Research Article

Global Asymptotic Stability in a Class of Reaction-Diffusion Equations with Time Delay

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We study a very general class of delayed reaction-diffusion equations in which the reaction term can be nonmonotone and spatially nonlocal. By using a fluctuation method, combined with the careful analysis of the corresponding characteristic equations, we obtain some sufficient conditions for the global asymptotic stability of the trivial solution and the positive steady state to the equations subject to the Neumann boundary condition.

1. Introduction

There has been a growing interest in the dynamic behavior of spatial nonlocal and time-delayed population systems since the 1970s [1]. When the death function of such a system is linear many researchers used the theory of monotone semiflows, the comparison arguments, and the fluctuation method to study spreading speeds, traveling waves, and the global stability (see, e.g., [2–10]). However, researches on these problems become relatively rare for nonmonotone delayed reaction-diffusion systems in which the death function is nonlinear (see [11, 12]). The reason lies in the fact that it is difficult to establish an appropriate expression for solutions to study the solution semiflow under this case.

In this paper, we will investigate the global asymptotic stability of the positive steady state for the following time-delayed reaction-diffusion equation:

$$\frac{\partial w(t, x)}{\partial t} = d\Delta w(t, x) - f(w(t, x))$$

$$+ \int_{\Omega} k(\alpha, x, y) b(w(t - \tau, y)) \, dy,$$

$$\frac{\partial w(t, x)}{\partial \mathbf{n}} = 0, \quad t > 0, \ x \in \partial\Omega,$$

$$w(t, x) = \phi(t, x) \geq 0, \quad t \in [-\tau, 0], \ x \in \Omega,$$

$$(1)$$

where $d > 0$, $\alpha \geq 0$, $\tau \geq 0$, Δ denotes the Laplacian operator on $\mathbb{R}^m$, Ω is a bounded and open domain of $\mathbb{R}^m$ with a smooth boundary $\partial\Omega$, $\partial/\partial\mathbf{n}$ is the differentiation in the direction of the outward normal $\mathbf{n}$ to $\partial\Omega$, and the kernel function $k(\alpha, x, y)$ is given by

$$k(\alpha, x, y) = \begin{cases} \sum_{n=1}^{+\infty} e^{-\lambda_n \alpha} \varphi_n(x) \varphi_n(y), & \text{if } \alpha > 0, \\ \delta(x - y), & \text{if } \alpha = 0. \end{cases} \quad (2)$$

Here, $0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$ with $\lim_{n \to \infty} \lambda_n = +\infty$ is the eigenvalue of the linear operator $-\Delta$ subject to the homogeneous Neumann boundary condition on $\partial\Omega$, φ_n is the eigenvector corresponding to λ_n, $\{\varphi_n\}_{n=1}^{+\infty}$ is a complete orthonormal system in the space $L^2(\overline{\Omega})$, $\varphi_1(x) > 0$ for all $x \in \Omega$, and $\delta(x)$ is the Dirac function on $\mathbb{R}^m$ [10, 13]. Throughout

this paper, we assume that the functions f and b satisfy the following.

(A1) $b : \mathbb{R}^+ \to \mathbb{R}^+$ is Lipschitz continuous with $b(0) = 0$ and $b'(0) > 0$, and $b(w) \leq b'(0)w$ for all $w \geq 0$.

(A2) $f(w) = wg(w)$ for all $w \geq 0$, where $g : \mathbb{R}^+ \to \mathbb{R}^+$ is Lipschitz continuous with $g'(0) \geq 0$, $g(w) > 0$, and $g'(w) \geq 0$ for all $w > 0$.

(A3) There exists a positive number M such that, for all $w > M$, $\overline{b}(w) < f(w)$, where $\overline{b}(w) = \max_{u \in [0,w]} b(u)$.

In the monotone case, where the function $b(w)$ increases with $w > 0$, Xu and Zhao [12] studied the global dynamics of (1) and obtained some results on the uniqueness and global attractivity of a positive steady state by using the theory of monotone dynamical systems. In the case of $f(w) = \mu w$, Zhao [10] proved the global attractivity of the positive constant equilibrium for (1) by using a fluctuation method of Thieme and Zhao [14], where μ is a positive constant. In the case where $f(w) = \mu w$ and $\alpha = 0$, Yi and Zou [7] proved the global attractivity of the unique positive constant equilibrium for (1) by combining a dynamical systems argument and some subtle inequalities. In the case where $\Omega = [0, L]$ and $f(w) = \mu w$, (1) reduces to the equation derived in [3], where the numerical solutions are considered. A global convergence theorem was obtained in [11] for a special case of (1).

The aim of this paper is to establish some criteria to guarantee the global asymptotic stability of the trivial solution and the positive steady state for (1) by using a fluctuation method, combined with the careful analysis of the corresponding characteristic equations. The interesting thing is that main results obtained in this paper extend the related existing results.

The rest of this paper is organized as follows. We will present some preliminary results in Section 2. Our main results are presented and proved in Sections 3 and 4, where we obtain sufficient conditions to ensure the global asymptotic stability of the trivial solution and the positive steady state for (1) in a nonmonotone case. In Section 5, we provide four examples to illustrate the applicability of the main results.

2. Preliminaries

Firstly, we show that the kernel $k(\alpha, x, y)$ in (2) enjoys the following properties.

