (S_{q_k-1}x)(s) \right) - f \left( s, y(s), y(\theta(s)), (S_{q_k-1}y)(s) \right) \right| ds \, dt_k \, |\Omega| \\text{for all } t \in J,$$

where $\mathcal{X}$ is a contraction. Therefore, by Banach's contraction mapping principle, we conclude that $\mathcal{X}$ has a fixed point which is the unique solution of problem (I).

The second existence result is based on Krasnosel’skii’s fixed point theorem.

**Lemma 6** (Krasnosel’skii’s fixed point theorem [19]). Let $M$ be a closed, bounded, convex, and nonempty subset of a Banach space $X$. Let $A, B$ be the operators such that (a) $Ax + By \in M$ whenever $x, y \in M$; (b) $A$ is compact and continuous; (c) $B$ is a contraction mapping. Then there exists $z \in M$ such that $z = Az + Bz$.

We use the following notations:

$$\Lambda_3 = \frac{|\beta| + |\Omega|}{|\Omega|} \sum_{k=1}^{m+1} \left( L_1 + L_2 \right) \left( t_k - t_{k-1} \right) + \frac{\phi_1 L_3 \left( t_k - t_{k-1} \right)^2}{1 + q_{k-1}} \leq N,$$

$$\Lambda_4 = \frac{|\beta| + |\Omega|}{|\Omega|} \sum_{i=1}^{m} \sum_{k=1}^{i} \left| y_i \right| \left( t_{i+1} - t_i \right) + \frac{L_4 \left( t_k - t_{k-1} \right)^3}{1 + q_{k-1}} \leq mN.$$

**Theorem 7.** Let $f : J \times \mathbb{R} \to \mathbb{R}$ be a continuous function. Suppose that $(H_3)$ holds. In addition, we suppose the following:

$$(H_4) \ |f(t, z_1, z_2, z_3)| \leq \mu(t), \forall (t, z_1, z_2, z_3) \in J \times \mathbb{R}^3, \text{ and } \mu \in C(J, \mathbb{R}^+),$$

$$(H_5) \text{ there exists a constant } N > 0 \text{ such that } |I_k(x)| \leq N \text{ for all } x \in \mathbb{R}, \text{ for } k = 1, 2, \ldots, m.$$

Then the impulsive functional $q_k$-integrodifference boundary value problem (I) has at least one solution on $J$ provided that

$$\left( \frac{|\beta| + |\Omega|}{|\Omega|} \right) mL_4 + \frac{L_4 \sum_{i=1}^{m} \left| y_i \right| \left( t_{i+1} - t_i \right)}{1 + q_{k-1}} < 1.$$
and $\Lambda_3, \Lambda_4$ are defined by (32) and (33), respectively; we define the operators $\mathcal{A}_1$ and $\mathcal{A}_2$ on $B_R$ by

\[
(\mathcal{A}_1 x)(t) = \frac{\beta}{\Omega} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} f(s, x(s), x(\theta(s)), (S_{\eta_k} x)(s)) \, d\eta_k, s + \frac{1}{\Omega} \sum_{j=0}^{m} \left| \gamma_j \right| \int_{t_j}^{t_{j+1}} d\eta_j, s \, d\eta_j, u
\]

\[
+ \frac{1}{\Omega} \sum_{j=1}^{m} \int_{t_j}^{t_{j+1}} f(s, x(s), x(\theta(s)), (S_{\eta_j} x)(s)) \, d\eta_j, s
\]

\[
+ \sum_{0<j<k} \int_{t_j}^{t_{j+1}} f(s, x(s), x(\theta(s)), (S_{\eta_k} x)(s)) \, d\eta_k, s
\]

\[
+ \int_{t}^{t_1} f(s, x(s), x(\theta(s)), (S_{\eta_1} x)(s)) \, d\eta_1, s.
\]

\[
(\mathcal{A}_2 x)(t) = \frac{\beta}{\Omega} \sum_{k=1}^{m} I_k(x(t_k)) + \frac{1}{\Omega} \sum_{j=1}^{m} \sum_{i=1}^{j} \left| \gamma_i(t_{i+1} - t_i) I_k(x(t_k)) \right|
\]

\[
+ \sum_{0<j<k} I_k(x(t_k)).
\]

