Research Article
The Yang-Laplace Transform for Solving the IVPs with Local Fractional Derivative

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The IVPs with local fractional derivative are considered in this paper. Analytical solutions for the homogeneous and nonhomogeneous local fractional differential equations are discussed by using the Yang-Laplace transform.

1. Introduction

In recent years, the ordinary and partial differential equations have found applications in many problems in mathematical physics [1, 2]. Initial value problems (IVPs) for ordinary and partial differential equations have been developed by some authors in [3–6]. There are analytical methods and numerical methods for solving the differential equations, such as the finite element method [6], the harmonic wavelet method [7–9], the Adomian decomposition method [10–12], the homotopy analysis method [13, 14], the homotopy decomposition method [15, 16], the heat balance integral method [17, 18], the homotopy perturbation method [19], the variational iteration method [20], and other methods [21].

Recently, owing to limit of classical and fractional differential equations, the local fractional differential equations have been applied to describe nondifferentiable problems for the heat and wave in fractal media [22, 23], the structure relation in fractal elasticity [24], and Fokker-Planck equation in fractal media [25]. Some methods were utilized to solve the local fractional differential equations. For example, the local fractional variation iteration method was used to solve the heat conduction in fractal media [26, 27]. The local fractional decomposition method for solving the local fractional diffusion and heat-conduction equations was considered in [28, 29]. The local fractional series expansion method for solving the Schrödinger equation with the local fractional derivative was presented [30]. The Yang-Laplace transform structured in 2011 [22] was suggested to deal with local fractional differential equations [31, 32]. The coupling method for variational iteration method within Yang-Laplace transform for solving the heat conduction in fractal media was proposed in [33].

In this paper, our aim is to use the Yang-Laplace transform to solve IVPs with local fractional derivative. The structure of the paper is as follows. In Section 2, some definitions and properties for the Yang-Laplace transform are given. Section 3 is devoted to the solutions for the homogeneous and nonhomogeneous local fractional differential equations. Finally, conclusions are presented in Section 4.

2. Yang-Laplace Transform

In this section we show some definitions and properties for the Yang-Laplace transform.

The local fractional integral operator is defined as [22, 23, 26–33]

\[
\left( a^I f (x) \right)^{(\alpha)} = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) (dt)^\alpha,
\]

(1)

\[
= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j) (\Delta t_j)^\alpha,
\]
where $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_0, \Delta t_1, \ldots, \Delta t_j, \ldots\}$, $[t_j, t_{j+1}]$, $j = 0, \ldots, N - 1$, $t_0 = a$, $t_N = b$, is a partition of the interval $[a, b]$.

As the inverse operator of (1), the local fractional derivative operator is given by [22, 23, 26–33]

$$f^{(\alpha)}(x_0) = \frac{d^{\alpha} f(x)}{dx^{\alpha}} \bigg|_{x = x_0} = \lim_{x \to x_0} \frac{\Delta^\alpha \left(f(x) - f(x_0)\right)}{(x - x_0)^{\alpha}}, \quad (2)$$

with $\Delta^\alpha(f(x) - f(x_0)) \equiv \Gamma(1 + \alpha) \Delta(f(x) - f(x_0))$.

The Yang-Laplace transform is expressed by [22, 31–33]

$${\mathcal L}_\alpha \{f(x)\} = f^{L\alpha}_s(s) = \frac{1}{\Gamma(1 + \alpha)} \int_0^\infty E_\alpha (-s^{\alpha} x^\alpha) f(x) \, dx s^\alpha,$$

where $s^\alpha = \beta^\alpha + s^\alpha \alpha$ and $\Re(s^\alpha) = \beta^\alpha$.

Some properties for Yang-Laplace transform are presented as follows [21, 22, 22–33]:

$$\mathcal L_\alpha \{a f(x) + b g(x)\} = a \mathcal L_\alpha \{f(x)\} + b \mathcal L_\alpha \{g(x)\}, \quad (5)$$

$$\mathcal L_\alpha \{f^{(n\alpha)}(x)\} = s^n \mathcal L_\alpha \{f(x)\} - \sum_{k=0}^{n-1} s^{(k-1)\alpha} f^{(n-k)\alpha}(0), \quad (6)$$