Lemma 1. *For $\alpha > 0$, one has*

(i) $(\partial/\partial \mathbf{n})k(\alpha, x, y)|_{x \in \partial \Omega} = (\partial/\partial \mathbf{n})k(\alpha, x, y)|_{y \in \partial \Omega} = 0$,

(ii) $0 < k(\alpha, x, y) \leq C^*$, for all $x, y \in \Omega$, where C^* is a positive constant depending only on m and Ω,

(iii) $|\varphi_n(x)| \leq \sqrt{C^*} \exp[(1/2)\lambda_n \alpha]$, for all $x \in \Omega$, $n = 1, 2, \ldots$,

(iv) $\int_{\overline{\Omega}} k(\alpha, x, y)dy = 1$, for all $x \in \Omega$.

Proof. The verification of (i) is straightforward and is thus omitted. Part (ii) follows from [15, Lemma 3.2.1 and Theorem 4.4.6] since $k(\alpha, x, y)$ is a heat kernel of the heat equation

$(\Delta - (\partial/\partial \alpha))u(x, \alpha) = 0$. Part (iii) follows from $e^{-\lambda_n \alpha}(\varphi_n(x))^2 \leq k(\alpha, x, x) \leq C^*$, for all $x \in \Omega$, $n = 1, 2, \ldots$. And part (iv) follows from $\lambda_1 = 0$, $\varphi_1(x) \equiv \sqrt{1/\text{mes}(\Omega)}$, and $\int_{\overline{\Omega}} \varphi_n(y)dy = 0$ for all $n = 2, 3, \ldots$, where $\text{mes}(\Omega)$ is the measure of Ω. The proof is completed. $\square$

Let $\mathbb{X} = C(\overline{\Omega}, \mathbb{R})$ and $\mathbb{X}^+ = \{\phi \in \mathbb{X} \mid \phi(x) \geq 0, \ \forall x \in \overline{\Omega}\}$. Then $(\mathbb{X}, \mathbb{X}^+)$ is a strongly ordered Banach space. It is well known that the differential operator $A = d\Delta$ generates a C^0-semigroup $T(t)$ on $\mathbb{X}$. Moreover, the standard parabolic maximum principle (see, e.g., [16, Corollary 7.2.3]) implies that the semigroup $T(t) : \mathbb{X} \to \mathbb{X}$ is strongly positive in the sense that $T(t)(\mathbb{X}^+ \setminus \{0\}) \subset \text{Int}(\mathbb{X}^+)$, $\forall t > 0$.

Let $\mathbb{Y} = C([-\tau, 0], \mathbb{X})$ and $\mathbb{Y}^+ = C([-\tau, 0], \mathbb{X}^+)$. For the sake of convenience, we will identify an element $\phi \in \mathbb{Y}$ as a function from $[-\tau, 0] \times \overline{\Omega}$ to $\mathbb{R}$ defined by $\phi(s, x) = \phi(s)(x)$, and for each $s \in [-\tau, 0]$, we regard $f(\phi(s))$ as a function on $\overline{\Omega}$ defined by $f(\phi(s)) = f(\phi(s, \cdot))$. For any function $w(\cdot) : [-\tau, \sigma) \to \mathbb{X}$, where $\sigma > 0$, we define $w_t \in \mathbb{Y}$, $t \in [0, \sigma)$ by $w_t(s) = w(t + s)$, $\forall s \in [-\tau, 0]$. Define $F : \mathbb{Y}^+ \to \mathbb{X}$ by

$$F(\phi)(x) = -f(\phi(0, x)) + \int_{\overline{\Omega}} k(\alpha, x, y) b(\phi(-\tau, y)) dy,$$

$$\forall x \in \overline{\Omega}, \phi \in \mathbb{Y}^+. \tag{3}$$

Then we can rewrite (1) as an abstract functional equation:

$$\frac{dw(t)}{dt} = Aw(t) + F(w_t), \quad t \geq 0,$$

$$w_0 = \phi \in \mathbb{Y}^+. \tag{4}$$

Therefore, we can write (4) as an integral equation:

$$w(t) = T(t)\phi(0) + \int_0^t T(t - s) F(w_s) ds, \quad t \geq 0,$$

$$w_0 = \phi \in \mathbb{Y}^+, \tag{5}$$

whose solutions are called mild solutions for (1).

Since $T(t) : \mathbb{X} \to \mathbb{X}$ is strongly positive, we have

$$\lim_{h \to 0^+} \text{dist}(\phi(0) + hF(\phi), \mathbb{X}^+) = 0, \quad \forall \phi \in \mathbb{Y}^+. \tag{6}$$

By [17, Proposition 3 and Remark 2.4] (or [18, Corollary 8.1.3]), for each $\phi \in \mathbb{Y}^+$, (1) has a unique noncontinuable mild solution $w(t, \phi)$ with $w_0 = \phi$, and $w(t, \phi) \in \mathbb{X}^+$ for all $t \in (0, \sigma_\phi)$. Moreover, $w(t, \phi)$ is a classical solution of (1) for $t > \tau$ (see [18, Corollary 2.2.5]).

By the same arguments as in the proof of [12, Theorems 2.1 and 3.1], we have the following two lemmas.

