Oscillations of Difference Equations with Several Oscillating Coefficients

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Received 18 April 2014; Accepted 10 May 2014; Published 1 June 2014

Abstract and Applied Analysis

Volume 2014, Article ID 392097, 9 pages

http://dx.doi.org/10.1155/2014/392097

1. Introduction

In the present paper, we study the oscillatory behavior of the solutions of the difference equation

\[ \Delta x(n) + \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) = 0, \quad n \in \mathbb{N}_0, \]  

(E_R)

where \( \mathbb{N} \ni m \geq 2, p_i, 1 \leq i \leq m \) are real sequences with oscillating terms, and \( \{\tau_i(n)\}_{n \in \mathbb{N}_0}, 1 \leq i \leq m \) are sequences of integers such that

\[ \tau_i(n) \leq n - 1, \quad n \in \mathbb{N}_0, \quad \lim_{n \to \infty} \tau_i(n) = \infty, \quad 1 \leq i \leq m \]  

(1)

and the (dual) advanced difference equation

\[ \nabla x(n) - \sum_{i=1}^{m} p_i(n) x(\sigma_i(n)) = 0, \quad n \in \mathbb{N}, \]  

(E_A)

where \( \mathbb{N} \ni m \geq 2, p_i, 1 \leq i \leq m \) are real sequences with oscillating terms and \( \{\sigma_i(n)\}_{n \in \mathbb{N}}, 1 \leq i \leq m \), are sequences of integers such that

\[ \sigma_i(n) \geq n + 1, \quad n \in \mathbb{N}, \quad 1 \leq i \leq m. \]  

(2)

Here, \( \mathbb{N}_0 = \{0, 1, 2, \ldots\} \) and \( \mathbb{N} = \{1, 2, \ldots\} \). Also, as usual, \( \Delta \) denotes the forward difference operator \( \Delta x(n) = x(n + 1) - x(n) \) and \( \nabla \) denotes the backward difference operator \( \nabla x(n) = x(n) - x(n - 1) \).

Strong interest in \( (E_R) \) is motivated by the fact that it represents a discrete analogue of the differential equation (see [1] and the references cited therein)

\[ x'(t) + \sum_{i=1}^{m} p_i(t) x(\tau_i(t)) = 0, \quad t \geq 0, \]  

(3)

where, for every \( i \in \{1, \ldots, m\} \), \( p_i \) is an oscillating continuous real-valued function in the interval \( [0, \infty) \), and \( \tau_i \) is a continuous real-valued function on \( [0, \infty) \) such that

\[ \tau_i(t) \leq t, \quad t \geq 0, \quad \lim_{t \to \infty} \tau_i(t) = \infty, \]  

(4)

while, \( (E_A) \) represents a discrete analogue of the advanced differential equation (see [1] and the references cited therein)

\[ x'(t) - \sum_{i=1}^{m} p_i(t) x(\sigma_i(t)) = 0, \quad t \geq 1, \]  

(5)
where, for every $i \in \{1, \ldots, m\}$, $p_i$ is an oscillating continuous real-valued function in the interval $[1, \infty)$ and $\sigma_i$ is a continuous real-valued function on $[1, \infty)$ such that
\[ \sigma_i(t) \geq t, \quad t \geq 1. \tag{6} \]

By a solution of $(E_R)$, we mean a sequence of real numbers $\{x(n)\}_{n \geq -\omega}$ which satisfies $(E_R)$ for all $n \in \mathbb{N}_0$. Here,
\[ w = -\min_{n \geq 0} \{ \tau_j(n) : 1 \leq j \leq m \}. \tag{7} \]

It is clear that, for each choice of real numbers $c_{-\omega}, c_{-\omega+1}, \ldots, c_1, c_0$, there exists a unique solution $\{x(n)\}_{n \geq -\omega}$ of $(E_R)$ which satisfies the initial conditions $x(-\omega) = c_{-\omega}$, $x(-\omega+1) = c_{-\omega+1}, \ldots, x(-1) = c_1$, and $x(0) = c_0$.

A solution of the advanced difference equation $(E_A)$, we mean a sequence of real numbers $\{x(n)\}_{n \in \mathbb{N}}$ which satisfies $(E_A)$ for all $n \in \mathbb{N}$.

A solution $\{x(n)\}_{n \geq -\omega}$ of $(E_R)$ $(E_A)$ is called oscillatory, if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be nonoscillatory.