$$\lim_{x \to 0^+} f(x) = \lim_{s \to +\infty} s^\alpha F(s),$$

$$\lim_{x \to 0^-} f(x) = \lim_{s \to -\infty} s^\alpha F(s),$$

$$\mathcal L_\alpha \{f(ax)\} = \frac{1}{a^\alpha} f^{L\alpha}_s \left(\frac{s}{a}\right), \quad a > 0, \quad (9)$$

$$\mathcal L_\alpha \{x^{k\alpha} f(x)\} = (-1)^k \frac{d^{k\alpha} f^{L\alpha}_s(s)}{ds^{k\alpha}}, \quad (10)$$

$$\mathcal L_\alpha \{f(x - c)\} = f^{L\alpha}_s(s) \mathcal E_\alpha \{-c^\alpha s^\alpha\}, \quad (11)$$

$$\mathcal L_\alpha \{f(x) \mathcal E_\alpha \{a^\alpha x^\alpha\}\} = f^{L\alpha}_s(s - c), \quad (12)$$

$$\mathcal L_\alpha \{x^{k\alpha} \mathcal E_\alpha \{a^\alpha x^\alpha\}\} = \frac{\Gamma(1 + k\alpha)}{(s - c)^{(k+1)\alpha}}, \quad (13)$$

$$\mathcal L_\alpha \{\sin_a \{a^\alpha x^\alpha\}\} = \frac{s^\alpha}{s^{2\alpha} + c^{2\alpha}}, \quad (14)$$

$$\mathcal L_\alpha \{\cos_a \{a^\alpha x^\alpha\}\} = \frac{s^\alpha}{s^{2\alpha} + c^{2\alpha}}, \quad (15)$$

$$\mathcal L_\alpha \{x^\alpha\} = \frac{\Gamma(1 + \alpha)}{s^{\alpha + 1}}, \quad (16)$$

3. IVPs with Local Fractional Derivatives

In this section we handle the homogeneous and non-homogeneous IVPs with local fractional derivative.

3.1. Homogeneous IVPs with Local Fractional Derivative

**Example 1.** The homogeneous IVPs with local fractional derivative are expressed by

$$\frac{d^{2\alpha} y}{dx^{2\alpha}} + 2 y = 0. \quad (17)$$

The initial boundary conditions are presented as

$$y(0) = 1, \quad y^{(\alpha)}(0) = 0. \quad (18)$$

From (6) we have

$$\mathcal L_\alpha \{y^{(\alpha)}(x)\} = s^{2\alpha} \mathcal L_\alpha \{y(x)\} - y(0), \quad (19)$$

$$\mathcal L_\alpha \{y^{(2\alpha)}(x)\} = s^{2\alpha} \mathcal L_\alpha \{y(x)\} - s^{2\alpha} y(0) - f^{(\alpha)}(0). \quad (20)$$

Hence, making use of (19) and (20), (19) can be written as

$$s^{2\alpha} \mathcal L_\alpha \{y(x)\} - s^{2\alpha} y(0) - f^{(\alpha)}(0) - \{s^{2\alpha} \mathcal L_\alpha \{y(x)\} - y(0)\} + 2 \mathcal L_\alpha \{y(x)\} = 0. \quad (21)$$

Hence, we obtain

$$\mathcal L_\alpha \{y(x)\} = \frac{1}{s^{\alpha} + 2} y(0) = \frac{1}{s^{\alpha} + 2}. \quad (22)$$

So, making use of (13), we get the solution of (17):

$$y(x) = \mathcal E_\alpha \{-2s^\alpha\}. \quad (23)$$

The solution of (17) for $\alpha = \ln 2 / \ln 3$ is shown in Figure 1.

**Example 2.** Let us consider the homogeneous IVPs with local fractional derivative in the form

$$\frac{d^{2\alpha} y}{dx^{2\alpha}} - y = 0 \quad (24)$$

subject to initial boundary conditions

$$y(0) = 0, \quad y^{(\alpha)}(0) = 0, \quad (25)$$

$$y^{(2\alpha)}(0) = 0, \quad y^{(3\alpha)}(0) = 1. \quad (26)$$

From (6) we have

$$\mathcal L_\alpha \{y^{(4\alpha)}(x)\} = s^{4\alpha} \mathcal L_\alpha \{y(x)\} - s^{3\alpha} y(0) - s^{2\alpha} y^{(\alpha)}(0) - s^{\alpha} y^{(2\alpha)}(0), \quad (27)$$