Lemma 2. *Let (A1)–(A3) hold. Then, for each $\phi \in \mathbb{Y}^+$, a unique solution $w(t, \phi)$ of (1) globally exists on $[-\tau, \infty)$, $\limsup_{t \to \infty} w(t, x, \phi) \leq M$ uniformly for $x \in \overline{\Omega}$, and the solution semiflow $\Phi(t) = w_t(\cdot) : \mathbb{Y}^+ \to \mathbb{Y}^+$, $t \geq 0$, admits a connected global attractor.*

Lemma 3. *Let (A1)–(A3) hold, and let $w(t, x, \phi)$ be the solution of (1) with $\phi \in \mathbb{Y}^+$. Then the following two statements are valid.*

(i) *If $b'(0) < g(0)$, then for any $\phi \in \mathbb{Y}^+$, we have $\limsup_{t \to \infty} w(t, x, \phi) = 0$ uniformly for $x \in \overline{\Omega}$.*

(ii) *If $b'(0) > g(0)$, then (1) admits at least one spatially homogeneous steady state $w^* \in (0, M)$, and there exists $\eta > 0$ such that for any $\phi \in \mathbb{Y}^+$ with $\phi(0, \cdot) \not\equiv 0$ we have $\liminf_{t \to \infty} w(t, x, \phi) \geq \eta$ uniformly for $x \in \overline{\Omega}$.*

Note that in case (ii) above, the function $S(w) = b(w) - f(w)$ satisfies $S(0) = 0$, $S'(0) > 0$, and $S(M) \leq 0$. Therefore, there exists at least one positive number $w^* \in (0, M]$ such that $S(w^*) = 0$, and hence, w^* is a spatially homogeneous steady state of (1).

3. Global Attractivity

In this section, we establish the global attractivity of the positive and spatially homogeneous steady state w^* for (1) by the fluctuation method used in [10, Theorem 3.1].

Motivated by [10, Section 3], we assume further that the functions $f(w)$ and $b(w)$ satisfy the following.

(A4) $b'(0) > g(0)$, $(b(w)/f(w))$ is strictly decreasing for $w \in (0, M]$, and $f(w)$ and $b(w)$ have the property (P) that, for any $u, v \in (0, M]$ satisfying $u \leq w^* \leq v$, $f(u) \geq b(v)$, and $f(v) \leq b(u)$, we have $u = v$.

Note that if $b(w)$ is nondecreasing for $w \in [0, M]$, then $f(w)$ and $b(w)$ have the property (P). Indeed, for any $0 < u \leq w^* \leq v \leq M$ with $f(u) \geq b(v)$ and $f(v) \leq b(u)$, we have

$$f(w^*) \leq f(v) \leq b(u) \leq b(w^*) \leq b(v) \leq f(u) \leq f(w^*), \tag{7}$$

which implies that $u = v = w^*$. Combining this observation and [10, Lemma 3.1] with μw replaced by $f(w)$, where $\mu > 0$, we then have the following result.

Lemma 4. *Either of the following two conditions is sufficient for the property (P) in condition (A4) to hold.*

(P1) *$b(w)$ is nondecreasing for $w \in [0, M]$.*

(P2) *$f(w)b(w)$ is strictly increasing for $w \in (0, M]$.*

Now we are in a position to prove our main result in this section.

Theorem 5. *Assume that (A1)–(A4) hold, and let $w(t, x, \phi)$ be the solution of (1) with $\phi \in \mathbb{Y}^+$. Then for any $\phi \in \mathbb{Y}^+$ with $\phi(0, \cdot) \not\equiv 0$, we have $\lim_{t \to \infty} w(t, x, \phi) = w^*$ uniformly for $x \in \overline{\Omega}$.*

In order to prove Theorem 5, we will need the following lemma.

Lemma 6. *Assume that (A1)–(A3) hold, and let $w(t, x) \equiv w(t, x, \phi)$ be the solution of (1) with $\phi \in \mathbb{Y}^+$. Then $w(t, x)$ satisfies*

$$
\begin{aligned}
& w(t, x) \\
&= e^{-\gamma t} \int_{\Omega} k(dt, x, y) \phi(0, y) \, dy \\
&\quad + \int_0^t e^{-\gamma s} \int_{\Omega} k(ds, x, y) \\
&\qquad \times \left[\gamma w(t - s, y) - f(w(t - s, y)) \right. \\
&\qquad\qquad \left. + \int_{\Omega} k(\alpha, y, z) b(w(t - s - \tau, z)) \, dz \right] dy \, ds,
\end{aligned}
\tag{8}
$$

where $\gamma = \max_{w \in [0, M]} f'(w)$ and the kernel function k is given in (2).

Proof. Let

$$
\begin{aligned}
H(t, x) &\equiv \gamma w(t, x) - f(w(t, x)) \\
&\quad + \int_{\Omega} k(\alpha, x, y) b(w(t - \tau, y)) \, dy.
\end{aligned}
\tag{9}
$$

Since $\mathbb{X} \subset L^2(\overline{\Omega})$, for each $t \geq 0$, there exist real numbers $a_n(t)$ and $b_n(t)$, $n = 1, 2, \ldots$, such that

$$w(t, x) = \sum_{n=1}^{+\infty} a_n(t) \varphi_n(x), \tag{10}$$

$$H(t, x) = \sum_{n=1}^{+\infty} b_n(t) \varphi_n(x). \tag{11}$$

Therefore, by (10), (11), and (1), we have

$$
\begin{aligned}
a_n(0) &= \int_{\Omega} \phi(0, y) \varphi_n(y) \, dy, \\
b_n(s) &= \int_{\Omega} H(s, y) \varphi_n(y) \, dy,
\end{aligned}
\tag{12}
$$

$$\frac{da_n(t)}{dt} = -(d\lambda_n + \gamma) a_n(t) + b_n(t), \quad n = 1, 2, \ldots. \tag{13}$$

