Research Article
Eigenvalue Based Approach for Global Consensus in Multiagent Systems with Nonlinear Dynamics

Wei Qian¹ and Lei Wang²

¹ School of Electrical Engineering and Automation, Henan Polytechnic University, Jiaozuo, Henan 454000, China
² School of Mathematics and Systems Science and LMIB, Beihang University, Beijing 100191, China

Correspondence should be addressed to Wei Qian; qwei@hpu.edu.cn
Received 25 March 2014; Accepted 4 May 2014; Published 14 May 2014

Academic Editor: Zidong Wang

Copyright © 2014 W.Qian and L.Wang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

This paper addresses the global consensus of nonlinear multiagent systems with asymmetrically coupled identical agents. By employing a Lyapunov function and graph theory, a sufficient condition is presented for the global exponential consensus of the multiagent system. The analytical result shows that, for a weakly connected communication graph, the algebraic connectivity of a redefined symmetric matrix associated with the directed graph is used to evaluate the global consensus of the multiagents system with nonlinear dynamics under the common linear consensus protocol. The presented condition is quite simple and easily verified, which can be effectively used to design consensus protocols of various weighted and directed communications. A numerical simulation is also given to show the effectiveness of the analytical result.

1. Introduction
Cooperative collective behavior in networked systems of autonomous agents has received a great deal of attention in the past decade. This is partially due to the growing interest in understanding group behaviors in nature, such as flocking and swarming, and also due to its broad applications of multiagent systems in many areas in sensor networks [1] and mobile robots [2], to name a few. In all cases, the aim is to control a group of agents connected through a communication network. Consensus problem that enables all agents to reach an agreement on a certain value of interest is a fundamental issue of controlling multiagent systems.

With the fundamental “nearest neighbor rule,” various models have been introduced to study the consensus problem. In [3], Vicsek et al. proposed a simple model for the phase transition of a group of autonomous agents and demonstrated by simulations the headings of agents converge to a common value. Jadbabaie et al. [4] provided a theoretical explanation for the collective behavior observed in [3] by using graph theory. In particular, Olfati-Saber and Murray presented a general framework of the consensus problem for multiagent systems with fixed or switching topologies and established the relationship between the algebraic connectivity and the convergence for a balanced directed network [5]. Moreover, some more relaxable conditions were presented for consensus with switching topology [6, 7]. Thereafter, many researchers extended these earlier works and proposed different protocols for agents that process second-order and higher-order dynamics in linear multiagent systems [8–18]. Recently, the consensus problem in multiagent systems with nonlinear dynamics has been investigated [19–25]. In particular, a passivity-based design tool was introduced to reach the velocity consensus among agents in [19]. A finite-time consensus algorithm was proposed to achieve consensus by using nonsmooth gradient flows [21]. By constructing a Lyapunov function, sufficient conditions in the form of generalized algebraic connectivity were established for reaching local and global consensus in [26].

In this paper, we present a sufficient condition of global consensus for multiagent systems with nonlinear dynamics by using a Lyapunov function and some graph theory techniques. A simple linear protocol is designed to generalize the tools developed for undirected graph in order to make
Abstract and Applied Analysis
them applicable to directed graphs. Similar to the idea in [26],
the algebraic connectivity of a redefined symmetric matrix
associated with the directed graph is used to evaluate the
consensus of the considered system. The obtained result is
quite simple and powerful, especially for those multiagent
systems containing spanning trees.

The rest of this paper is organized as follows. Section 2
presents some preliminaries. In Section 3, we derive a suffi-
cient condition to guarantee the global consensus under the
designed protocol. A numerical validation is given to show
the effectiveness of the presented result in Section 4. Section 5
concludes the investigation.

