Oscillation Criteria of Even Order Delay Dynamic Equations with Nonlinearities Given by Riemann-Stieltjes Integrals

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Received 7 September 2013; Revised 29 January 2014; Accepted 4 February 2014; Published 16 March 2014

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We study the oscillatory properties of the following even order delay dynamic equations with nonlinearities given by Riemann-Stieltjes integrals:

\[ p(t)x^{\Delta^{n-1}}(t) + x^{\Delta^{n-1}}(t) = f(t, x(\delta(t))) + \int_{a}^{\sigma(b)} k(t, s)x(g(t, s))\text{sgn}(x(g(t, s)))\Delta \xi(s), \]

where \( n \geq 2 \), \( t \in [t_0, \infty) \), \( a, b \in \mathbb{T}_t \), \( \mathbb{T}_t \) is a time scale which is unbounded above, \( \sigma(a) = a \), \( \sigma(b) = b \), and \( \delta(t) \) is strictly increasing and satisfying \( 0 < \theta(a) < \alpha < \theta(b) \), \( k(t, s) \geq 0 \), \( g(t, s) \geq 0 \), and \( \text{sgn}(x(g(t, s))) = x(g(t, s)) / |x(g(t, s))| \).

1. Introduction

In this paper, we consider the following even order delay dynamic equations with nonlinearities of the form:

\[ \left( p(t)x^{\Delta^{n-1}}(t) + x^{\Delta^{n-1}}(t) \right)^{\Delta} + f(t, x(\delta(t))) + \int_{a}^{\sigma(b)} k(t, s)x(g(t, s))\text{sgn}(x(g(t, s)))\Delta \xi(s) = 0, \]

where \( n \geq 2 \), \( t \in [t_0, \infty) \), \( a, b \in \mathbb{T}_t \), and \( \mathbb{T}_t \) is a time scale which is unbounded above, and the following are satisfied:

\( H_1 \): \( a, b \in \mathbb{T}_t \) is another time scale, \( C_{rd}(\mathbb{T}_t, \mathbb{S}) \) denotes the collection of all functions \( f : \mathbb{T}_t \rightarrow \mathbb{S} \) which are right-dense continuous on \( \mathbb{T}_t \); \( H_2 \): \( p(t) \in C_{rd}([t_0, \infty) \cap \mathbb{T}_t, (0, \infty)) \), \( p^{\Delta}(t) \geq 0 \), \( P(t) := \int_{a}^{\sigma(b)} (\theta(s)^{\Delta}) \Delta s \leq 0 \), \( \lim_{s \rightarrow \infty} P(t) = \infty \), and \( \theta(t) \in C_{rd}([a, b]_{\mathbb{T}_t}, [0, \infty)) \) is a strictly increasing and satisfying \( 0 < \theta(a) < \alpha < \theta(b) \); \( H_3 \): \( \delta(t) \in C_{rd}([t_0, \infty) \cap \mathbb{T}_t, [0, \infty) \cap \mathbb{T}_t) \), \( \lim_{t \rightarrow \infty} \delta(t) = \infty \), and \( \delta^{\Delta}(t) > 0 \) is right-dense continuous on \( [t_0, \infty) \) and \( \bar{T} := \delta(T) \in \mathbb{T} \) is a time scale, \( \delta(\sigma(t)) = \sigma(\delta(t)) \) for all \( t \in [t_0, \infty) \), where \( \sigma(t) \) is the forward jump operator on \( [t_0, \infty) \).

By a solution of (1), we mean a function \( x(t) \) such that \( x(t) \in C_{rd}^{\infty}([t_\sigma, \infty) \cap \mathbb{T}_t) \), \( p(t)x^{\Delta^{n-1}}(t) + x^{\Delta^{n-1}}(t) \in C_{rd}^{\infty}([t_\sigma, \infty) \cap \mathbb{T}_t) \), and \( x(t) \) satisfies (1) for all \( t \geq t_\sigma \) and \( \delta(t) \leq t \) for \( t \in [t_0, \infty) \) and \( \lim_{t \rightarrow \infty} \delta(t) = \infty \). A nontrivial solution of (1) is called oscillatory if it has arbitrary large zeros; otherwise, it is called nonoscillatory.
Equation (1) is said to be oscillatory if all its solutions are oscillatory.