(36)

For any $x, y \in B_R$, we have

\[
\|\mathcal{A}_1 x + \mathcal{A}_2 y\| = \|\mu\left[ \left| \beta \right| + \left| \Omega \right|^2 \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_k} d\eta_k, s + \frac{1}{\Omega} \sum_{j=0}^{m} \left| \gamma_j \right| \int_{t_j}^{t_{j+1}} d\eta_j, s \, d\eta_j, u
\]

\[
+ \frac{1}{\Omega} \sum_{j=1}^{m} \int_{t_j}^{t_{j+1}} f(s, x(s), x(\theta(s)), (S_{\eta_j} x)(s)) \, d\eta_j, s
\]

\[
+ \sum_{0<j<k} \int_{t_j}^{t_{j+1}} f(s, x(s), x(\theta(s)), (S_{\eta_k} x)(s)) \, d\eta_k, s
\]

\[
+ \frac{m \left| \beta \right| \left| \Omega \right|}{\left| \Omega \right|} \sum_{j=1}^{m} \sum_{i=1}^{j} \left| \gamma_i(t_{i+1} - t_i) + mN \right|
\]

\[
\leq R.
\]

(37)

This implies that $\mathcal{A}_1 x + \mathcal{A}_2 y \in B_R$.

To show that $\mathcal{A}_2$ is a contraction, for $x, y \in PC(J, \mathbb{R})$, we have

\[
\|\mathcal{A}_2 x - \mathcal{A}_2 y\| \leq \left| \beta \right| \sum_{k=1}^{m} \int_{t_k}^{t_{k+1}} f(s, x(s), x(\theta(s)), (S_{\eta_k} x)(s)) \, d\eta_k, s
\]

\[
+ \frac{1}{\Omega} \sum_{j=1}^{m} \int_{t_j}^{t_{j+1}} f(s, x(s), x(\theta(s)), (S_{\eta_j} x)(s)) \, d\eta_j, s
\]

\[
+ \sum_{0<j<k} \int_{t_j}^{t_{j+1}} f(s, x(s), x(\theta(s)), (S_{\eta_k} x)(s)) \, d\eta_k, s
\]

\[
\leq \left[ \left| \beta \right| + \left| \Omega \right|^2 \right] mL_4 + \frac{L_4}{\left| \Omega \right|} \sum_{j=1}^{m} \left| \gamma_j \right| \left| t_{j+1} - t_j \right| \|x - y\|.
\]

(38)

From (34), it follows that $\mathcal{A}_2$ is a contraction.

Next, the continuity of $f$ implies that operator $\mathcal{A}_2$ is continuous. Further, $\mathcal{A}_1$ is uniformly bounded on $B_R$ by

\[
\|\mathcal{A}_1 x\| \leq \|\mu\| \Lambda_3.
\]

(39)

Now we will prove the compactness of $\mathcal{A}_1$. Setting $\sup_{(s, z_1, z_2, z_3) \in J^2 \times D^2} |f(s, z_1, z_2, z_3)| = f^* < \infty$, then for each $\tau_1, \tau_2 \in (t_i, t_{i+1})$ for some $l \in \{0, 1, \ldots, m\}$ with $\tau_2 > \tau_1$, we have

\[
\left| (\mathcal{A}_1 x)(\tau_2) - (\mathcal{A}_1 x)(\tau_1) \right| = \left| \int_{\tau_1}^{\tau_2} f(s, x(s), x(\theta(s)), (S_{\eta_l} x)(s)) \, d\eta_l, s
\]

\[
- \int_{\tau_1}^{\tau_2} f(s, x(s), x(\theta(s)), (S_{\eta_l} x)(s)) \, d\eta_l, s \right| \leq |\tau_2 - \tau_1| f^*.
\]