2. Preliminaries
Suppose that the multiagent system under consideration
consists of \( N \) identical agents, each of which evolves as a
nonlinear behavior and is described by the equation
\[
\dot{x}_i(t) = f(x_i(t)) + Bu_i(t), \quad 1 \leq i \leq N, \tag{1}
\]
where \( x_i(t) \in \mathbb{R}^n \) and \( u_i \in \mathbb{R}^p \) denote the state and the control
input of agent \( i \), respectively, \( f : \mathbb{R}^n \to \mathbb{R}^n \) is continuously
differentiable, describing the intrinsic nonlinear dynamics
of the isolated agent, and constant matrix \( B \in \mathbb{R}^{n \times p} \) is the
input matrix. The consensus protocol in multiagent system
(1) considered in [27, 28] is chosen as
\[
u_i(t) = -cK \sum_{j=1}^{N} l_{ij} x_j(t), \tag{2}
\]
where \( c \) is the overall coupling strength, \( K \in \mathbb{R}^{n \times n} \) is the
feedback gain matrix to be determined, and the commu-
nication topology among agents is represented by digraph
\( \mathcal{G} \) and described in a matrix form by the Laplacian
matrix \( L = (l_{ij}) \in \mathbb{R}^{N \times N} \). The Laplacian matrix of digraph \( \mathcal{G} \) is defined
as follows: if there is a directed connection from agent \( i \) to
agent \( j \) (\( j \neq i \)), then \( l_{ij} = 1 \); otherwise, \( l_{ij} = 0 \), and the
diagonal entries of matrix \( L \) are defined by \( l_{ii} = -\sum_{j=1,j \neq i}^{N} l_{ij} \).

It is noted that network (1) is said to be a multiagent system
with asymmetrically coupled identical agents if the Laplacian
matrix \( L \) is not assumed to be symmetric and irreducible.

Clearly, under the diffusive condition, if a consensus
is achieved, the solution \( s(t) \) of system (1) is expected to be
a possible trajectory of an individual agent satisfying
\( \dot{s}(t) = f(s(t)) \), where \( s(t) \) can be an equilibrium or a periodic
or chaotic orbit. Let
\[
X(t) = \left( x_1^T(t), x_2^T(t), \ldots, x_N^T(t) \right)^T \in \mathbb{R}^{nN}, \tag{3}
\]
\[
F(X) = \left( f_1^T(x_1), \ldots, f_N^T(x_N) \right)^T \in \mathbb{R}^{nN};
\]
then the multiagent system (1) with protocol (2) can be
written in a block form as
\[
\dot{X}(t) = F(X) - c(L \otimes BK)X(t), \tag{4}
\]
where the notation \( \otimes \) represents the Kronecker product. For
the subsequent analysis, we decompose \( L \) into \( L = L^+ + L^- \),
where symmetric matrix \( L^+ = (l_{ij}^+_{ij})_{N \times N} \) and antisymmetric
matrix \( L^- = (l_{ij}^-_{ij})_{N \times N} \) satisfy the zero-row-sum condition with
nondiagonal entries
\[
l_{ij}^+ = l_{ji}^+ = \frac{1}{2}(l_{ij} + l_{ji}),
\]
\[
l_{ij}^- = -l_{ji}^- = \frac{1}{2}(l_{ij} - l_{ji}). \tag{5}
\]

Throughout this paper, two important hypotheses are
introduced.

Hypothesis 1 (H1). Suppose that the directed graph \( \mathcal{G} \) is
weakly connected. That is, there exists a path between any
two agents in graph \( \mathcal{G} \) if one replaces all the directed edges of
graph \( \mathcal{G} \) with undirected edges.

Noting that all nondiagonal entries of matrix \( L^+ \) are
nonnegative, then \( L^+ \) can be regarded as the Laplacian matrix
associated with an undirected graph \( \mathcal{G} \). Moreover, under
(H1), \( L^+ \) has an eigenvalue 0 with multiplicity 1 and the
eigenvector \( \xi = (1/ \sqrt{N})(1, 1, \ldots, 1)^T \in \mathbb{R}^N \) and the algebraic
connectivity \( (\text{second smallest eigenvalue}) \), denoted by
\( \lambda_2(L^+) \), is positive [29].