If \( n \) is a quotient of odd positive integers, \( f(t, u) = q(t)u^n \), \( k(t, s) = 0 \), then (1) simplifies to the even order dynamic equation

\[
\left( a(t) \left( x^{\Delta^{-1}}(t) \right)^n \right)^{\Delta} + q(t) (x(t))^n = 0. \tag{2}
\]

If \( f(t, u) = q(t)|u|^{n-1}u, \, \mathbb{T}_1 = \mathbb{N}, \, a = 1, \, b = k \) for \( n \in \mathbb{N}, \) and \( \xi(s) = s; \, \hat{\theta}(s) = \alpha_s, \) \( s = 1, 2, \ldots, k \) satisfying \( \alpha_1 > \alpha_2 > \cdots > \alpha_m > \alpha > \alpha_{m+1} > \cdots > \alpha_k; \) \( k(t, s) = p_j(t), \, s = 1, 2, \ldots, k; g(t, s) = \delta(t), \) then (1) reduces to

\[
\left( r(t) \left| x^{\Delta^{-1}}(t) \right|^{n-1} x^{\Delta^{-1}}(t) \right)^{\Delta} + p(t) |x(\delta(t))|^{\alpha-1} x(\delta(t)) + \sum_{j=1}^{m} p_j(t) |x(\delta(t))|^{\alpha-1} x(\delta(t)) = 0. \tag{3}
\]

Lemma 1 (Kiguradze’s Lemma [16, Theorem 5]). Let \( n \in \mathbb{N}, \) \( f \in C^n_{\text{rd}}(\mathbb{T}, \mathbb{R}), \) and \( \sup \mathbb{T} = \infty. \) Suppose that \( f \) is either positive or negative and \( f^{\Delta} \) is not identically zero and is either nonnegative or nonpositive on \( [t_0, \infty)_\mathbb{T} \) for some \( t_0 \in \mathbb{T}. \) Then there exist \( t_1 \in [t_0, \infty)_\mathbb{T}, \) \( m \in (0, n) \) such that

\[
(1)^{\alpha-m} f(t) f^{\Delta}(t) \geq 0 \text{ holds for all } t \in [t_1, \infty)_\mathbb{T} \text{ with }
\]

(i) \( f(t) f^{\Delta}(t) > 0 \) holds for all \( t \in [t_1, \infty)_\mathbb{T} \) and all \( j \in [0, m)_\mathbb{Z}; \)

(ii) \( (1)^{\alpha-m} f(t) f^{\Delta}(t) > 0 \) holds for all \( t \in [t_1, \infty)_\mathbb{T} \) and all \( j \in [m, n)_\mathbb{Z}. \)

In order to present the next lemma, we use the Taylor monomials (see [15, section 1.6]) \( h_{n}(t, s) \) which are defined recursively by

\[
h_{n+1}(t, s) = \int_s^t h_n(\tau, s) \Delta \tau, \quad t, s \in \mathbb{T}, n \geq 1, \tag{4}
\]

where \( h_0(t, s) = 1. \)

Lemma 2 (see [1]). Let \( sup \mathbb{T} = \infty \) and \( f \in C^n_{\text{rd}}(\mathbb{T}, \mathbb{R}) (n \geq 2). \) Moreover, suppose that Kiguradze’s theorem holds with \( m \in (0, n) \) and \( f^{\Delta}(t) \leq 0 \) on \( \mathbb{T}. \) Then there exists a sufficiently large \( t_1 \in \mathbb{T} \) such that

\[
f^{\Delta}(t) \geq h_{n-1}(t, t_1) f^{\Delta n}(t) \quad \forall t \in [t_1, \infty)_\mathbb{T}. \tag{5}
\]

Lemma 3 (see [1]). Assume that the conditions of Lemma 2 hold. Then

\[
f(t) \geq h_{m}(t, t_1) f^{\Delta m}(t) \quad \forall t \in [t_1, \infty)_\mathbb{T}. \tag{6}
\]

Lemma 4 (see [17]). Suppose that \( (H_2) \) holds. Let \( x : \mathbb{T} \to \mathbb{R}. \) If \( x^{\Delta} \) exists for all sufficiently large \( t \in \mathbb{T}, \) then \( x(\delta(t))^{\Delta} = x^{\Delta}(\delta(t))^{\Delta}(t) \) for all sufficiently large \( t \in \mathbb{T}. \)