(40)

As $\tau_1 \to \tau_2$, the right hand side above tends to zero independently on $x$. Therefore, the operator $\mathcal{A}_1$ is equicontinuous. Since $\mathcal{A}_1$ maps bounded subsets into relatively compact subsets, it follows that $\mathcal{A}_1$ is relatively compact on $B_R$. Hence, by the Arzelá-Ascoli theorem, $\mathcal{A}_1$ is compact on $B_R$. Thus, all the assumptions of Lemma 6 are satisfied. Hence, by
the conclusion of Lemma 6, the impulsive functional \( q_k \)-
integrodifference boundary value problem (1) has at least one
solution on \( J \).

Our third existence result is based on Leray-Schauder
degree theory. Before proving the result, we set

\[
\Lambda_5 = \frac{|\beta| + |\Omega|}{|\Omega|} \sum_{k=1}^{m} \left[ \xi_1 (t_k - t_{k-1}) + \xi_2 \xi_3 \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} \right] \\
+ \sum_{k=1}^{m} \left[ \xi_1 \frac{(t_{i+1} - t_i)^2}{1 + q_i} + \xi_2 \xi_3 \frac{(t_{i+1} - t_i)^3}{1 + q_i} \right] \\
+ \sum_{i=1}^{m} \left[ \xi_1 (t_{i+1} - t_i) \xi_4 + \frac{|\beta| + |\Omega|}{|\Omega|} \xi_4 \right] \\
+ \sum_{i=1}^{m} \sum_{k=1}^i \left[ \xi_4 \xi_2 (t_{i+1} - t_i) \frac{(t_{i+1} - t_i)^2}{1 + q_i} + \xi_1 (t_{i+1} - t_i) \frac{(t_{i+1} - t_i)^2}{1 + q_i} \right] \\
+ \sum_{i=1}^{m} \sum_{k=1}^i \left[ \xi_4 \xi_2 (t_{i+1} - t_i) Q_3 + \frac{|\beta| + |\Omega|}{|\Omega|} Q_3 \right] \\
+ \sum_{i=1}^{m} \sum_{k=1}^i \left[ \xi_4 \xi_2 (t_{i+1} - t_i) (t_{i+1} - t_i) Q_3 + \frac{|\beta| + |\Omega|}{|\Omega|} Q_3 \right],
\]

(41)

\[
\Lambda_6 = \frac{|\beta| + |\Omega|}{|\Omega|} \sum_{k=1}^{m} \left[ \xi_1 (t_k - t_{k-1}) + \xi_2 \xi_3 \frac{(t_k - t_{k-1})^2}{1 + q_{k-1}} \right] + Q_1 (t_k - t_{k-1}) \\
+ \sum_{k=1}^{m} \left[ \xi_1 \frac{(t_{i+1} - t_i)^2}{1 + q_i} + \xi_2 \xi_3 \frac{(t_{i+1} - t_i)^3}{1 + q_i} \right] + Q_1 (t_{i+1} - t_i) \\
+ \sum_{i=1}^{m} \sum_{k=1}^i \left[ \xi_4 \xi_2 (t_{i+1} - t_i) Q_3 + \frac{|\beta| + |\Omega|}{|\Omega|} Q_3 \right] \\
+ \sum_{i=1}^{m} \sum_{k=1}^i \left[ \xi_4 \xi_2 (t_{i+1} - t_i) (t_{i+1} - t_i) Q_3 + \frac{|\beta| + |\Omega|}{|\Omega|} Q_3 \right],
\]

(42)

**Theorem 8.** Assume that \( f : \mathbb{R}^3 \rightarrow \mathbb{R} \) and \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R} \) are continuous functions. In addition we suppose the following.