In the last few decades, the oscillatory behavior of all solutions of difference equations has been extensively studied when the coefficients $p_i(n)$ are nonnegative. See, for example, [2–20] and the references cited therein. However, for the general case when $p_i(n)$ are allowed to oscillate, it is difficult to study the oscillation of $(E_R)$ $(E_A)$, since the difference $\Delta x(n)[Vx(n)]$ of any nonoscillatory solution of $(E_R)$ $(E_A)$ is always oscillatory. Thus, a small number of papers are dealing with this case. See, for example, [1, 21–32] and the references cited therein.

For (3) and (5) with oscillating coefficients, Fukagai and Kusano [1] established the following theorems.

\textbf{Theorem 1} (see [1, Theorem 3(i)]). Assume (4) and that there is a continuous nondecreasing function $\tau^*(t)$ such that $\tau_1(t) \leq \tau^*(t) \leq t$ for $t \geq 0$, $1 \leq i \leq m$. Suppose moreover that there is a sequence $\{t_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} t_n = \infty$, the intervals $\bigcup_{n \in \mathbb{N}} [t_n, (\tau^*)^n(t_n)]$ are disjoint, and
\[ p_i(t) \geq 0 \quad \forall t \in \bigcup_{n \in \mathbb{N}} [(\tau^*)^n(t_n), t_n], \quad 1 \leq i \leq m. \tag{8} \]

If there is a constant $c$ such that
\[ \int_{\tau^*(t)}^t \frac{1}{\tau^*(s)} \sum_{i=1}^m p_i(s) ds > c > \frac{1}{e}, \quad \forall t \in \bigcup_{n \in \mathbb{N}} [(\tau^*)^{n-1}(t_n), t_n], \tag{9} \]
then all solutions of (3) oscillate.

\textbf{Theorem 2} (see [1, Theorem 3(ii)]). Assume (6) and that there is a continuous nondecreasing function $\sigma_*(t)$ such that $t \leq \sigma_*(t) \leq \sigma_i(t)$ for $t \geq 0$, $1 \leq i \leq m$. Suppose moreover that there is a sequence $\{t_n\}_{n \in \mathbb{N}}$ such that $\lim_{n \to \infty} t_n = \infty$, the intervals $\bigcup_{n \in \mathbb{N}} [t_n, (\sigma_*)^n(t_n)]$ are disjoint, and
\[ p_i(t) \geq 0 \quad \forall t \in \bigcup_{n \in \mathbb{N}} [(\sigma_*)^{n-1}(t_n), t_n], \quad 1 \leq i \leq m. \tag{10} \]

If there is a constant $c$ such that
\[ \int_{\sigma_*(t)}^t \frac{1}{\sigma_*(s)} \sum_{i=1}^m p_i(s) ds > c > \frac{1}{e}, \quad \forall t \in \bigcup_{n \in \mathbb{N}} [(\sigma_*)^{n-1}(t_n), t_n], \tag{11} \]
then all solutions of (5) oscillate.

For $(E_R)$ and $(E_A)$ with oscillating coefficients, recently, Bohner et al. [21, 23] established the following theorems.

\textbf{Theorem 3} (see [23, Theorem 2.4]). Assume (1) and that the sequences $\tau_i$ are increasing for all $i \in \{1, \ldots, m\}$. Suppose also that for each $i \in \{1, \ldots, m\}$ there exists a sequence $\{n_i(j)\}_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} n_i(j) = \infty$ and
\[ p_k(n) \geq 0 \quad \forall n \in \bigcup_{i=1}^m \left\{ \tau \left( \sigma(n_i(j)), n_i(j) \right) \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m, \tag{12} \]
where
\[ \tau(n) = \max_{1 \leq i \leq m} \tau_i(n), \quad n \in \mathbb{N}_0. \tag{13} \]
If, moreover,
\[ \limsup_{j \to \infty} \sum_{i=1}^m p_i(q) > 1, \tag{14} \]
then all solutions of $(E_R)$ oscillate.