Lemma 5 (see [15]). Assume that \( x(t) \) is \( \Delta \)-differentiable and eventually positive or eventually negative; then

\[
\left( x^{\Delta}(t) \right)^{\Delta} = \alpha \left\{ \int_0^1 \left[ (1-h) x(t) + h x(\sigma(t)) \right]^{\alpha-1} dh \right\} x^{\Delta}(t). \tag{7}
\]

Lemma 6 (see [18]). Suppose \( X \) and \( Y \) are nonnegative; then

\[
\gamma XY^{\gamma-1} - X^{\gamma} \leq (y-1)^{Y^{\gamma}}, \quad y > 1, \tag{8}
\]

where equality holds if and only if \( X = Y. \)

Lemma 7 (see [19]). Let \( u(t) \in C_{\text{rd}}([a, b], \mathbb{R}) \) and \( \eta(t) \in L^2_{\text{rd}}([a, b]) \) satisfy \( u(t) \geq 0 \) \((\neq 0), \) \( \eta(t) > 0 \) on \([a, b], \) and

\[
\int_a^b \eta(s) \Delta \xi(s) = 1. \tag{9}
\]

Then

\[
\int_a^b \eta(s) u(s) \Delta \xi(s) \geq \exp \left( \int_a^b \eta(s) \ln[ u(s) ] \Delta \xi(s) \right). \tag{10}
\]
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3. Main Results

Theorem 8. Assume that \((H_1)-(H_6)\) hold. If there exist a function \(\phi(t) \in C^1_{\mathcal{A}}([T_0, \infty), (0, \infty))\) and a function \(\eta(t) \in L^1_{\mathcal{A}}[a, b]\) satisfying \(\eta(s) > 0\) on \([a, b]_T\) and for all \(t_1 \in [T_0, \infty)_T\),
\[
\int_{a}^{\sigma(b)} \eta(s) \Delta \xi(s) = 1, \tag{11}
\]
\[
\int_{a}^{\sigma(b)} \eta(s) \theta(s) \Delta \xi(s) = \alpha, \tag{12}
\]
\[
\int_{t_1}^{\infty} \left( p^{-1}(\gamma) \right) \int_{t}^{\infty} \left( q(\zeta) + \int_{a}^{\sigma(b)} k(\zeta, s) \Delta \xi(s) \right) \Delta \gamma \right)^{1/\alpha} \Delta \gamma = \infty, \tag{13}
\]
where
\[
M(t) = \phi(t) q(t) + \phi(t) \exp \left( \int_{a}^{\sigma(b)} \eta(s) \ln \left( \eta^{-1}(s) k(t, s) \right) \Delta \xi(s) \right),
\]
\[
\phi^\Delta(t) = \max \left\{ \phi^\Delta(t), 0 \right\}, \tag{15}
\]
then (1) is oscillatory.

Proof. Suppose that (1) has a nonoscillatory solution \(x(t)\), then there exists \(T_0(T) \geq t_0\) such that \(x(t) \neq 0\) for all \(t \in [T_0, \infty)_T\). Without loss of generality, we assume that \(x(t) > 0\), \(x(\delta(t)) > 0\), and \(x(g(t, s)) > 0\) for \(t \in [T_0, \infty)_T\), \(s \in [a, b]_T\), because a similar analysis holds for \(x(t) < 0\), \(x(\delta(t)) < 0\), and \(x(g(t, s)) < 0\). From (1) and \((H_2), (H_3)\), we have
\[
\left( p(t) \left( x^\Delta(t) \right)^{\alpha} x^{\Delta^{-1}(t)} \right)^{\Delta} \leq 0, \quad t \in [T_0, \infty)_T. \tag{16}
\]
Therefore \(p(t)x^{\Delta^{-1}(t)} \leq x^\Delta(t)\) is a nonincreasing function and \(x^\Delta(t)\) is eventually of one sign on \([T_0, \infty)_T\).

We claim that
\[
x^{\Delta^{-1}}(t) > 0 \text{ or } x^{\Delta^{-1}}(t) = 0, \quad t \in [T_0, \infty)_T. \tag{17}
\]
Otherwise, if there exists a \(t_1(T) \geq T_0\) such that \(x^{\Delta^{-1}}(t) < 0\) for \(t \in [t_1, \infty)_T\), then from (16), for some positive constant \(K\), we have
\[
-p(t) \left( x^{\Delta^{-1}}(t) \right)^{\alpha} \leq -K, \quad t \in [t_1, \infty)_T. \tag{18}
\]
integrating the above inequality from \(t_1\) to \(t\), we have
\[
x^{\Delta^{-2}}(t) \leq x^{\Delta^{-2}}(t_1) - K^{1/\alpha} \left( P(t) - P(t_1) \right). \tag{20}
\]
Letting \(t \to \infty\), from \((H_3)\), we get \(\lim_{t \to \infty} x^{\Delta^{-2}}(t) = -\infty\). Analogously, we have \(\lim_{t \to \infty} x^{\Delta^{-2}}(t) = \lim_{t \to \infty} x^{\Delta^{-4}}(t) = \cdots = \lim_{t \to \infty} x^{\Delta}(t) = \lim_{t \to \infty} x(t) = -\infty\), which contradicts the fact that \(x(t) > 0\) for \([T_0, \infty)_T\). Thus, we have proved (17).