\((H_1)\) There exist constants \( \xi_1, \xi_2 > 0 \) and \( Q_1 \geq 0 \) such that

\[
|f(t, z_1, z_2, z_3)| \leq \xi_1 |z_1| + \xi_2 |z_2| + Q_1 \quad \forall (t, z_1, z_2, z_3) \in J \times \mathbb{R}^3.
\]

\((H_2)\) There exist constants \( \xi_3 > 0 \) and \( Q_2 \geq 0 \) such that

\[
|\phi(t, z)| \leq \xi_3 |z| + Q_2 \quad \forall (t, z) \in J^2 \times \mathbb{R}.
\]

\((H_3)\) There exist constants \( \xi_4 > 0 \) and \( Q_3 \geq 0 \) such that

\[
|I_k(z)| \leq \xi_4 |z| + Q_3 \quad \forall z \in \mathbb{R}, k = 1, 2, \ldots, m.
\]

If \( \Lambda_5 < 1 \),

(46)

where \( \Lambda_5 \) is given by (41), then the impulsive functional \( q_k \)-
integrodifference boundary value problem (1) has at least one
solution on \( J \).

**Proof.** We define an operator \( \mathcal{K} : PC(J, \mathbb{R}) \rightarrow PC(J, \mathbb{R}) \) as in (22) and consider the fixed point problem:

\[
x = \mathcal{K} x.
\]

(47)

We are going to prove that there exists a fixed point \( x \in PC(J, \mathbb{R}) \) satisfying (47). It is sufficient to show that \( \mathcal{K} : B_{\rho} \rightarrow PC(J, \mathbb{R}) \) satisfies

\[
x \neq \lambda \mathcal{K} x, \quad \forall x \in \partial B_{\rho}, \forall \lambda \in [0, 1],
\]

(48)

where \( B_{\rho} = \{ x \in PC(J, \mathbb{R}) : \max_{t \in J} |x(t)| < \rho, \rho > 0 \} \). We define

\[
H(\lambda, x) = \lambda \mathcal{K} x, \quad x \in PC(J, \mathbb{R}), \lambda \in [0, 1].
\]

(49)

It is easy to see that the operator \( \mathcal{K} \) is continuous, uniformly bounded, and equicontinuous. Then, by the Arzelà-Ascoli Theorem, a continuous map \( h_{\lambda} \) defined by \( h_{\lambda}(x) = x - H(\lambda, x) \) is completely continuous. If (48) is true, then the following Leray-Schauder degrees are well defined and by the homotopy invariance of topological degree, it follows that

\[
\deg(h_{\lambda}, B_{\rho}, 0) = \deg(I - \lambda \mathcal{K}, B_{\rho}, 0) = \deg(h_{1}, B_{\rho}, 0)
\]

(50)

\[
= \deg(h_{0}, B_{\rho}, 0) = \deg(I, B_{\rho}, 0) = 1 \neq 0, \quad 0 \in B_{\rho},
\]

where \( I \) denotes the identity operator. By the nonzero property of Leray-Schauder degree, \( h_{\lambda}(x) = x - \mathcal{K} x = 0 \) for at least one \( x \in B_{\rho} \). In order to prove (48), we assume that \( x = \lambda \mathcal{K} x \) for some \( \lambda \in [0, 1] \). Then

\[
|\mathcal{K} x(t)| \leq \sup_{t \in J} \left( \frac{\beta \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} f(s, x(s), x(\theta(s)), (S_{q_k-1} x)(s)) \, dq_k s}{\Omega} \right) + \frac{\beta \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} f(s, x(s), x(\theta(s)), (S_{q_k-1} x)(s)) \, dq_k s}{\Omega} \right) \times \int_{t_{i+1}}^{t_i} \int_{t_{i+1}}^{t_i} f(s, x(s), x(\theta(s)), (S_{q_k-1} x)(s)) \, dq_k s \, dq_k u \]

\[
+ \frac{1}{\Omega} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} f(s, x(s), x(\theta(s)), (S_{q_k-1} x)(s)) \, dq_k s \, dq_k u \]