\textbf{Theorem 4} (see [23, Theorem 3.4]). Assume (2) and that the sequences $\sigma_i$ are increasing for all $i \in \{1, \ldots, m\}$. Suppose also that for each $i \in \{1, \ldots, m\}$ there exists a sequence $\{n_i(j)\}_{j \in \mathbb{N}}$ such that $\lim_{j \to \infty} n_i(j) = \infty$ and
\[ p_k(n) \geq 0 \quad \forall n \in \bigcup_{i=1}^m \left\{ \sigma \left( \sigma(n_i(j)), n_i(j) \right) \cap \mathbb{N} \right\} \neq \emptyset, \quad 1 \leq k \leq m, \tag{15} \]
where
\[ \sigma(n) = \min_{1 \leq i \leq m} \sigma_i(n), \quad n \in \mathbb{N}. \tag{16} \]
If, moreover,
\[ \limsup_{j \to \infty} \sum_{i=1}^m p_i(q) > 1, \tag{17} \]
then all solutions of $(E_A)$ oscillate.
Theorem 5 (see [21, Theorem 2.1]). Assume (1) and that the sequences τᵢ are increasing for all i ∈ {1, ..., m}. Suppose also that for each i ∈ {1, ..., m} there exists a sequence {nᵢ(𝑗)} ∈ ℤ⁺ such that
\[
\lim_{j \to \infty} nᵢ(𝑗) = \infty,
\]
\[
p_k(n) \geq 0 \quad \forall n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j \in \mathbb{N}} [\tau_i(n_i(𝑗)), n_i(𝑗)] \cap \mathbb{N} \right\} \quad \neq \emptyset,
\]
\[
\sum_{n \to \infty} \left| x(n(𝑗)) \right| > 0
\]
then all solutions of (Eᵣ) oscillate.

Theorem 6 (see [21, Theorem 3.1]). Assume (2) and that the sequences σᵢ are increasing for all i ∈ {1, ..., m}. Suppose also that for each i ∈ {1, ..., m} there exists a sequence {nᵢ(𝑗)} ∈ ℤ⁺ such that
\[
\lim_{j \to \infty} nᵢ(𝑗) = \infty,
\]
\[
p_k(n) \geq 0 \quad \forall n \in \bigcap_{i=1}^{m} \left\{ \bigcup_{j \in \mathbb{N}} [n_i(𝑗), \sigma_i(n_i(𝑗))] \cap \mathbb{N} \right\} \quad \neq \emptyset,
\]
\[
\sum_{n \to \infty} \left| x(n(𝑗)) \right| > 0
\]
then all solutions of (Eᵣ) oscillate.

In the present paper, the authors study further (Eᵣ) [Eᵣ] and derive new sufficient oscillation conditions when neither (14) [(17)] nor (20) [(23)] is satisfied (cf. [6–8] and the references cited therein in the case of the equations (Eᵣ) [Eᵣ]) with nonnegative coefficients pᵢ, 1 ≤ i ≤ m). Examples illustrating the results are also given.

2. Retarded Equations

In this section, we present new sufficient conditions for the oscillation of all solutions of (Eᵣ) when the conditions (14) and (20) are not satisfied, under the assumption that the sequences τᵢ are increasing for all i ∈ {1, ..., m}. To that end, the following lemma provides a useful tool.

Lemma 7. Assume (1) holds, the sequences τᵢ are increasing for all i ∈ {1, ..., m} and (x(n))) ∈ ℤ⁺ is a nonoscillatory solution of (Eᵣ). Suppose also that for each i ∈ {1, ..., m} there exists a sequence {nᵢ(𝑗)} ∈ ℤ⁺, such that
\[
\lim_{j \to \infty} nᵢ(𝑗) = \infty,
\]
\[
\alpha := \liminf_{j \to \infty} \left| \sum_{q=\tau(n(𝑗))}^{n(𝑗)+1} \left| x(n(𝑗)) \right| \right| \geq \frac{\alpha^2}{4(1-\alpha)},
\]
\[\text{where } n(𝑗) = \min\{nᵢ(𝑗) : 1 \leq i \leq m\}.
\]

If 0 < α < 1, then
\[
\lim_{j \to \infty} \left( x(\tau(n(𝑗))) \right) \geq \frac{\alpha^2}{4(1-\alpha)}.
\]

Proof. Since the solution (x(n))) ∈ ℤ⁺ of (Eᵣ) is nonoscillatory, it is either eventually positive or eventually negative. As {x(n)}) ∈ ℤ⁺ is also a solution of (Eᵣ), we may restrict ourselves only to the case where x(n) > 0 eventually.