so that

$$s^{2\alpha} \mathcal L_\alpha \{y(x)\} - s^{3\alpha} y(0) - s^{2\alpha} y^{(\alpha)}(0) - s^{\alpha} y^{(2\alpha)}(0) - f^{(3\alpha)}(0) - \mathcal L_\alpha \{y(x)\} = 0.$$
Hence, (27) can be written as
\[
\tilde{L}_\alpha \{ y(x) \} - 1 - \tilde{L}_\alpha \{ y(x) \} = 0,
\]
which leads to
\[
\tilde{L}_\alpha \{ y(x) \} = \frac{1}{s^{2\alpha} - 1}.
\]
Therefore, we get
\[
y(x) = \frac{1}{3} \left( \frac{1}{s^\alpha - 1} - \frac{1}{s^\alpha + 1} \right) - \frac{1}{2} \frac{1}{s^{2\alpha} + 1}.
\]
so that
\[
\tilde{L}_\alpha \{ y(x) \} = \frac{3}{4} \left( \frac{1}{s^\alpha - 1} - \frac{1}{s^\alpha + 1} \right) - \frac{1}{2} \frac{1}{s^{2\alpha} + 1}.
\]
So,
\[
y(x) = \frac{3}{4} E_\alpha (-x^\alpha) - \frac{3}{4} E_\alpha (x^\alpha) - \frac{1}{2} \sin\alpha (x^\alpha).\]

The exact solution of (31) for \( \alpha = \ln 2/\ln 3 \) is shown in Figure 3.

Example 4. The non-homogeneous IVPs with local fractional derivative are
\[
\frac{d^{2\alpha} y}{d^{2\alpha} x} + y = E_\alpha (x^\alpha).
\]
The initial boundary conditions are
\[
y(0) = 1, \quad y^{(\alpha)}(0) = 0.
\]
In view of (6), we give
\[
\tilde{L}_\alpha \{ y(x) \} = \frac{1}{(s^\alpha + 1) (s^{2\alpha} + 1)} + \frac{s^\alpha}{s^{2\alpha} + 1}.
\]
So, we obtain
\[
y(x) = \cos\alpha (x^\alpha) + \frac{1}{\Gamma (1 + \alpha)} \int_0^x E_\alpha (x - t)^\alpha \sin\alpha (t^\alpha) (dt)^\alpha
\]
\[
= \cos\alpha (x^\alpha) + \frac{1}{\Gamma (1 + \alpha)}
\]
\[
\times \int_0^x E_\alpha (t^\alpha) (\sin\alpha (x^\alpha) \cos\alpha (t^\alpha) - \cos\alpha (x^\alpha) \sin\alpha (t^\alpha)) (dt)^\alpha.
\]
$\cos_\alpha(x^\alpha)$

+ $\sin_\alpha(x^\alpha) \left\{ \frac{1}{\Gamma(1+\alpha)} \int_0^x E_\alpha(t^\alpha) \cos_\alpha(t^\alpha) \, dt^\alpha \right\}$

- $\cos_\alpha(x^\alpha) \left\{ \frac{1}{\Gamma(1+\alpha)} \int_0^x E_\alpha(t^\alpha) \sin_\alpha(t^\alpha) \, dt^\alpha \right\}$

= $\cos_\alpha(x^\alpha)$

+ $\sin_\alpha(x^\alpha) \left[ E_\alpha(x^\alpha) \left( \cos_\alpha(x^\alpha) + \sin(x^\alpha) \right) - 1 \right]$

- $\cos_\alpha(x^\alpha) \left[ E_\alpha(x^\alpha) \left( \sin_\alpha(x^\alpha) - \cos_\alpha(x^\alpha) \right) + 1 \right]$

= $\frac{1}{2} \left[ \cos_\alpha(x^\alpha) - \sin_\alpha(x^\alpha) + E_\alpha(x^\alpha) \right]$. (39)

The exact solution of (36) for $\alpha = \ln 2/\ln 3$ is shown in Figure 4.

4. Conclusions

In this work we have used the Yang-Laplace transform to handle the homogeneous and non-homogeneous IVPs with local fractional derivative. Some illustrative examples of approximate solutions for local fractional IVPs are discussed. The nondifferentiable solutions for fractal dimension $\alpha = \ln 2/\ln 3$ are shown graphically. The obtained results illustrate that the Yang-Laplace transform is an efficient mathematical tool to solve the homogeneous and non-homogeneous IVPs with local fractional derivative.

Conflict of Interests

The authors declare that there is no conflicts of interests regarding publication of this paper.

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