Hypothesis 2 (H2). Suppose that there exist a symmetric
matrix \( P \in \mathbb{R}^{n \times n} > 0 \) and constants \( \alpha > 0 \) and \( \beta > 0 \) such
that, for all \( i, j = 1, \ldots, N \),
\[
\begin{align*}
(f(x_i) - f(x_j))^T PX_{ij}(t) - \alpha X_{ij}^T(t) PBB^T PX_{ij}(t) \\
+ \beta X_{ij}^T(t) X_{ij}(t) \leq 0,
\end{align*}
\]
where \( X_{ij}(t) = x_i(t) - x_j(t) \).

As shown in [30], (H2) is solvable for many coupled
limit-cycle or chaotic systems, such as Hindmarsh-Rose
neuron models, Lorenz systems, and coupled double-scrolls.
Considering a special case that \( f(x_i) = Ax_i \), then (6) reduces
to the following linear matrix inequality:
\[
A^T P + PA - 2\alpha PBB^T P + 2\beta I_N \leq 0, \tag{7}
\]
or equivalently written as
\[
A^T P + PA - 2\alpha PBB^T P < 0. \tag{8}
\]

Recalling the result in [27], a necessary and sufficient
condition for the existence of a \( P > 0 \) to (8) is that
\((A, B)\) is stabilizable. For nonlinear vector-valued function \( f \)
satisfying the Lipschitz condition, a feasible operation is to
convert inequality (6) into some linear matrix inequalities;
see (35)-(36) for details.

3. Main Results
For a multiagent system of coupled identical agents, a typical
approach that handles the consensus issue is to investigate
the stability of errors between the dynamics of an individual agent and the average dynamics [25]; that is,

\[ y_j(t) = x_j(t) - \frac{1}{N} \sum_{i=1}^{N} x_i(t), \]  

(9)

or equivalently written in a block form as

\[ Y(t) = (\Pi \otimes I_N) X(t), \]  

(10)

where \( Y(t) = (y_1^T(t), \ldots, y_N^T(t))^T \in \mathbb{R}^{nN} \) is the error vector, \( \Pi = I_N - \xi \xi^T \), and \( I_N \) is an \( N \times N \) identity matrix. Then we have the following results.

**Theorem 1.** Suppose that (H1) and (H2) hold. If

\[ \lambda_2(\tilde{L}) \geq \frac{\alpha}{c}, \]  

(11)

then the multiagent system (1) achieves global exponential consensus under the designed controller (2) and \( K = B^TP \), where \( \lambda_2(\tilde{L}) \) is the algebraic connectivity of matrix \( \tilde{L} = (\tilde{l}_{ij})_{N \times N} \) which is a zero-row-sum symmetric matrix with nondiagonal entries

\[ \tilde{l}_{ij} = l_{ij} - \frac{1}{N} (l_{ii} + l_{jj}), \quad \forall i \neq j. \]  

(12)

**Proof.** Solving (H2) to obtain a symmetric matrix \( P > 0 \), then one can choose the Lyapunov function as

\[ V(t) = \frac{1}{2} X^T(t) (\Pi \otimes P) X(t). \]  

(13)

Differentiating \( V(t) \) along the trajectory of system (4) gives

\[ \dot{V}(t) = X^T(t) (\Pi \otimes P) F(X) \]

\[ - cX^T(t) (\Pi \otimes P) (L \otimes BK) X(t) \]

\[ \triangleq S_1(t) - cS_2(t), \]  

(14)

where the term \( S_1(t) \) satisfies the following equality:

\[ S_1(t) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i^T(t) P \left[ f(x_i) - f(x_j) \right]. \]  

(15)

Renaming the sum \( S_1(t) \) \( j \) by \( i \) and vice versa yields

\[ S_1(t) = \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} x_i^T(t) P \left[ f(x_i) - f(x_j) \right]. \]  

(16)