By (12), it is obvious that there exists j₀ ∈ ℤ⁺ such that
\[
p_k(n) \geq 0 \quad \forall n \in \bigcap_{i=1}^{m} [\tau(\tau(nᵢ(𝑗₀))), nᵢ(𝑗₀)] \cap \mathbb{N},
\]
\[1 \leq k \leq m,
\]
\[
x(\tau_k(n)) > 0 \quad \forall n \in \bigcap_{i=1}^{m} [\tau(\tau(nᵢ(𝑗₀))), nᵢ(𝑗₀)] \cap \mathbb{N},
\]
\[1 \leq k \leq m.
\]

Also, by (24) we have
\[
\sum_{i=1}^{m} \left| \sum_{q=\tau(nᵢ(𝑗₀))}^{nᵢ(𝑗₀)+1} p_i(q) \right| \geq \alpha - \epsilon,
\]
\[\text{where } \epsilon \text{ is an arbitrary real number with } 0 < \epsilon < \alpha.
\]

In view of (26) and (27), (Eᵣ) gives
\[
x(n+1) - x(n) = \sum_{i=1}^{m} \left( p_i(n) x(\tau_i(n)) \right) \leq 0,
\]
\[\text{for every } n \in \bigcap_{i=1}^{m} [\tau(\tau(nᵢ(𝑗₀))), nᵢ(𝑗₀)] \cap \mathbb{N}.
\]

Assume that 0 < α < 1, where α is defined by (24). From inequality (28), it is clear that there exists n* such that
\[
\sum_{i=1}^{m} \left| \sum_{q=\tau(nᵢ(𝑗₀))}^{nᵢ(𝑗₀)+1} p_i(q) \right| < \frac{\alpha - \epsilon}{2}, \quad \sum_{i=1}^{m} \left| \sum_{q=\tau(nᵢ(𝑗₀))}^{nᵢ(𝑗₀)+1} p_i(q) \right| \geq \frac{\alpha - \epsilon}{2}.
\]

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This is because in the case where \( p_i(q) < (\alpha - \varepsilon)/2 \), there exists \( n^*(j_0) > n(j_0) \) such that (30) is satisfied, while in the case where \( p_i(q) \geq (\alpha - \varepsilon)/2 \), then \( n^*(j_0) = n(j_0) \), and, therefore,

\[
\sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n^*(j_0)-1} p_i(q) = \sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)-1} p_i(q) \quad (31)
\]

(by which we mean) \( 0 < \frac{\alpha - \varepsilon}{2} \),

\[
\sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n^*(j_0)-1} p_i(q) > \sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)-1} p_i(q) < \frac{\alpha - \varepsilon}{2} \quad (32)
\]

That is, in both cases (30) is satisfied.

Now, we will show that \( \tau(n^*(j_0)) \leq n(j_0) - 1 \). Indeed, in the case where \( p_i(n(j_0)) \geq (\alpha - \varepsilon)/2 \), since \( n^*(j_0) = n(j_0) \), it is obvious that \( \tau(n^*(j_0)) = \tau(n(j_0)) \leq n(j_0) - 1 \). In the case where \( p_i(n(j_0)) < (\alpha - \varepsilon)/2 \), then \( n^*(j_0) > n(j_0) \). Assume, for the sake of contradiction, that \( \tau(n^*(j_0)) > n(j_0) - 1 \) and then

\[
\sum_{i=1}^{m} \sum_{q=\tau(n^*(j_0))}^{n^*(j_0)-1} p_i(q) = \sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)-1} p_i(q) > (\alpha - \varepsilon) - \frac{\alpha - \varepsilon}{2} = \frac{\alpha - \varepsilon}{2} \quad (33)
\]

which contradicts (28). Thus, in both cases, we have \( \tau(n^*(j_0)) \leq n(j_0) - 1 \). Therefore,

\[
\sum_{i=1}^{m} \sum_{q=\tau(n^*(j_0))}^{n^*(j_0)-1} p_i(q) = \sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)-1} p_i(q) - \sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n^*(j_0)-1} p_i(q) > (\alpha - \varepsilon) - \frac{\alpha - \varepsilon}{2} = \frac{\alpha - \varepsilon}{2} \quad (34)
\]