By using the variation of constants method, we obtain

$$a_n(t) = \left[a_n(0) + \int_0^t e^{(d\lambda_n + \gamma)s} b_n(s) \, ds \right] e^{-(d\lambda_n + \gamma)t}, \tag{14}$$

$$n = 1, 2, \ldots.$$

Thus, by (10), (12), and (14), we further get

$$w(t, x)$$

$$= \sum_{n=1}^{+\infty} \left[a_n(0) + \int_0^t e^{(d\lambda_n + \gamma)s} b_n(s)\,ds \right] e^{-(d\lambda_n + \gamma)t} \varphi_n(x)$$

$$= e^{-\gamma t} \sum_{n=1}^{+\infty} \int_\Omega \phi(0, y) e^{-d\lambda_n t} \varphi_n(x) \varphi_n(y)\,dy$$

$$\quad + \int_0^t e^{-\gamma(t-s)} \sum_{n=1}^{+\infty} \int_\Omega e^{-d\lambda_n(t-s)} H(s, y) \varphi_n(x) \varphi_n(y)\,dy\,ds$$

$$= e^{-\gamma t} \int_\Omega \phi(0, y) k(dt, x, y)\,dy$$

$$\quad + \int_0^t e^{-\gamma(t-s)} \int_\Omega H(s, y) k(d(t-s), x, y)\,dy\,ds$$

$$= e^{-\gamma t} \int_\Omega \phi(0, y) k(dt, x, y)\,dy$$

$$\quad + \int_0^t e^{-\gamma s} \int_\Omega H(t-s, y) k(ds, x, y)\,dy\,ds. \tag{15}$$

Therefore, (8) follows immediately from (9) and (15). The proof is completed. □

Proof of Theorem 5. For any given $\phi \in \mathbb{Y}^+$ with $\phi(0, \cdot) \not\equiv 0$, let $\omega(\phi)$ be the omega limit set of the positive orbit through ϕ for the solution semiflow $\Phi(t)$. By Lemma 1, we get $\omega(\phi) \subset \mathbb{A} \subseteq \mathbb{Y}_{[0,M]}$, where $\mathbb{A}$ is the global attractor of the solution semiflow $\Phi(t)$ and

$$\mathbb{Y}_{[0,M]} \equiv \left\{ \phi \in \mathbb{Y} \mid 0 \le \phi(\theta, x) \le M, \forall (\theta, x) \in [-\tau, 0] \times \overline{\Omega} \right\}. \tag{16}$$

Note that $\mathbb{A}$ is a maximal compact invariant set of the solution semiflow $\Phi(t)$. Thus, it is sufficient to prove the global attractivity of w^* for all $\phi \in \mathbb{Y}_{[0,M]}$ with $\phi(0, \cdot) \not\equiv 0$.

Let $\phi \in \mathbb{Y}_{[0,M]}$ be given such that $\phi(0, \cdot) \not\equiv 0$. Then it follows from Lemma 6 that

$$w(t, x)$$

$$= e^{-\gamma t} \int_\Omega k(dt, x, y) \phi(0, y)\,dy$$

$$\quad + \int_0^t e^{-\gamma s} \int_\Omega k(ds, x, y)$$

$$\quad\quad \times \left[\gamma w(t-s, y) - f(w(t-s, y)) \right.$$

$$\quad\quad\quad \left. + \int_\Omega k(\alpha, y, z) b(w(t-s-\tau, z))\,dz \right] dy\,ds, \tag{17}$$

where $w(t, x) \equiv w(t, x, \phi)$ is the solution of (1) starting from the initial function ϕ. Following [19], we define a function $h : [0, M] \times [0, M] \to \mathbb{R}$ by

$$h(u, v) = \begin{cases} \min \{ b(w) \mid u \le w \le v \}, & \text{if } u \le v, \\ \max \{ b(w) \mid v \le w \le u \}, & \text{if } v \le u. \end{cases} \tag{18}$$

Then $h(u, v)$ is nondecreasing in $u \in [0, M]$ and nonincreasing in $v \in [0, M]$. Moreover, $b(w) = h(w, w)$, $\forall w \in [0, M]$, and $h(u, v)$ is continuous in $(u, v) \in [0, M] \times [0, M]$ (see [20, Section 2]). Therefore, by (17), we have

$$w(t, x)$$

$$= e^{-\gamma t} \int_\Omega k(dt, x, y) \phi(0, y)\,dy$$

$$\quad + \int_0^t e^{-\gamma s} \int_\Omega k(ds, x, y)$$

$$\quad\quad \times \left[\gamma w(t-s, y) - f(w(t-s, y)) \right.$$

$$\quad\quad\quad + \int_\Omega k(\alpha, y, z)$$

$$\quad\quad\quad\quad \left. \times h(w(t-s-\tau, z), w(t-s-\tau, z))\,dz \right] dy\,ds. \tag{19}$$

Let

$$w^\infty(x) \equiv \limsup_{t \to \infty} w(t, x), \qquad w_\infty(x) \equiv \liminf_{t \to \infty} w(t, x),$$

$$\forall x \in \overline{\Omega}. \tag{20}$$

Then Lemmas 2 and 3 imply that

$$M \ge w^\infty(x) \ge w_\infty(x) \ge \eta > 0, \quad \forall x \in \overline{\Omega}. \tag{21}$$