\[
+ \frac{1}{\Omega} \sum_{k=1}^{m} \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t_k} f(s, x(s), x(\theta(s)), (S_{q_k-1} x)(s)) \, dq_k s \, dq_k u
\]
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\[ + \frac{1}{\Omega} \sum_{j=1}^{m} \sum_{i=0}^{i} y_j (t_{i+1} - t_i) I_k (x(t)) \]

\[ + \int_{t_{k-1}}^{t_{k}} \int_{t_{i+1}}^{t_{i}} f (s, x(s), \theta(s), (S_{q_i}, x)(s)) \, d_{q_i} s \]

\[ + \sum_{0 \leq i \leq k} \left( \int_{t_{k-1}}^{t_{k}} f (s, x(s), \theta(s), (S_{q_i}, x)(s)) \, d_{q_i} s \right) \]

\[ + I_k (x(t)) \right) \}

\[ \leq \left| \frac{\beta + |\Omega|}{|\Omega|} \right| + \left( \left| \frac{\beta}{|\Omega|} \right| \sum_{k=1}^{m} I_k (x(t_k)) \right) \]

\[ \times \left| \frac{m+1}{\Omega} \int_{t_{k-1}}^{t_{k}} \left( \xi_1 ||x|| + Q_2 \right) d_{q_i} s \right| \]

\[ + \frac{1}{|\Omega|} \sum_{i=0}^{i} |y_i| (t_{i+1} - t_i) \]

\[ \times \left| \frac{m+1}{\Omega} \sum_{k=1}^{m+1} \int_{t_{k-1}}^{t_{k}} \left( \xi_1 ||x|| + Q_2 \right) d_{q_i} s \right| \]

\[ + \left| \frac{\beta}{|\Omega|} \sum_{k=1}^{m} I_k (x(t_k)) \right| \]

\[ \times \left| \frac{m+1}{\Omega} \int_{t_{k-1}}^{t_{k}} \left( \xi_1 ||x|| + Q_2 \right) d_{q_i} s \right| \]

\[ \times \left| \frac{\beta + |\Omega|}{|\Omega|} \right| \sum_{k=1}^{m} I_k (x(t_k)) \]

which implies that

\[ \left| x ||x|| \right| \leq \frac{\Lambda_\delta}{1 - \Lambda_\delta}. \] (52)

If \( \rho = \Lambda_\delta / (1 - \Lambda_\delta) + 1 \), inequality (48) holds. This completes the proof. \( \Box \)

4. Examples

In this section, we will give some examples to illustrate our main results.

Example 1. Consider the following boundary value problem for nonlinear first-order impulsive functional \( q_k \)-integrodifference equation:

\[ D^{(1/2)} \sin((k+1)/6) \pi x \] (t)

\[ = \frac{t^2 \sin \pi t}{2(t+4)^2} |x| + \frac{3tx(t/2)}{2(t+3)^2} \]

\[ + \frac{t^2}{2(e^t + 1)^2} \int_{t_k}^{t_{k+1}} 2t - s \, x(s) \, d_{(1/2)} \sin((k+1)/6)^\pi s, \]

\[ t \in J, \quad t \neq t_k, \]

\[ \Delta x(t_k) = \frac{|x(t_k)|}{2(k+3) + |x(t_k)|}, \quad t_k = \frac{k}{5}, \quad k = 1, 2, \ldots, 4, \]

\[ x(0) = \frac{1}{3} x(1) + \sum_{i=0}^{4} \left( \frac{1}{i + 2} \right) \int_{t_i}^{t_{i+1}} x(s) \, d_{(1/2)} \sin((k+1)/6)^\pi s. \] (53)