Substituting (6) into (16) obtains

\[ S_1(t) \leq \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \alpha X_{ij}^T(t) \beta X_{ij}(t) \]

\[ - \frac{1}{2N} \sum_{i=1}^{N} \sum_{j=1}^{N} \beta X_{ij}^T(t) X_{ij}(t) \]

\[ = \alpha X^T(t) \left[ (\Pi \otimes (PB^TP)) X(t) \right. \]

\[ - \beta X^T(t) (\Pi \otimes I_n) X(t). \]

(17)

And the term \( S_2(t) \) satisfies the equation

\[ S_2(t) = X^T(t) \left( I_L \otimes (PBK) \right) X(t) \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} (I_{ik} - I_{jk}) x_i^T(t) PBK x_k(t). \]  

(18)

By the diffusive condition of the Laplacian matrix \( L \), we have

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} I_{ik} x_i^T(t) PBK x_k(t) = 0. \]  

(19)

Also by renaming \( j \) by \( i \) and vice versa, one obtains

\[ \sum_{i=1}^{N} \sum_{j=1}^{N} I_{ik} x_i^T(t) PBK x_k(t) \]

\[ = \sum_{i=1}^{N} \sum_{j=1}^{N} I_{ik} x_i^T(t) PBK x_k(t). \]  

(20)

Substituting (19)-(20) into \( S_2(t) \) we obtain

\[ S_2(t) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ I_{ik}^+ + I_{ik}^- \right] x_i^T(t) PBK x_k(t) \]

\[ \triangleq S_{21}(t) + S_{22}(t). \]  

(21)

Note that

\[ S_{21}(t) = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} \left[ I_{ik}^+ + I_{ik}^- \right] X_{ij}^T(t) PBK X_{ki}(t) \]

\[ = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} I_{ik}^+ X_{ij}^T(t) PBK X_{ki}(t) \]

\[ + \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{N} I_{ik}^- X_{ij}^T(t) PBK X_{ki}(t). \]  

(22)

Renaming the second sum in \( S_{21}(t) \) \( k \) by \( i \) and vice versa, we get

\[ S_{21}(t) = \frac{1}{N} \sum_{i=1}^{N} \sum_{k=1}^{N} I_{ik}^+ X_{ik}^T(t) PBK X_{ik}(t) = S_{21}(t). \]  

(23)
Thus,
\[ S_{21}(t) = \frac{1}{2} \sum_{i,j=1}^{N} l_{ij} x_{ij}(t) PBK X_{ij}(t). \]  

And, for the sum \( S_{22}(t) \), we have
\[ S_{22}(t) = \frac{1}{N} \sum_{i,j=1}^{N} \sum_{k=1}^{N} l_{ki} X_{ik}^T(t) PBK X_{kj}(t) \]
\[ = \frac{1}{N} \sum_{i,j=1}^{N} \sum_{k=1}^{N} l_{ki} X_{ik}^T(t) PBK X_{kj}(t) \]
\[ + \frac{1}{N} \sum_{i,j=1}^{N} \sum_{k=1}^{N} l_{kj} X_{kj}^T(t) PBK X_{ik}(t). \]  

Renaming the second sum in the above equation \( i \) by \( k \) and vice versa, we have
\[ S_{22}(t) = \frac{1}{N} \sum_{i,j=1}^{N} \sum_{k=1}^{N} l_{ij} X_{ij}^T(t) PBK X_{ij}(t) \]
\[ = \frac{1}{N} \sum_{i,j=1}^{N} \sum_{k=1}^{N} \left( l_{ik} + l_{jk} \right) X_{ij}^T(t) PBK X_{ij}(t). \]  

Then \( S_2(t) \) can be rewritten as
\[ S_2(t) = -\frac{1}{2} \sum_{i,j=1}^{N} \left( l_{ij} - \frac{1}{N} \left( l_{ii} + l_{jj} \right) \right) X_{ij}^T(t) PBK X_{ij}(t). \]  