Summing up (E) from \( n(j_0) \) to \( n^*(j_0) \), and using the fact that the function \( x \) is decreasing and the function \( \tau \) (as defined by (13)) is increasing, we have

\[
x(n(j_0)) = x(n^*(j_0) + 1) + \sum_{i=1}^{m} p_i(q) x(\tau_i(q)) \geq x(n^*(j_0) + 1) + \sum_{i=1}^{m} p_i(q) x(\tau(q)) \quad (35)
\]

or

\[
x(n(j_0)) \geq x(n^*(j_0) + 1) + \tau(n^*(j_0)) \sum_{i=1}^{m} p_i(q) \quad (36)
\]

which, in view of (30), gives

\[
x(n(j_0)) \geq x(n^*(j_0) + 1) + \frac{\alpha - \varepsilon}{2} x(\tau(n^*(j_0))) \quad (37)
\]

Summing up (E) from \( \tau(n^*(j_0)) \) to \( n(j_0) - 1 \), and using the same arguments, we have

\[
x(\tau(n^*(j_0))) = x(n(j_0)) + \sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)-1} p_i(q) x(\tau_i(q)) \geq x(n(j_0)) + \sum_{i=1}^{m} \sum_{q=\tau(n^*(j_0))}^{n(j_0)-1} p_i(q) x(\tau(q)) \quad (38)
\]

or

\[
x(\tau(n^*(j_0))) \geq x(n(j_0)) + \tau(n^*(j_0)) \sum_{i=1}^{m} p_i(q) \quad (39)
\]

which, in view of (34), gives

\[
x(\tau(n^*(j_0))) > x(n(j_0)) + \frac{\alpha - \varepsilon}{2} x(\tau(n(j_0) - 1)) \quad (40)
\]

Combining inequalities (37) and (40), we obtain

\[
x(n(j_0)) > x(n^*(j_0) + 1) + \frac{\alpha - \varepsilon}{2} x(\tau(n^*(j_0))) \quad (41)
\]

or

\[
\left(1 - \frac{\alpha - \varepsilon}{2}\right) x(n(j_0)) > x(n^*(j_0) + 1) + \left(\frac{\alpha - \varepsilon}{2}\right)^2 x(\tau(n(j_0) - 1)) \quad (42)
\]

Thus

\[
x(n(j_0)) > \frac{(\alpha - \varepsilon)^2}{2[2 - (\alpha - \varepsilon)]} x(\tau(n(j_0) - 1)) \quad (43)
\]

In view of (43), inequality (42) gives

\[
x(n(j_0)) > \frac{(\alpha - \varepsilon)^2}{2[2 - (\alpha - \varepsilon)]} x(\tau(n^*(j_0))) \quad (44)
\]

or

\[
x(n(j_0)) > \frac{(\alpha - \varepsilon)^2}{2[2 - (\alpha - \varepsilon)]} x(\tau(n(j_0) - 1)) \quad (45)
\]

which, in view of (40) becomes

\[
x(n(j_0)) > \frac{(\alpha - \varepsilon)^2}{2[2 - (\alpha - \varepsilon)]} x(\tau(n(j_0) - 1)) \quad (45)
\]
Thus
\[
\frac{x(n(j_0))}{x(\tau(n(j_0)) - 1))} > (\alpha - \varepsilon)^2 \quad \frac{1 - (\alpha - \varepsilon)}{4}, \quad (46)
\]
or
\[
\frac{x(n(j_0) + 1)}{x(\tau(n(j_0)))} > (\alpha - \varepsilon)^2 \quad \frac{1 - (\alpha - \varepsilon)}{4}, \quad (47)
\]
Hence,
\[
\liminf_{j_0 \to -\infty} \frac{x(n(j_0) + 1)}{x(\tau(n(j_0)))} \geq (\alpha - \varepsilon)^2 \quad \frac{1 - (\alpha - \varepsilon)}{4}, \quad (48)
\]
which, for arbitrarily small values of \(\varepsilon\), implies (25).

The proof of the lemma is complete. \(\square\)

**Theorem 8.** Assume that (1) holds, the sequences \(\tau_i\) are increasing for all \(i \in \{1, \ldots, m\}\) and \(\tau\) is defined by (13). Suppose also that for each \(i \in \{1, \ldots, m\}\) there exists a sequence \(\{n_i(j)\}_{j \in \mathbb{N}}\) such that \(\lim_{j \to \infty} n_i(j) = \infty\), (12) and define \(\alpha\) by (24), where \(n(j) = \min\{n_i(j) : 1 \leq i \leq m\}\).

If \(0 < \alpha < 1\), and
\[
\limsup_{j \to \infty} \sum_{i=1}^{m} n^{(i)}(q) > 1 - \frac{\alpha^2}{4(1 - \alpha)} , \quad (49)
\]
then all solutions of (E\(_g\)) oscillate.

