Research Article

Commutator Theorems for Fractional Integral Operators on Weighted Morrey Spaces

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Let L be the infinitesimal generator of an analytic semigroup on $L^2(\mathbb{R}^n)$ with Gaussian kernel bounds, and let $L^{-\alpha/2}$ be the fractional integrals of L for $0 < \alpha < n$. For any locally integrable function b , the commutators associated with $L^{-\alpha/2}$ are defined by $[b, L^{-\alpha/2}](f)(x) = b(x)L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x)$. When $b \in \text{BMO}(\omega)$ (weighted BMO space) or $b \in \text{BMO}$, the authors obtain the necessary and sufficient conditions for the boundedness of $[b, L^{-\alpha/2}]$ on weighted Morrey spaces, respectively.

1. Introduction and Main Results

Morrey [1] introduced the classical Morrey spaces to investigate the local behavior of solutions to second order elliptic partial differential equations. Chiarenza and Frasca [2] established the boundedness of the Hardy-Littlewood maximal operator, the fractional operator, and a singular integral operator on the Morrey spaces. On the other hand, Coifman and Fefferman [3] and Muckenhoupt [4] studied the boundedness of these operators on weighted L^p spaces. Motivated by these works, Komori and Shirai [5] introduced the following weighted Morrey space and investigated the boundedness of classical operators in harmonic analysis, that is, the Hardy-Littlewood maximal operator, a Calderón-Zygmund operator, the fractional integral operator, and so forth.

Let $1 \leq p < \infty$ and $0 \leq k < 1$. Then for two weights μ and ν , the weighted Morrey space is defined by

$$L^{p,k}(\mu, \nu) = \{f \in L^p_{\text{loc}}(\mu) : \|f\|_{L^{p,k}(\mu, \nu)} < \infty\}, \quad (1)$$

where

$$\|f\|_{L^{p,k}(\mu, \nu)} = \sup_Q \left(\frac{1}{\nu(Q)^k} \int_Q |f(x)|^p \mu(x) dx \right)^{1/p}, \quad (2)$$

and the supremum is taken over all balls Q in $\mathbb{R}^n$.

If $\mu = \nu$, then we have the classical Morrey space $L^{p,k}(\mu)$ with measure μ . When $k = 0$, then $L^{p,k}(\mu, \nu) = L^p(\mu)$ is the Lebesgue space with measure μ .

Suppose that L is a linear operator on $L^2(\mathbb{R}^n)$ which generates an analytic semigroup e^{-tL} with a kernel $p_t(x, y)$ satisfying a Gaussian upper bound, that is,

$$|p_t(x, y)| \leq \frac{C}{t^{n/2}} e^{-c(|x-y|^2/t)} \quad (3)$$

for $x, y \in \mathbb{R}^n$ and all $t > 0$.

For $0 < \alpha < n$, the fractional integral $L^{-\alpha/2}$ of the operator L is defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty e^{-tL}(f) \frac{dt}{t^{-\alpha/2+1}}(x). \quad (4)$$

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the classical fractional integral I_α which plays important roles in many fields. It is well known that I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$ for all $p > 1$, $1/q = 1/p - \alpha/n > 0$ and is also of weak type $(1, n/(n-\alpha))$.

Let $1 \leq p < \infty$ and ω be a weight function. A locally integrable function b is said to be in $BMO_p(\omega)$ if

$$\|b\|_{BMO_p(\omega)} = \sup_Q \left(\frac{1}{\omega(Q)} \int_Q |b(x) - b_Q|^p \omega(x)^{1-p} dx \right)^{1/p} \leq C < \infty, \quad (5)$$

where $b_Q = (1/|Q|) \int_Q b(y) dy$ and the supremum is taken over all balls $Q \in \mathbb{R}^n$.

Let $\omega \in A_1$; García-Cuerva [6] proved that the spaces $BMO_p(\omega)$ coincide, and the norms of $\|\cdot\|_{BMO_p(\omega)}$ are equivalent with respect to different values provided that $1 \leq p < \infty$.