Since \( X_{ii}(t) = 0 \), then
\[ S_2(t) = -\frac{1}{2} \sum_{i,j=1}^{N} \left| l_{ij} \right| X_{ij}^T(t) PBK X_{ij}(t) \]
\[ = \frac{1}{2} \sum_{i,j=1}^{N} \left| l_{ij} \right| X_{ij}^T(t) PBK X_{ij}(t) \]
\[ = \frac{1}{2} \sum_{i,j=1}^{N} \left| l_{ij} \right| X_{ij}^T(t) PBK X_{ij}(t) \]
\[ = X^T(t) \left( \bar{L} \odot \left( PBB^T P \right) \right) X(t). \]  

Substituting \( S_1(t) \) and \( S_2(t) \) into the derivative of \( V(t) \) gives
\[ \dot{V}(t) \leq X^T(t) \left[ \left( \Pi (\alpha I_N - c\bar{L}) \Pi \right) \odot \left( PBB^T P \right) \right] X(t) \]
\[ - \beta X^T(t) \left( \Pi \otimes I_n \right) X(t). \]  

Notice that \( \bar{L} \) is a symmetric matrix; then there exists an orthogonal matrix \( U \in R^{N \times N} \) such that \( U \bar{L} U^T = \Lambda = \text{diag}(\lambda_1(\bar{L}), \lambda_2(\bar{L}), \ldots, \lambda_N(\bar{L})) \) with \( \lambda_1(\bar{L}) = 0 \). Therefore, the derivative of \( V(t) \) satisfies
\[ \dot{V}(t) \leq Y^T(t) \left[ (\alpha I_N - c\Lambda) \odot \left( PBB^T P \right) \right] Y(t) \]
\[ - \beta X^T(t) \left( \Pi \otimes I_n \right) X(t) \]
\[ = \sum_{i=1}^{N} (\alpha - c\lambda_i(\bar{L})) y_i^T(t) PBB^T P y_i(t) \]
\[ - \beta X^T(t) \left( \Pi \otimes I_n \right) X(t) \]
\[ \leq \left( \alpha - c\lambda_2(\bar{L}) \right) \sum_{i=1}^{N} y_i^T(t) PBB^T P y_i(t) \]
\[ - \beta X^T(t) \left( \Pi \otimes I_n \right) X(t) \]
\[ \leq -\beta \sum_{i=1}^{N} (x_i(t) - \bar{x}(t))^T (x_i(t) - \bar{x}(t)), \]
where \( Y(t) = ((\Pi U) \otimes I_n) X(t) = (y_1^T(t), \ldots, y_N^T(t))^T \in R^{Nn} \) with \( y_i(t) = (\xi_i(t) \Pi \otimes I_n)(x_i(t) - \bar{x}(t)) = 0 \), \( \bar{x}(t) = (1/N) \sum_{i=1}^{N} x_i(t) = (\bar{\xi}^T \otimes I_n) x(t). \)

It is obvious that \( \dot{V}(t) \leq 0 \) and \( \dot{\bar{V}}(t) = 0 \) if only \( x_i(t) = \bar{x}(t) \). By LaSalle’s invariance principle, it follows that \( x_i(t) = \bar{x}(t) \) exponentially converges to zero for all \( i \) as time approaches infinity. Hence, the multiai system (4) achieves global exponential consensus under the designed protocol. The proof is thus completed.

**Remark 2.** In terms of graph, the symmetrization operation in (12) amounts to replacing the edge directed from agent \( i \) to agent \( j \) by an undirected edge corresponding to the coupling coefficients and node strengths of agents \( i \) and \( j \). As a result, the consensus issue of a nonlinear multiai system with a weighted directed communication graph \( \mathcal{G} \) can be evaluated by the algebraic connectivity of a symmetric matrix associated with asymmetric Laplacian matrix \( L \). From the above proof, the symmetrization operation in (12) can also deal with a time-varying Laplacian matrix.

**Remark 3.** Consider a special case of node balance; that is, \( \sum_{i=1}^{N} l_{ii} = 0 \). Then we have \( \bar{L} = L^+ \) and the following result.

**Corollary 4.** Suppose that (H1) and (H2) hold. If
\[ c \lambda_2(L^+) \geq \alpha, \]  
then multiai system (1) with node balance achieves global exponential consensus under the designed controller (2) and \( K = B^T P \).