On the other hand, note that $\gamma = \max_{w \in [0,M]} f'(w)$. Therefore, the function $\gamma w - f(w)$ is nondecreasing in $w \in [0, M]$. Thus, by Fatou's lemma and (19), we further get

$$w^\infty(x)$$

$$\le \int_0^\infty e^{-\gamma s} \int_\Omega k(ds, x, y)$$

$$\quad \times \left[\gamma w^\infty(y) - f(w^\infty(y)) \right.$$

$$\quad\quad \left. + \int_\Omega k(\alpha, y, z) h(w^\infty(z), w_\infty(z))\,dz \right] dy\,ds. \tag{22}$$

Let

$$w^\infty \equiv \sup_{x \in \overline{\Omega}} w^\infty(x), \qquad w_\infty \equiv \inf_{x \in \overline{\Omega}} w_\infty(x). \tag{23}$$

Then

$$M \ge w^\infty \ge w_\infty \ge \eta > 0. \tag{24}$$

Moreover, it follows from Lemma 1 that

$$\int_\Omega k\,(ds, x, y)\,dy = 1, \qquad \int_\Omega k\,(\alpha, x, y)\,dy = 1,$$
$$\forall s \geq 0, \quad x \in \overline{\Omega}. \tag{25}$$

Therefore, by (22), we have

$$w^\infty \leq [\gamma w^\infty - f(w^\infty) + h(w^\infty, w_\infty)] \int_0^\infty e^{-\gamma s}ds$$
$$= \frac{1}{\gamma}[\gamma w^\infty - f(w^\infty) + h(w^\infty, w_\infty)]. \tag{26}$$

Thus,

$$f(w^\infty) \leq h(w^\infty, w_\infty). \tag{27}$$

Similarly, we have

$$f(w_\infty) \geq h(w_\infty, w^\infty). \tag{28}$$

By (18), we may find $u, v \in [w_\infty, w^\infty] \subset (0, M]$ such that

$$h(w^\infty, w_\infty) = b(u), \qquad h(w_\infty, w^\infty) = b(v). \tag{29}$$

It then follows from (27) and (28) that

$$b(u) \geq f(w^\infty) \geq f(u), \qquad b(v) \leq f(w_\infty) \leq f(v), \tag{30}$$

and hence,

$$\frac{b(v)}{f(v)} \leq 1 = \frac{b(w^*)}{f(w^*)} \leq \frac{b(u)}{f(u)}. \tag{31}$$

This, together with the strict monotonicity of $b(w)/f(w)$ for $w \in (0, M]$, implies that $u \leq w^* \leq v$. Moreover, by (27) and (28), we also have

$$b(u) \geq f(w^\infty) \geq f(v), \qquad b(v) \leq f(w_\infty) \leq f(u). \tag{32}$$

Therefore, the property (P) implies that

$$u = v = w^*. \tag{33}$$

Thus, by (30), we obtain

$$w^\infty = w_\infty = w^*. \tag{34}$$

Since

$$w^\infty \geq w^\infty(x) \geq w^\infty(x) \geq w_\infty, \quad x \in \overline{\Omega}, \tag{35}$$

we further get

$$w^\infty(x) = w_\infty(x) = w^*, \quad x \in \overline{\Omega}. \tag{36}$$

This implies that

$$\lim_{t\to\infty} w(t, x) = w^*, \quad x \in \overline{\Omega}. \tag{37}$$

It remains to prove that $\lim_{t\to\infty} w(t, x) = w^*$ uniformly for $x \in \overline{\Omega}$. For any $\psi \in \omega(\phi)$, there exists a sequence $t_n \to \infty$ such that $\Phi(t_n)\phi \to \psi$ in $\mathbb{Y}$ as $n \to \infty$. Therefore, we have

$$\lim_{n\to\infty} w(t_n + \theta, x, \phi) = \psi(\theta, x) \tag{38}$$

uniformly for $(\theta, x) \in [-\tau, 0] \times \overline{\Omega}$. By (37), we further get

$$\psi(\theta, x) = w^*, \quad \forall (\theta, x) \in [-\tau, 0] \times \overline{\Omega}. \tag{39}$$

Thus, we obtain $\omega(\phi) = \{w^*\}$, which implies that $w(t, \cdot, \phi)$ converges to w^* in $\mathbb{X}$ as $t \to \infty$. The proof is completed. $\square$

4. Global Asymptotic Stability

In this section, we establish the global asymptotic stability of the trivial solution and the positive and spatially homogeneous steady state w^* for (1) by the careful analysis of the corresponding characteristic equations. To this end, we first give the following formal definitions of stability (see, e.g., [18, Remark 2.1.3]).

Definition 7. Let $w = \widehat{w}$ be a steady state of the abstract equation (4). It is called stable if for any $\varepsilon > 0$ there exists $\delta > 0$ such that the solution $w(t, \phi)$ of (4) with $\|\phi - \widehat{w}\|_\mathbb{Y} < \delta$ satisfies $\|w(t, \phi) - \widehat{w}\|_\mathbb{X} < \varepsilon$, for all $t \geq 0$. It is called unstable if it is not stable. It is asymptotically stable if it is stable and there exists $\delta_0 > 0$ such that the solution $w(t, \phi)$ of (4) with $\|\phi - \widehat{w}\|_\mathbb{Y} < \delta_0$ satisfies $\lim_{t\to+\infty}\|w(t, \phi) - \widehat{w}\|_\mathbb{X} = 0$. It is globally asymptotically stable if it is stable and any solution $w(t, \phi)$ of (4) with arbitrary $\phi \in \mathbb{Y}$ satisfies $\lim_{t\to+\infty}\|w(t, \phi) - \widehat{w}\|_\mathbb{X} = 0$.