So from (16) and (17) and Lemma 5, we obtain
\[
\left( \left( p(t) \left( x^\Delta(t) \right)^{\alpha} \right)^{\Delta} \right) \leq 0,
\]
\[
x(t) \geq x(\delta(t)) \geq x(T_0) := c > 0,
\]
\[
(x(g(t), s)) \geq x(\delta(t)) \geq x(T_0) := c > 0, \tag{22}
\]
for any \(s \in [a, b]_T\).

For the case \(n = 2\), from Lemmas 1 and 2, we get \(m = 1\). For the case \(n \geq 4\), we claim that \(m = n - 1\). Otherwise, we obtain \(m \leq n - 3\). Therefore, it follows from (ii) of Lemma 1 that \(x^{\Delta^{-2}}(t) < 0, x^{\Delta^{-3}}(t) > 0\) on \([T_0, \infty)_T\). From (1), we have
\[
\left( p(t) \left( x^\Delta(t) \right)^{\alpha} \right) \leq -q(t) c^\alpha - \int_{a}^{\sigma(b)} k(t, s) c^\theta(s) \Delta \xi(s),
\]
\[
\leq -C_1 \left( q(t) + \int_{a}^{\sigma(b)} k(t, s) \right), \tag{23}
\]
where
\[
C_1 = \min \left\{ \frac{1}{c^\alpha}, \frac{1}{d(\theta(b))}, \quad 0 < c < 1, \right.
\]
\[
\min \left\{ \frac{1}{c^\alpha}, \frac{1}{d(\theta(b))}, \quad c \geq 1. \right. \tag{24}
\]
Integrating (23) from \( t \geq T_0 \) to \( v \geq t \) and from (17) we obtain
\[
p(t)\left(x^{\Delta^{-1}}(t)\right)^\alpha \geq p(v)\left(x^{\Delta^{-1}}(v)\right)^\alpha + c_1 \int_T^v q(\zeta) + \int_a^{\sigma(b)} k(\zeta, s) \Delta \xi(s) \Delta \zeta
\]
\[
\geq c_1 \int_T^v q(\zeta) + \int_a^{\sigma(b)} k(\zeta, s) \Delta \xi(s) \Delta \zeta.
\]
(25)

Letting \( v \to \infty \), we have
\[
x^{\Delta^{-1}}(t) \geq (c_1)^{1/\alpha}
\]
\[
\times \left[ p^{-1}(t) \int_T^\infty q(\zeta) + \int_a^{\sigma(b)} k(\zeta, s) \Delta \xi(s) \Delta \zeta \right]^{1/\alpha}.
\]
(26)

Integrating both sides of the last inequality from \( T_0 \) to \( t \) and from \( x^{\Delta^{-1}}(t) < 0 \), we get
\[
-x^{\Delta^{-1}}(T_0)
\geq x^{\Delta^{-1}}(t) - x^{\Delta^{-1}}(T_0)
\geq (c_1)^{1/\alpha}
\]
\[
\times \int_{T_0}^t \left[ p^{-1}(y) \int_y^\infty \left( q(\zeta) + \int_a^{\sigma(b)} k(\zeta, s) \Delta \xi(s) \Delta \zeta \right) \right]^{1/\alpha} \Delta y.
\]
(27)

Letting \( t \to \infty \), we get
\[
\int_{T_0}^\infty \left[ p^{-1}(y) \int_y^\infty \left( q(\zeta) + \int_a^{\sigma(b)} k(\zeta, s) \Delta \xi(s) \Delta \zeta \right) \right]^{1/\alpha} \Delta y
\leq -x^{\Delta^{-1}}(T_0) (c_1)^{-1/\alpha} < \infty,
\]
(28)

which contradicts (13). Thus, we have \( m = n - 1 \), so from Lemma 2, there exists a sufficiently large \( t_1 \in \mathbb{T} \) \( \geq T_0 \) such that
\[
x^{\Delta}(\delta(t)) \geq h_{n-2}(\delta(t), t_1) x^{\Delta^{-1}}(\delta(t)) > 0 \quad \forall t \in [t_1, \infty) \cap \mathbb{T}.
\]
(29)