**Proof.** Assume, for the sake of contradiction, that \(\{x(n)\}_{n \geq w}\) is an eventually positive solution of (E\(_g\)). Then there exists \(j_0 \in \mathbb{N}\) such that
\[
p_k(n) \geq 0 \quad \forall n \in \bigcap_{k=1}^{m} \{ \tau(\tau(n_k(j_0)))\} \cap \mathbb{N},
\]
\[
1 \leq k \leq m, \quad (50)
\]
\[
x(n_k) > 0 \quad \forall n \in \bigcap_{k=1}^{m} \{ \tau(\tau(n_k(j_0)))\} \cap \mathbb{N},
\]
\[
1 \leq k \leq m.
\]

Therefore, by (E\(_g\)) we have
\[
x(n + 1) - x(n) = \sum_{i=1}^{m} p_i(n) x(\tau_i(n)) \leq 0, \quad (51)
\]
for every \(n \in \bigcap_{k=1}^{m} \{ \tau(\tau(n_k(j_0)))\} \cap \mathbb{N}\). This guarantees that the sequence \(x\) is decreasing on \(\bigcap_{k=1}^{m} \{ \tau(\tau(n_k(j_0)))\}\), \(n_k(j_0)\) and \(\mathbb{N}\).

Summing up (E\(_g\)) from \(\tau(n(j_0))\) to \(n(j_0)\), and using the fact that the function \(x\) is decreasing and the function \(\tau\) (as defined by (13)) is increasing, we obtain
\[
x(\tau(n(j_0))) = x(n(j_0)) + 1 + \sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q) \times x(\tau_i(q))
\]
\[
\geq x(n(j_0) + 1) + x(\tau(n(j_0)))
\]
\[
\times \sum_{i=1}^{m} \sum_{q=\tau(n(j_0))}^{n(j_0)} p_i(q) , \quad (52)
\]
Consequently,
\[
\sum_{i=1}^{m} \sum_{q=n(j_0)}^{n(j_0) + 1} p_i(q) \leq 1 - \frac{x(n(j_0) + 1)}{x(\tau(n(j_0)))} , \quad (53)
\]
which gives
\[
\limsup_{j \to -\infty} \sum_{i=1}^{m} \sum_{q=n(j_0)}^{n(j_0) + 1} p_i(q) \leq 1 - \limsup_{j \to -\infty} \frac{x(n(j_0) + 1)}{x(\tau(n(j_0)))} , \quad (54)
\]
Assume that \(0 < \alpha < 1\) and (49) holds. Then by Lemma 7, inequality (25) is fulfilled, and so (54) leads to
\[
\limsup_{j \to -\infty} \sum_{i=1}^{m} \sum_{q=n(j_0)}^{n(j_0) + 1} p_i(q) \leq 1 - \frac{\alpha^2}{4(1 - \alpha)} , \quad (55)
\]
which contradicts condition (49).

The proof of the theorem is complete. \(\square\)

### 3. Advanced Equations

Oscillation of all solutions of (E\(_A\)) is described by the theorem below. Note that the proof is an easy modification of the proof of Theorem 8 and hence is omitted.

**Theorem 9.** Assume (2) holds, the sequences \(\sigma_i\) are increasing for all \(i \in \{1, \ldots, m\}\) and \(\sigma\) is defined by (16). Suppose also that for each \(i \in \{1, \ldots, m\}\) there exists a sequence \(\{n_i(j)\}_{j \in \mathbb{N}}\) such that \(\lim_{j \to \infty} n_i(j) = \infty\), (15) and
\[
\alpha := \liminf_{j \to \infty} \sum_{i=1}^{m} \sigma(n^{(i)}(j)), \quad (56)
\]
where \(n(j) = \max\{n_i(j) : 1 \leq i \leq m\}\).

If \(0 < \alpha < 1\) and
\[
\limsup_{j \to \infty} \sum_{i=1}^{m} \sigma(n^{(i)}(j)) \leq 1 - \frac{\alpha^2}{4(1 - \alpha)} , \quad (57)
\]
then all solutions of (E\(_A\)) oscillate.

**Remark 10.** When \(\alpha \to 0\), then the conditions (49) and (57) reduce to the conditions (14) and (17), respectively. However the improvement is clear when \(\alpha \to 1/e\). The lower bound in (49) and (57) is \(0.946475699\). That is, when \(0 < \alpha < 1/e\), our conditions (49) and (57) essentially improve (14) and (17).
The significance of the results is illustrated in the following examples.