Set \( J = [0, 1], q_k = (1/2) \sin((k+1)\pi)/6 \) for \( k = 0, 1, \ldots, 4, y_i = 1/(i + 2) \) for \( i = 0, 1, \ldots, 4, m = 4, T = 1, \theta(t) = t/2, \)

\[ f (t, x, x(t), (S_{q_i}, x)) \]

\[ = \frac{t^2 \sin \pi t}{2(t+4)^2} |x| + \frac{3tx(t/2)}{2(t+3)^2} \]

\[ + \frac{t^2}{2(e^t + 1)^2} \int_{t_k}^{t_{k+1}} 2t - s \, x(s) \, d_{(1/2)} \sin((k+1)/6)^\pi s, \] (54)

and \( I_k(x) = |x(t_k)|/(2(k+3) + |x(t_k)|) \).
Since
\[ |\phi(t, s, y) - \phi(t, s, z)| \leq \frac{1}{2} |y - z|, \]
\[ |f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)| \leq \frac{1}{9} |y_1 - z_1| + \frac{1}{6} |y_2 - z_2| + \frac{1}{8} |y_3 - z_3|, \]
\[ |I_k(x) - I_k(z)| \leq \frac{1}{8} |y - z|, \]
then \((H_1)-(H_3)\) are satisfied with \(\phi_0 = 1/2, L_1 = 1/9, L_2 = 1/6, L_3 = 1/8, \) and \(L_4 = 1/8.\) We can show that \(\Delta 1 = 0.9517257476 < 1.\) Hence, by Theorem 5, the boundary value problem (53) has a unique solution on [0, 1].

**Example 2.** Consider the following boundary value problem for nonlinear first-order impulsive functional \(q_k\)-integrodifference equation:
\[
D_{k+1/\sqrt{3}} x(t) = \frac{t^2 \cos nt}{(4t^3)^2} x + 2 + (t + 2)^2 \sin^2 \left( x \left( \frac{2t}{3} \right) \right) + \frac{5 \cos nt}{3(e^t + 4)^2} \int_{t_i}^{t_i+1} x(s) d_{k+1/\sqrt{3}} s, \quad t \in J = [0, 1], \quad t \neq t_k, \]
\[
\Delta x(t_k) = \frac{|x(t_k)|}{5(k + 4) + 3 \sin nx(t_k)}, \quad t_k = \frac{k}{9}, \quad k = 1, 2, \ldots, 8, \]
\[
2x(0) = \frac{1}{4} x(1) + \sum_{i=0}^{8} \left( \frac{i + 1}{i + 4} \right) \int_{i}^{i+1} x(s) d_{i+1} s. \]

Set \(q_k = (k + 1)/(\sqrt{3} t_i)\) for \(k = 0, 1, \ldots, 8, \) \(y_i = (i + 1)/(i + 4)\) for \(i = 0, 1, \ldots, 8, \) \(m = 8, \) \(T = 1, \)
\[
f(t, x, x(\theta), (S_{q_k} x)) = \frac{t^2 \cos nt}{(4t^3)^2} x + 2 + (t + 2)^2 \sin^2 \left( x \left( \frac{2t}{3} \right) \right) + \frac{5 \cos nt}{3(e^t + 4)^2} \int_{t_i}^{t_i+1} x(s) d_{k+1/\sqrt{3}} s, \]
\[
\theta(t) = \frac{2t}{3}, \quad \text{and} \quad I_k(x) = \left( |x(t_k)|/(5k + 5) + |x(t_k)| \right) + 3 \sin nx(t_k). \]

Since
\[
|\phi(t, s, x)| \leq (1/4)|x|, \quad |\phi(t, s, x)| \leq (1/25)|x| + 3, \quad \text{then} \quad (H_5)-(H_6) \quad \text{are satisfied with} \quad \xi_1 = 1/18, \xi_2 = 1/15, \xi_3 = 1/4, \xi_4 = 1/25, Q_1 = 9, Q_2 = 0, \quad \text{and} \quad Q_3 = 3. \quad \text{We can show that} \quad \Delta 5 = 0.9134190736 < 1. \quad \text{Hence, by Theorem 8, the boundary value problem (56) has at least one solution on [0, 1].} \]

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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