Let $\widehat{w}$ be a spatially homogeneous steady state for (1) (e.g., the trivial solution and w^*). Define $G : \mathbb{Y}^+ \to \mathbb{X}$ by

$$G(\phi)(x) = -f'(\widehat{w})\phi(0, x)$$
$$+ b'(\widehat{w})\int_{\overline{\Omega}} k(\alpha, x, y)\phi(-\tau, y)\,dy, \tag{40}$$
$$\forall x \in \overline{\Omega}, \quad \phi \in \mathbb{Y}^+,$$

where $f'(\widehat{w}) = (df(w)/dw)|_{w=\widehat{w}}$ and $b'(\widehat{w}) = (db(w)/dw)|_{w=\widehat{w}}$. Note that $k(\alpha, x, y)$ is given in (2). Then we can write the linearized equation of (1) at $w = \widehat{w}$ as the following abstract functional equation

$$\frac{dw(t)}{dt} = Aw(t) + G(w_t), \quad t \geq 0, \tag{41}$$
$$w_0 = \phi \in \mathbb{Y}^+,$$

where A can be referred to Section 2.

For each complex number λ we define the $\mathbb{X}$-valued linear operator $\Theta(\lambda)$ by

$$\Theta(\lambda)u = Au - \lambda u + G(e^{\lambda \cdot}u), \quad u \in \text{Dom}(A), \tag{42}$$

where $e^{\lambda \cdot}u \in \mathbb{Y}$ is defined by (note that we use $\mathbb{Y}$ to denote its complexification here)

$$(e^{\lambda \cdot}u)(\theta) = e^{\lambda \theta}u, \quad \theta \in [-\tau, 0]. \tag{43}$$

We will call λ a characteristic value of (41) if there exists $u \in \text{Dom}(A) \setminus \{0\}$ solving the characteristic equation $\Theta(\lambda)u = 0$ (see, e.g., [18]). Since $\text{Dom}(A) \subset \mathbb{X} \subset L^2(\overline{\Omega})$, for any $u \in \text{Dom}(A) \setminus \{0\}$, there exist complex numbers a_n, $n = 1, 2, \ldots,$ such that

$$u(x) = \sum_{n=1}^{+\infty} a_n \varphi_n(x). \tag{44}$$

Therefore, by (2), (42), and (44), we have

$$
\begin{aligned}
\Theta\left(\lambda\right) & u\left(x\right) \\
& = d\Delta u\left(x\right) - \lambda u\left(x\right) - f'\left(\widehat{w}\right) u\left(x\right) \\
& \quad + b'\left(\widehat{w}\right) \int_{\overline{\Omega}} e^{-\tau\lambda} u\left(y\right) k\left(\alpha, x, y\right) dy \\
& = \sum_{n=1}^{+\infty} a_n \left[-d\lambda_n - \lambda - f'\left(\widehat{w}\right) + b'\left(\widehat{w}\right) e^{-\tau\lambda} e^{-\lambda_n \alpha}\right] \varphi_n\left(x\right).
\end{aligned}
\tag{45}
$$

Thus, the characteristic value λ of (41) satisfies at least one of the following equations:

$$
\lambda = -d\lambda_n - f'\left(\widehat{w}\right) + b'\left(\widehat{w}\right) e^{-\lambda_n \alpha} e^{-\tau\lambda}, \quad n = 1, 2, \ldots. \tag{46}
$$

Lemma 8. *Assume that (A1)–(A3) hold, and let β be the smallest real number such that if λ is a characteristic value of (41), then $\operatorname{Re}\lambda \le \beta$. One has the following:*

   (i) *if $b'\left(\widehat{w}\right) > f'\left(\widehat{w}\right)$, then $\beta > 0$,*

   (ii) *if $-f'\left(\widehat{w}\right) \le b'\left(\widehat{w}\right) < f'\left(\widehat{w}\right)$, then $\beta < 0$,*

   (iii) *if $b'\left(\widehat{w}\right) = f'\left(\widehat{w}\right)$, then $\beta = 0$.*

Proof. (i) If $b'\left(\widehat{w}\right) > f'\left(\widehat{w}\right)$, then, by (46) and [21, Proposition 4.6], there exists at least one characteristic value λ of (41) such that $\operatorname{Re}\lambda > 0$. Therefore, $\beta > 0$.

(ii) If $-f'\left(\widehat{w}\right) \le b'\left(\widehat{w}\right) < f'\left(\widehat{w}\right)$, then since $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \le \cdots$, we have

$$
\begin{aligned}
-\left[d\lambda_n + f'\left(\widehat{w}\right)\right] e^{\lambda_n \alpha} & \le b'\left(\widehat{w}\right) \\
& < \left[d\lambda_n + f'\left(\widehat{w}\right)\right] e^{\lambda_n \alpha}, \quad n = 1, 2, \ldots.
\end{aligned}
\tag{47}
$$

Therefore, by (46) and [21, Proposition 4.6], all the characteristic values of (41) have negative real parts. Thus, it follows from [18, Theorem 3.1.10] that $\beta < 0$.