Define
\[
w(t) = \frac{p(t) x^{\Delta^{-1}}(t)^\alpha}{x^{\alpha} \left(\delta(t)\right)^{1/\alpha}} \quad \text{for} \quad t \in [t_1, \infty) \cap \mathbb{T}.
\]
(30)

Obviously, \( w(t) > 0 \). From (1), \((H_3),(30)\) and \( x^{\Delta}(t) > 0 \), it follows that
\[
w^{\Delta}(t) = \frac{\phi(t)}{x^{\alpha}(\delta(t))} \left( p(t) x^{\Delta^{-1}}(t)^\alpha \right)^\Delta
+ \frac{\phi^{\Delta}(t) x^{\alpha} \left(\delta(t)\right) - \phi(t) \left(x^{\alpha} \left(\delta(t)\right)^{1/\alpha}\right)^\Delta}{x^{\alpha} \left(\delta(t)\right)^{1/\alpha}}
\times p \left(\sigma(t)\right) \left(x^{\Delta^{-1}} \left(\sigma(t)\right)^\alpha\right).
\]
(31)

Now we consider the following two cases.

In the first case \( \alpha \geq 1 \). By \((H_4),(30)\), (29)–(32), \( x^{\Delta}(t) > 0 \), \( x^{\Delta^{-1}}(t) \leq 0 \), and the fact that
\[
p^{1/(\alpha)}(t) x^{\Delta^{-1}}(t) \geq p(\sigma(t)) \left(x^{\Delta^{-1}}(\sigma(t))\right)^{1/\alpha},
\]
we obtain
\[
w^{\Delta}(t)
\leq -\phi(t) q(t) - \frac{\phi(t)}{x^{\alpha} \left(\delta(t)\right)^{1/\alpha}} \int_a^{\sigma(b)} k(t, s) x(g(t, s))(g(t, s)) \Delta \xi(s)
\]
\[
+ \frac{\phi^{\Delta}(t) \phi(\sigma(t)) - \phi(t) \left(x^{\alpha} \left(\delta(t)\right)^{1/\alpha}\right) \delta^{\Delta}(t)}{x^{\alpha} \left(\delta(t)\right)^{1/\alpha}} \omega(\sigma(t))
\]
\[
\leq -\phi(t) q(t) - \frac{\phi(t)}{x^{\alpha} \left(\delta(t)\right)^{1/\alpha}} \int_a^{\sigma(b)} k(t, s) x(g(t, s))(g(t, s)) \Delta \xi(s)
\]
\[
+ \frac{\phi^{\Delta}(t) \phi(\sigma(t)) - \phi(t) \left(x^{\alpha} \left(\delta(t)\right)^{1/\alpha}\right) \delta^{\Delta}(t)}{x^{\alpha} \left(\delta(t)\right)^{1/\alpha}} \omega(\sigma(t))
\]
(33)
\[ \begin{align*}
&= -\phi(t)q(t) - \frac{\phi(t)}{x^\alpha(\delta(t))} \int_a^{\sigma(b)} k(t, s) \left( x(g(t, s)) \right)^{\delta(s) \Delta \xi(s)} \\ &+ \frac{\phi^\Delta(t)}{\phi(\sigma(t))} \omega(\sigma(t)) - \frac{\alpha \phi(t) h_{n-2}(\delta(t), t_1) \delta^\Delta(t)}{\phi^{(\alpha+1)/\alpha}(\sigma(t))} \frac{\alpha^1}{\alpha}(t) \\ &\times \omega^{(\alpha+1)/\alpha}(\sigma(t)).
\end{align*} \]

(34)

In the second case \(0 < \alpha < 1\). By \((H_4)\) and Lemmas 4 and 5, we get

\[ \begin{align*}
&(x^\alpha(\delta(t)))^\Delta = \alpha \left\{ \int_0^1 [(1-h) x(\delta(t)) + h x(\sigma(\sigma(t)))]^{-1} dt \right\} \\ &\times (x(\delta(t)))^\Delta \\
&\geq \alpha(x(\delta(\sigma(t))))^{\frac{1}{\Delta}} x^\Delta(\delta(t)) \delta^\Delta(t).
\end{align*} \]