**Example 1.** Consider the retarded difference equation
\[
\Delta x(n) + p_1(n) x(n-2) + p_2(n) x(n-3) + p_3(n) x(n-4) = 0, \quad n \in \mathbb{N}_0,
\]
(58)
where \(p_1(n), p_2(n),\) and \(p_3(n)\) are oscillating coefficients, as shown in Figure 1.

In view of (13), it is obvious that \(\tau(n) = n - 2\). Observe that for \(n_1(j) = 20j + 9, \quad j \in \mathbb{N}\),
we have \(p_1(n) > 0\) for every \(n \in A\), where
\[
A = \bigcup_{j \in \mathbb{N}} [\tau(\tau(n_1(j))), n_1(j)] \cap \mathbb{N}
\]
(60)

For \(n_2(j) = 20j + 8, \quad j \in \mathbb{N}\), we have \(p_2(n) > 0\) for every \(n \in B\), where
\[
B = \bigcup_{j \in \mathbb{N}} [\tau(\tau(n_2(j))), n_2(j)] \cap \mathbb{N}
\]
(62)

and, for \(n_3(j) = 20j + 9, \quad j \in \mathbb{N}\), we have \(p_3(n) \geq 0\) for every \(n \in C\), where
\[
C = \bigcup_{j \in \mathbb{N}} [\tau(\tau(n_3(j))), n_3(j)] \cap \mathbb{N}
\]
(64)

Therefore,
\[
p_1(n) > 0, \quad p_2(n) > 0, \quad p_3(n) \geq 0, \quad \forall n \in A \cap B \cap C
\]

\[
= \bigcup_{j \in \mathbb{N}} [20j + 5, 20j + 8] \cap \mathbb{N} \neq 0.
\]

Observe that
\[
n(j) = \min \{n_i(j) : 1 \leq i \leq 3\} = 20j + 8, \quad j \in \mathbb{N}.
\]
(66)

Now,
\[
\alpha = \liminf_{j \to \infty} \frac{1}{m} \sum_{i=1}^{m} \sum_{q=\tau(n(j))}^{n(j)-1} p_1(q)
\]
\[
= \liminf_{j \to \infty} \left[ \sum_{q=20j+6}^{20j+7} p_1(q) + \sum_{q=20j+6}^{20j+7} p_2(q) + \sum_{q=20j+6}^{20j+7} p_3(q) \right]
\]
\[
= \frac{7}{100} + \frac{6}{100} + 2 \cdot \frac{2}{100} = 0.3,
\]
(67)

Observe that
\[
0.968 > 1 - \frac{\alpha^2}{4(1 - \alpha)} = 0.967857142;
\]
(68)

that is, condition (49) of Theorem 8 is satisfied and, therefore, all solutions of equation (58) oscillate.

On the other hand,
\[
0.968 < 1.
\]
(69)
Observe that $p_1(n) > 0$ for every $n \in A'$, where

$$B' = \bigcup_{j \in \mathbb{N}} [r_2(r_2(n_2(j))), n_2(j)] \cap \mathbb{N}$$

$$= \bigcup_{j \in \mathbb{N}} [20j + 2, 20j + 8] \cap \mathbb{N},$$

and $p_3(n) \geq 0$ for every $n \in C'$, where

$$C' = \bigcup_{j \in \mathbb{N}} [r_3(r_3(n_3(j))), n_3(j)] \cap \mathbb{N}$$

$$= \bigcup_{j \in \mathbb{N}} [20j + 1, 20j + 9] \cap \mathbb{N}.$$  

Therefore,

$$p_1(n) > 0, \quad p_2(n) > 0, \quad p_3(n) \geq 0$$

$$\forall n \in A' \cap B' \cap C'$$  

(72)

Also,

$$\liminf_{j \to \infty} 2 \sum_{i=1}^{2} \frac{n_1(j) - 1}{\sigma(n_1(j))} p_i(q)$$

$$= \liminf_{j \to \infty} \left[ \sum_{q=20j+7}^{20j+8} p_1(q) + \sum_{q=20j+5}^{20j+8} p_2(q) + \sum_{q=20j+5}^{20j+8} p_3(q) \right]$$

$$= 2 \cdot \frac{7}{100} + 3 \cdot \frac{6}{100} + 2 \cdot \frac{2}{100} = 0.36 < \frac{1}{e}.$$  

(73)

Therefore none of the conditions (14) and (20) is satisfied.