It is noted that \( \lambda_2(L^+) > 0 \) under (H1). Then the consensus of system (1) can be guaranteed by a sufficiently large coupling strength \( c \).
4. An Example

In this section, a multiagent system consisting of five identical agents will be constructed to demonstrate the efficiency of the result proposed in the previous section. The agent dynamics can be described by

\[ f(x_i) = Ax_i + g(x_i), \tag{32} \]

where \( x_i = (x_{i1}, x_{i2}, x_{i3}, x_{i4})^T \), \( g(x_i) = (0, 0, 0, -(1/3) \sin(x_{i3}))^T \), and

\[
A = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-48.6 & -1.25 & 48.6 & 0 \\
0 & 0 & 0 & 10 \\
1.95 & 0 & -1.95 & 0
\end{pmatrix}. \tag{33}
\]

The input matrix is defined as \( B = (0, 20, 0, 0)^T \) and the Laplacian matrix is chosen as

\[
L = \begin{pmatrix}
2 & -2 & 0 & 0 & 0 \\
0 & 2 & -2 & 0 & 0 \\
0 & 0 & 4 & -4 & 0 \\
0 & 0 & 0 & 2 & -2 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{34}
\]

By calculation, we have \( \lambda_2(L) = 0.1728 \). Also notice that

\[
\begin{align*}
(g(x_i) - g(x_j))^T P X_{ij}(t) & \leq \left\| (g(x_i) - g(x_j))^T P Q^{-1/2} \right\| \cdot \left\| X_{ij}(t) Q^{1/2} \right\| \\
& \leq \gamma \left\| X_{ij}^T P Q^{-1/2} \right\| \cdot \left\| X_{ij}(t) Q^{1/2} \right\| \\
& \leq \frac{1}{2} X_{ij}^T(t) \left( \gamma^2 P Q^{-1} P + Q \right) X_{ij}(t),
\end{align*}
\]

where \( \gamma = 1/3 \) and \( Q \in \mathbb{R}^{n \times n} \) is an arbitrary positive definite matrix to be determined. Substituting (35) into (H2) gives the following linear matrix inequality:

\[ A^T P + PA - 2\alpha P B B^T P + \gamma^2 P Q^{-1} P + Q + 2\beta I_n \leq 0. \tag{36} \]

Solving (36) gives

\[
P = \begin{pmatrix}
18.2759 & 0.3099 & -10.8809 & 28.4167 \\
0.3099 & 0.0228 & -0.1837 & 0.4840 \\
-10.8809 & -0.1837 & 8.4982 & -14.1034 \\
28.4167 & 0.4840 & -14.1034 & 75.9731
\end{pmatrix}. \tag{37}
\]

and \( \alpha = 0.57 \) by setting \( \beta = 0.01 \) and \( Q = I_4 \). Then, according to Theorem 1, the global consensus is achieved if \( c > 3.3 \). The state responses are depicted in Figure 1 with \( c = 3.5 \).

5. Conclusions

This paper has investigated the global consensus problem of multiagent systems with asymmetrically coupled nonidentical agents. By employing a Lyapunov function, a criterion of global exponential consensus has been derived under the designed consensus protocol. The presented framework is quite simple and powerful, without assuming the symmetry or irreducibility of the Laplacian matrix, just evaluating the algebraic connectivity of a symmetric matrix derived by the Laplacian matrix. A numerical simulation is given to show the effectiveness of the analytical result.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgments**

This work is supported by the National Natural Science Foundations of China nos. 6104119, 61004106, and 61340015, the Science and Technology Innovation Talents Project of Henan University under Grant 3HASTIT044, the Young Core Instructor Foundation from Department of Education of Henan Province under Grant 2011GGJS-054, and the Fundamental Research Funds for the Central Universities nos. YWF-13-T-RSC-023 and YWF-13-A02-17.
References


Submit your manuscripts at http://www.hindawi.com