(35)

From \((H_4)\), (29)–(31), (35), \(x^\Delta(t) > 0\), \(x^\Delta(t) \leq 0\), and the fact that

\[ \frac{p^{1/\alpha}(t) x^{\Delta^{-1}}(t)}{x(\delta(\sigma(t)))} \geq \frac{p^{1/\alpha}(\sigma(t)) x^{\Delta^{-1}}(\sigma(t))}{x(\delta(\sigma(t)))} = \left( \frac{\omega(\sigma(t))}{\phi(\sigma(t))} \right)^{1/\alpha}, \]

(36)

we have

\[ w^\Delta(t) \]

\[ \leq -\phi(t)q(t) - \frac{\phi(t)}{x^\alpha(\delta(t))} \int_a^{\sigma(b)} k(t, s) \left( x(g(t, s)) \right)^{\delta(s) \Delta \xi(s)} \\ + \frac{\phi^\Delta(t)}{\phi(\sigma(t))} \omega(\sigma(t)) - \frac{\alpha \phi(t) h_{n-2}(\delta(t), t_1) \delta^\Delta(t)}{\phi^{(\alpha+1)/\alpha}(\sigma(t))} \frac{\alpha^1}{\alpha}(t) \\ \times \omega^{(\alpha+1)/\alpha}(\sigma(t)). \]

(37)

Therefore, for \(\alpha > 0\), from (34) and (37), we get

\[ w^\Delta(t) \]

\[ \leq -\phi(t)q(t) - \frac{\phi(t)}{x^\alpha(\delta(t))} \int_a^{\sigma(b)} k(t, s) \left( x(g(t, s)) \right)^{\delta(s) \Delta \xi(s)} \\ + \frac{\phi^\Delta(t)}{\phi(\sigma(t))} \omega(\sigma(t)) - \frac{\alpha \phi(t) h_{n-2}(\delta(t), t_1) \delta^\Delta(t)}{\phi^{(\alpha+1)/\alpha}(\sigma(t))} \frac{\alpha^1}{\alpha}(t) \\ \times \omega^{(\alpha+1)/\alpha}(\sigma(t)). \]

(38)

On the other hand, by (11) and (12), we have

\[ \int_a^{\sigma(b)} \eta(s) [(\theta(s) - \alpha] \Delta \xi(s) = 0. \]

(39)

Therefore, by \((H_4)\), Lemma 7, (39), and \(x^\Delta(t) > 0\), we have for \(t \in [t_1, \infty) \)

\[ \frac{1}{x^\alpha(\delta(t))} \int_a^{\sigma(b)} k(t, s) \left( x(g(t, s)) \right)^{\delta(s) \Delta \xi(s)} \]

\[ \leq \int_a^{\sigma(b)} k(t, s) \left( x(\delta(t)) \right)^{\delta(s) \Delta \xi(s)} \]

\[ \geq \exp \left( \int_a^{\sigma(b)} \eta(s) \eta^{-1}(s) k(t, s) \left[ x(\delta(t)) \right]^\delta(s)^{\alpha} \Delta \xi(s) \right) \]

\[ = \exp \left( \int_a^{\sigma(b)} \eta(s) \ln \left( \eta^{-1}(s) k(t, s) \left[ x(\delta(t)) \right]^\delta(s)^{\alpha} \Delta \xi(s) \right) \right) \]

\[ \times \exp \left( \ln(x(\delta(t))) \right) \int_a^{\sigma(b)} \eta(s) [\theta(s) - \alpha] \Delta \xi(s) \right) \]

\[ = \exp \left( \int_a^{\sigma(b)} \eta(s) \ln \left( \eta^{-1}(s) k(t, s) \Delta \xi(s) \right) \right). \]

(40)

Substituting (40) into (38) we obtain

\[ w^\Delta(t) \leq -M(t) + \frac{\phi^\Delta(t)}{\phi(\sigma(t))} \omega(\sigma(t)) - \frac{\alpha \phi(t) h_{n-2}(\delta(t), t_1) \delta^\Delta(t)}{\phi^{(\alpha+1)/\alpha}(\sigma(t))} \frac{\alpha^1}{\alpha}(t) \]

\[ \times \omega^{(\alpha+1)/\alpha}(\sigma(t)). \]

(41)

where \(M(t)\) and \(\phi^\Delta(t)\) are defined by (15).