Example 2. Consider the advanced difference equation

$$\nabla x(n) - p_1(n) x(n+1) - p_2(n) x(n+2) = 0, \quad n \in \mathbb{N},$$

(74)

where $p_1(n)$ and $p_2(n)$ are oscillating coefficients, as shown in Figure 2.

In view of (16), it is obvious that $\sigma(n) = n + 1$. Observe that for

$$n_1(j) = 16j, \quad j \in \mathbb{N},$$

(75)

we have $p_1(n) \geq 0$ for every $n \in A$, where

$$A = \bigcup_{j \in \mathbb{N}} [n_1(j), \sigma(n_1(j))] \cap \mathbb{N} = \bigcup_{j \in \mathbb{N}} [16j, 16j + 2] \cap \mathbb{N}.$$  

(76)

Also, for

$$n_2(j) = 16j + 1, \quad j \in \mathbb{N},$$

(77)

we have $p_2(n) > 0$ for every $n \in B$, where

$$B = \bigcup_{j \in \mathbb{N}} [n_2(j), \sigma(n_2(j))] \cap \mathbb{N}$$

$$= \bigcup_{j \in \mathbb{N}} [16j + 1, 16j + 3] \cap \mathbb{N}.$$  

(78)

Therefore,

$$p_1(n) > 0, \quad p_2(n) > 0$$

$$\forall n \in A \cap B$$

(79)

Observing that

$$n(j) = \max \{n_i(j) : 1 \leq i \leq 2 \} = 16j + 1, \quad j \in \mathbb{N},$$

(80)

we have

$$\alpha = \liminf_{j \to \infty} 2 \sum_{i=1}^{2} \frac{\sigma(n(j))}{\sigma(n(j))} p_i(q)$$

$$= \liminf_{j \to \infty} \left[ \sum_{q=16j+1}^{16j+2} p_1(q) + \sum_{q=16j+2}^{16j+2} p_2(q) \right]$$

$$= \frac{15}{100} + \frac{12}{100} = 0.27.$$  

(81)

Also

$$\limsup_{j \to \infty} 2 \sum_{i=1}^{2} \frac{\sigma(n(j))}{\sigma(n(j))} p_i(q)$$

$$= \limsup_{j \to \infty} \left[ \sum_{q=16j+1}^{16j+2} p_1(q) + \sum_{q=16j+2}^{16j+2} p_2(q) \right]$$

(82)

$$= 2 \cdot \frac{15}{100} + \frac{12}{100} + \frac{556}{1000} = 0.976.$$
Observe that
\[ 0.976 > 1 - \frac{\alpha^2}{4(1 - \alpha)} = 0.975034246; \quad (83) \]
that is, condition (57) of Theorem 9 is satisfied and, therefore, all solutions of equation (74) oscillate.

On the other hand,
\[ 0.976 < 1. \quad (84) \]
Observe that \( p_1(n) \geq 0 \) for every \( n \in A' = A \) and \( p_2(n) \geq 0 \) for every \( n \in B' \), where
\[
\begin{align*}
B' &= \bigcup_{j \in \mathbb{N}} \left[ n_2(j), \sigma_2(n_2(j)) \right] \cap \mathbb{N} \\
&= \bigcup_{j \in \mathbb{N}} \left[ 16j + 1, 16j + 5 \right] \cap \mathbb{N}. \quad (85)
\end{align*}
\]
Therefore,
\[
\begin{align*}
p_1(n) > 0, \quad p_2(n) > 0 \\
\forall n \in A' \cap B' \quad (86)
\end{align*}
\]
Also,
\[
\liminf_{j \to \infty} \sum_{i=1}^{m} \sum_{q=i-\tau(j)+1}^{\tau(j)} p_i(q) \\
= \liminf_{j \to \infty} \left[ \sum_{q=16j+1}^{16j+1} p_1(q) + \sum_{q=16j+2}^{16j+3} p_2(q) \right] \quad (87)
\]
\[
= \frac{15}{100} + \frac{12}{100} + \frac{9}{100} = 0.36 < \frac{1}{\varepsilon}.
\]
Therefore none of the conditions (17) and (23) is satisfied.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

The authors would like to thank the referees for the constructive remarks which improved the presentation of the paper.

References


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