(iii) If $b'\left(\widehat{w}\right) = f'\left(\widehat{w}\right)$, then $\lambda = 0$ is a characteristic value of (41). Therefore, $\beta \ge 0$. If $\beta > 0$, then there exists at least one characteristic value of (41) $\lambda^{(0)}$ and a positive number n such that $\operatorname{Re}\lambda^{(0)} > 0$ and

$$
\lambda^{(0)} = -d\lambda_n - f'\left(\widehat{w}\right) + b'\left(\widehat{w}\right) e^{-\lambda_n \alpha} e^{-\tau\lambda^{(0)}}. \tag{48}
$$

Let $\lambda^{(0)} = x^{(0)} + iy^{(0)}$, where $x^{(0)}$ and $y^{(0)}$ both are real numbers. Then $x^{(0)} > 0$. By (48), we have

$$
x^{(0)} = -d\lambda_n - f'\left(\widehat{w}\right) + f'\left(\widehat{w}\right) e^{-\lambda_n \alpha} e^{-\tau x^{(0)}} \cos\left(\tau y^{(0)}\right), \tag{49}
$$

and hence, $\cos(\tau y^{(0)}) > 0$. This implies that

$$
\begin{aligned}
x^{(0)} & \le -f'\left(\widehat{w}\right) + f'\left(\widehat{w}\right) e^{-\tau x^{(0)}} \cos\left(\tau y^{(0)}\right) \\
& = f'\left(\widehat{w}\right) \left[e^{-\tau x^{(0)}} \cos\left(\tau y^{(0)}\right) - 1\right].
\end{aligned}
\tag{50}
$$

But, since $x^{(0)} > 0$, we have $e^{-\tau x^{(0)}} \cos(\tau y^{(0)}) < 1$. Therefore,

$$
x^{(0)} \le f'\left(\widehat{w}\right) \left[e^{-\tau x^{(0)}} \cos\left(\tau y^{(0)}\right) - 1\right] < 0, \tag{51}
$$

contradicting $x^{(0)} > 0$. This contradiction proves $\beta = 0$. The proof is completed. $\qquad\square$

Now we are ready to summarize our main results on the global stability. By Definition 7, Lemmas 3 and 8, Theorem 5, [18, Corollary 3.1.11], and the principle of linearized stability (see, e.g., [21]), we obtain the following.

Theorem 9. *Assume that (A1)–(A3) hold. Then the following two statements are valid.*

   (i) *If $b'(0) < g(0)$, then the zero solution of (1) is globally asymptotically stable in $\mathbb{Y}^+$.*

   (ii) *If $b'(0) > g(0)$, then the zero solution of (1) is unstable, and (1) admits at least one spatially homogeneous steady state $w^* \in (0, M]$.*

Theorem 10. *Assume that (A1)–(A3) hold, and $b'(0) > g(0)$. Then the following two statements for the positive and spatially homogeneous steady state w^* of (1) are valid.*

   (i) *If $b'(w^*) > f'(w^*)$, then w^* is unstable.*

   (ii) *If $-f'(w^*) \le b'(w^*) < f'(w^*)$ and (A4) hold, then w^* is globally asymptotically stable in $\mathbb{Y}^+ \setminus \{0\}$.*

5. Examples

In this section, we present four examples to illustrate the feasibility of our main results.

Example 1. Consider the equation resulting from letting $f(w) = \mu w$ in (1); that is,

$$
\begin{aligned}
\frac{\partial w\left(t, x\right)}{\partial t} & = d\Delta w\left(t, x\right) - \mu w\left(t, x\right) \\
& \quad + \int_{\Omega} k\left(\alpha, x, y\right) b\left(w\left(t - \tau, y\right)\right) dy, \\
\frac{\partial w\left(t, x\right)}{\partial \mathbf{n}} & = 0, \quad t > 0, \, x \in \partial\Omega, \\
w\left(t, x\right) & = \phi\left(t, x\right) \ge 0, \quad t \in [-\tau, 0], \, x \in \Omega,
\end{aligned}
\tag{52}
$$

where μ is a positive constant.

In this case, we now formulate the following assumptions to replace (A2)–(A4):

  (A2′) There exists a positive number M such that, for all $w > M$, $\overline{b}(w) < \mu w$, where $\overline{b}(w) = \max_{u \in [0, w]} b(u)$.

  (A3′) $b'(0) > \mu$, $(b(w)/w)$ is strictly decreasing for $w \in (0, M]$, and $b(w)$ has the property (P') that, for any $u, v \in (0, M]$ satisfying $u \le w^* \le v$, $\mu u \ge b(v)$, and $\mu v \le b(u)$, we have $u = v$.

By applying Theorems 9 and 10, we then obtain the following results for (52).

Theorem 11. *Assume that (A1) and (A2$'$) hold. Then the following two statements are valid.*

   (i) *If $b'(0) < \mu$, then the zero solution of (52) is globally asymptotically stable in $\mathbb{Y}^+$.*

   (ii) *If $b'(0) > \mu$, then the zero solution of (52) is unstable, and (52) admits at least one spatially homogeneous steady state $w^* \in (0, M]$.*

Theorem 12. *Assume that (A1) and (A2$'$) hold, and $b'(0) > \mu$. Then the following two statements for the positive and spatially homogeneous steady state w^* of (52) are valid.*

   (i) *If $b'(w^*) > \mu$, then w^* is unstable.*

   (ii) *If $-\mu \leq b'(w^*) < \mu$ and (A3$'$) hold, then w^* is globally asymptotically stable in $\mathbb{Y}^+ \setminus \{0\}$.*

Remark 13. It is easy to see that (52) is discussed in [10] and some partial results of Theorems 11 and 12 have been obtained [10].