Taking \(a = (\phi^\Delta(t))/(\phi(\sigma(t)))\), \(b = \alpha \phi(t) h_{n-2}(\delta(t), t_1) \delta^\Delta(t)/\phi^{(\alpha+1)/\alpha}(\sigma(t)) \), by Lemma 6 and (41), we obtain

\[ w^\Delta(t) \leq -M(t) + \frac{\phi^\Delta(t)}{\phi(\sigma(t))} \omega(\sigma(t)) - \frac{\alpha \phi(t) h_{n-2}(\delta(t), t_1) \delta^\Delta(t)}{\phi^{(\alpha+1)/\alpha}(\sigma(t))} \frac{\alpha^1}{\alpha}(t), \]

(42)
Integrating above inequality (42) from \( t_1 \) to \( t > t_1 \), we have

\[
\int_{t_1}^{t} \left( M(s) - \frac{1}{(\alpha + 1)^{\alpha + 1}} \right) \Delta s - \int_{t_1}^{t} \left( \frac{p(s)}{[\phi(s) h_{n-1}(\sigma(s), t_1) \Delta(\sigma(s), t_1)]^{\alpha}} \right) \Delta s \leq w(t_1) + \int_{t_1}^{t} M(s) \Delta s - w(t) \leq w(t_1) + \int_{t_0}^{t} M(s) \Delta s.
\]

(43)

Since \( w(t) > 0 \) for \( t > t_1 \), we have

\[
\int_{t_0}^{t} \left( M(s) - \frac{1}{(\alpha + 1)^{\alpha + 1}} \right) \Delta s - \int_{t_0}^{t} \left( \frac{p(s)}{[\phi(s) h_{n-1}(\sigma(s), t_1) \Delta(\sigma(s), t_1)]^{\alpha}} \right) \Delta s \leq w(t_1) + \int_{t_0}^{t_1} M(s) \Delta s - w(t) \leq w(t_1) + \int_{t_0}^{t_1} M(s) \Delta s.
\]

(44)

Taking upper limit of both sides of the inequality (44) as \( t \to \infty \), the right-hand side is always bounded, which contradicts condition (14). This completes the proof of Theorem 8.

**Theorem 9.** Assume that \((H_1)-(H_4)\) and (13) hold for all \( t_1 \in [t_0, \infty)_T \). If there exist a function \( \phi(t) \in C_T([t_0, \infty)_T, (0, \infty)) \) and a function \( \eta(t) \in L_T[a, b] \) such that \( \eta(s) > 0 \) on \([a, b]_T\) and (11), (12) hold,

\[
\lim_{t \to \infty} \int_{t_0}^{t} \left( \frac{\phi(t) q(t) - 1}{(\alpha + 1)^{\alpha + 1}} \right) \Delta t = \infty,
\]

(45)

where \( (\phi(t))_+ = \max\{\phi(t), 0\} \), then (1) is oscillatory.

**Proof.** The proof is in fact a simpler version of the proof of Theorem 8. We need only to note that \( (\phi(t)/x^{\alpha}(\sigma(t))) \int_a^{(b)} k(t, s)(x(g(t, s)))^{\theta(s)} \Delta(\sigma(s), t_1) \) is positive in (37).

**Theorem 10.** Assume that \((H_1)-(H_4)\) and (13) hold for all \( t_1 \in [t_0, \infty)_T \). If there exist a function \( \phi(t) \in C_T([t_0, \infty)_T, (0, \infty)) \) and a function \( \eta(t) \in L_T[a, b] \) such that \( \eta(s) > 0 \) on \([a, b]_T\), and (11) and (12) hold,

\[
\lim_{t \to \infty} \int_{t_0}^{t} \left( Q(t) - \frac{p(t)(\phi(t))_+}{[h_{n-1}(\sigma(t), t_1)]^{\alpha}} \right) \Delta t = \infty,
\]

(46)

where

\[
Q(t) = \phi(t) q(t)
\]

\[+ \phi(t) \exp \left( \int_a^{(b)} \eta(s) \ln(\eta^{-1}(s) k(t, s)) \Delta(\sigma(s), t_1) \right),
\]

(47)

then (1) is oscillatory.

**Proof.** We proceed as in the proof of Theorem 8 to have (30). From (1) and \((H_3), (30), \) Lemmas 3–5, \( (p(t)(x^{\Delta^{-1}}(t)))^{\Delta} \leq 0 \), \( x^{\Delta}(t) \leq 0 \), and \( x^{\Delta}(t) > 0 \), it follows that

\[
w(t) = \frac{\phi(t)}{x^{\alpha}(\sigma(t))} \left( p(t) \left( x^{\Delta^{-1}}(t) \right)^{\alpha} \right)
\]

\[+ \frac{\phi((t) x^{\alpha}(\sigma(t))) - \phi(t) (x^{\alpha}(\sigma(t)))^{\Delta}}{x^{\alpha}(\sigma(t)) x^{\alpha}(\sigma(t))} \times \int_{t_0}^{t} k(t, s)(x(g(t, s)))^{\theta(s)} \Delta(\sigma(s), t_1)
\]

\[+ \phi(t) \exp \left( \int_a^{(b)} \eta(s) \ln(\eta^{-1}(s) k(t, s)) \Delta(\sigma(s), t_1) \right),
\]

(48)
From (40), we obtain
\[ w^\Delta(t) \leq -\phi(t) q(t) - \phi(t) \exp \left( \int_a^b \eta(s) \ln \left( \eta^{-1}(s) k(t, s) \right) \Delta \xi(s) \right) \]
\[ + \frac{\left( \phi^\Delta(t) \right)_+ \rho(t)}{[h_{n-1}(\delta(t), t_1)]^{\alpha}} \]
\[ = -Q(t) + \frac{\left( \phi^\Delta(t) \right)_+ \rho(t)}{[h_{n-1}(\delta(t), t_1)]^{\alpha}}. \]  
(49)

Integrating this inequality from \( t_1 \) to \( t > t_1 \), we have
\[ w(t) \leq w(t_1) - \int_{t_1}^{t} \left( Q(s) - \frac{\left( \phi^\Delta(s) \right)_+ \rho(s)}{[h_{n-1}(\delta(s), t_1)]^{\alpha}} \right) \Delta s \]
\[ \leq w(t_1) + \int_{t_1}^{t} Q(s) \Delta s \]
\[ - \int_{t_1}^{t} \left( Q(s) - \frac{\left( \phi^\Delta(s) \right)_+ \rho(s)}{[h_{n-1}(\delta(s), t_1)]^{\alpha}} \right) \Delta s. \]  
(50)

Taking upper limit of both sides of the inequality (50) as \( t \rightarrow \infty \) and using (46) we obtain a contradiction to the fact that \( w(t) > 0 \) on \( [t_1, \infty) \). This completes the proof of Theorem 9. \( \square \)

**Theorem 11.** Assume that (H1)–(H6) and (13) hold for all \( t_1 \in [t_0, \infty) \). If there exist a function \( \phi(t) \in C^1_{\text{rd}}([t_0, \infty) \), \( (0, \infty) \) and a function \( \eta(t) \in L^1_{\text{rd}}[a,b] \) such that \( \eta(s) > 0 \) on \( [a, b] \), and (11), (12) hold,
\[ \lim_{s \rightarrow \infty} \int_{t_1}^{t} \left( \phi(t) q(t) - \frac{\rho(t) \left( \phi^\Delta(t) \right)_+}{[h_{n-1}(\delta(t), t_1)]^{\alpha}} \right) \Delta t = \infty, \]  
(51)
where \( \left( \phi^\Delta(t) \right)_+ = \max(\phi^\Delta(t), 0) \), then (1) is oscillatory.

**Proof.** The proof is in fact a simpler version of the proof of Theorem 10. We need only to note that \( \phi(t) \exp\left( \int_a^b \eta(s) \ln(\eta^{-1}(s) k(t, s) \Delta \xi(s) \right) \) is positive in (49). \( \square \)

**Remark 12.** If we let \( \alpha \) be the ratio of positive odd integers, \( f(t, x(\delta(t))) = q(t)(x(t))^a, k(x, s) = 0, \) and \( \delta(t) = t \) and use the convention that \( \ln 0 = -\infty, e^{-\infty} = 0 \), then Theorem 8 reduces to [9, Theorems 2.3] and Theorem 10 reduces to [9, Theorems 2.2].

**Conflicts of Interest**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgments**

This research was supported by the National Natural Science Foundations of China (nos. 11171178 and 61104136), the Natural Science Foundation of Shandong Province of China (no. ZR2010FQ002), and the Foundation of Qufu Normal University (no. XJ201014).

**References**


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