Example 2. Consider the following Nicholson's blowfly equation resulting from letting $f(w) = \mu w$ and $b(w) = pwe^{-qw}$ in (1):

$$\frac{\partial w(t,x)}{\partial t} = d\Delta w(t,x) - \mu w(t,x)$$

$$+ \int_\Omega pw(t-\tau, y) e^{-qw(t-\tau, y)} k(\alpha, x, y)\, dy,$$

$$\frac{\partial w(t,x)}{\partial \mathbf{n}} = 0, \quad t > 0,\ x \in \partial\Omega,$$

$$w(t,x) = \phi(t,x) \geq 0, \quad t \in [-\tau, 0],\ x \in \Omega, \tag{53}$$

where p and q are two positive constants.

By the same arguments as in [10, Section 4], together with Theorems 11 and 12, we have the following results for (53).

Theorem 14. *(i) If $p < \mu$, then the zero solution of (53) is globally asymptotically stable in $\mathbb{Y}^+$.*
(ii) If $p > \mu$, then the zero solution of (53) is unstable, and (53) admits the unique positive constant equilibrium $w^* = (\ln(p/\mu))/q$.

Theorem 15. *If $\mu < p \leq e^2\mu$, the unique positive constant equilibrium $w^* = (\ln(p/\mu))/q$ of (53) is globally asymptotically stable in $\mathbb{Y}^+ \setminus \{0\}$.*

Remark 16. It is easy to see that the equation in [7] is a special case of $\alpha = 0$ of (53). Hence all main results of [7] are special cases of our Theorems 14 and 15.

Example 3. Consider the following Mackey-Glass equation resulting from letting $f(w) = \mu w^{l+1}$ and $b(w) = pw/(q + w^l)$ in (1):

$$\frac{\partial w(t,x)}{\partial t} = d\Delta w(t,x) - \mu w^{l+1}(t,x)$$

$$+ \int_\Omega \frac{pw(t-\tau, y)}{q + w^l(t-\tau, y)} k(\alpha, x, y)\, dy,$$

$$\frac{\partial w(t,x)}{\partial \mathbf{n}} = 0, \quad t > 0,\ x \in \partial\Omega,$$

$$w(t,x) = \phi(t,x) \geq 0, \quad t \in [-\tau, 0],\ x \in \Omega, \tag{54}$$

where l is a positive constant.

By the same arguments as in [10, Section 4], together with Theorems 9 and 10, we have the following results for (54).

Theorem 17. *The zero solution of (54) is always unstable, and (54) must admit the unique positive constant equilibrium $w^* = (Z_0)^{1/l}$, where*

$$Z_0 = \frac{1}{2}\left(-q + \sqrt{q^2 + \frac{4p}{\mu}}\right). \tag{55}$$

Theorem 18. *If $((pq + p(1-l)Z_0)/(q+Z_0)^2) > \mu(1+l)Z_0$, the unique positive constant equilibrium $w^* = (Z_0)^{1/l}$ is unstable, and if $-\mu(1+l)Z_0 \leq ((pq + p(1-l)Z_0)/(q+Z_0)^2) < \mu(1+l)Z_0$ it is globally asymptotically stable in $\mathbb{Y}^+ \setminus \{0\}$.*

Example 4. Consider the equation resulting from letting $f(w) = \mu w^2$ and $b(w) = pw(1 - (w/r))$ in (1); that is,

$$\frac{\partial w(t,x)}{\partial t}$$

$$= d\Delta w(t,x) - \mu w^2(t,x)$$

$$+ \int_\Omega pw(t-\tau, y)\left(1 - \frac{w(t-\tau, y)}{r}\right) k(\alpha, x, y)\, dy,$$

$$\frac{\partial w(t,x)}{\partial \mathbf{n}} = 0, \quad t > 0,\ x \in \partial\Omega,$$

$$w(t,x) = \phi(t,x) \geq 0, \quad t \in [-\tau, 0],\ x \in \Omega, \tag{56}$$

where $0 < r \leq +\infty$.

Clearly, $w^* = (p/(\mu + pr^{-1}))$, $\max_{w\geq0} b(w) = b(r/2)$, and (A1)–(A3) hold, where $r^{-1} = 0$ if $r = +\infty$. Moreover, $0 < w^* \leq r/2$ if $\mu \geq pr^{-1}$. Therefore, (A4) is satisfied if $\mu \geq pr^{-1}$. Thus, Theorems 9 and 10 imply the following results.

Theorem 19. *The zero solution of (56) is always unstable, and (56) must admit the unique positive constant equilibrium $w^* = (p/(\mu + pr^{-1}))$.*

Theorem 20. *If $\mu \geq pr^{-1}$, then the unique positive constant equilibrium $w^* = (p/(\mu + pr^{-1}))$ is globally asymptotically stable in $\mathbb{Y}^+ \setminus \{0\}$.*

Remark 21. It is easy to see that the equation in [11] is a special case of $r = +\infty$ of (56) and hence some partial results of Theorems 19 and 20 have been obtained [11].

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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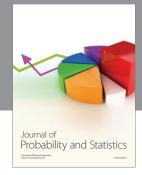
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[21] H. Smith, *An Introduction to Delay Differential Equations with Applications to the Life Sciences*, vol. 57 of *Texts in Applied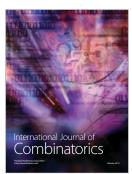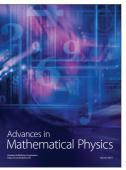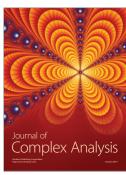 Math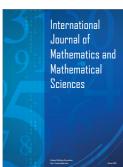ematics*, Springer, New York, NY, USA, 2011.