Let b be a locally integrable function on $\mathbb{R}^n$; we consider the commutator $[b, L^{-\alpha/2}]$ defined by

$$[b, L^{-\alpha/2}](f)(x) = b(x) L^{-\alpha/2}(f)(x) - L^{-\alpha/2}(bf)(x). \quad (6)$$

Chanillo [7] proved that the commutator $[b, I_\alpha]$ of the multiplication operator by $b \in BMO$ is bounded on L^p for $1 < p < \infty$.

Duong and Yan [8] proved that $[b, L^{-\alpha/2}]$ is bounded from L^p to L^q , where $b \in BMO$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \alpha < n$.

Mo and Lu [9] proved that the multilinear commutator generated by $\vec{b}$ and $L^{-\alpha/2}$ is bounded from L^p to L^q , where $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < \alpha < 1$, $\vec{b} = (b_1, \dots, b_m)$, $b_i \in BMO$, for $i = 1, \dots, m$.

Lu et al. [10] proved that $[b, I_\alpha]$ is bounded from L^p to L^q if and only if $b \in BMO$.

Wang [11] proved that $[b, I_\alpha]$ is bounded from $L^{p,k}(\omega)$ to $L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)$, where $b \in BMO(\omega)$, $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < k < p/q$, and $\omega^{q/p} \in A_1$.

Inspired by the above results, we study the boundedness properties of the commutator $[b, L^{-\alpha/2}]$ on weighted Morrey spaces in this work. The main theorems are stated as follows.

Theorem 1. Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 \leq k < p/q$, $\omega^{q/p} \in A_1$, and $r_\omega > ((1-k)/(p-q-k))$, where r_ω denotes the critical index of ω for the reverse Hölder condition. Then the following conditions are equivalent.

- (a) $b \in BMO(\omega)$.
- (b) $[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega)$ to $L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)$.

In particular, when $k = 0$ in Theorem 1, we get the following.

Corollary 2. Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $\omega^{q/p} \in A_1$, and $r_\omega > q/p$, where r_ω denotes the critical index of ω for the reverse Hölder condition. Then the following conditions are equivalent.

- (a) $b \in BMO(\omega)$.
- (b) $[b, L^{-\alpha/2}]$ is bounded from $L^p(\omega)$ to $L^q(\omega^{1-(1-\alpha/n)q})$.

Furthermore, if $L = -\Delta$ is the Laplacian, then the following conditions are equivalent.

- (a') $b \in BMO(\omega)$.
- (b') $[b, I_\alpha]$ is bounded from $L^p(\omega)$ to $L^q(\omega^{1-(1-\alpha/n)q})$.

Theorem 3. Let $0 < \alpha < n$, $0 \leq k < p/q$, $1/q = 1/p - \alpha/n$, and $1 < r, s < \infty$ such that $1 < rs < p < n/\alpha$, $\omega^{rs} \in A_{p/rs, q/rs}$. Then the following conditions are equivalent.

- (a) $b \in BMO$.
- (b) $[b, L^{-\alpha/2}]$ is bounded from $L^{p,k}(\omega^p, \omega^q)$ to $L^{q,kq/p}(\omega^q)$.

In particular, when $k = 0$ in Theorem 3, we obtain the following.

Corollary 4. Let $0 < \alpha < n$, $1/q = 1/p - \alpha/n$, and $1 < r, s < \infty$ such that $1 < rs < p < n/\alpha$, $\omega^{rs} \in A_{p/rs, q/rs}$. Then the following conditions are equivalent.

- (a) $b \in BMO$.
- (b) $[b, L^{-\alpha/2}]$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$.

Furthermore, if $L = -\Delta$ is the Laplacian, then the following conditions are equivalent.

- (a') $b \in BMO$.
- (b') $[b, I_\alpha]$ is bounded from $L^p(\omega^p)$ to $L^q(\omega^q)$.

Remark 5. It is easy to see that our results extend the results in [7, 8, 10, 11] significantly.

2. Prerequisite Material

Let us first recall some definitions.

Definition 6. The Hardy-Littlewood maximal operator M is defined by

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)| dy. \quad (7)$$

Let ω be a weight. The weighted maximal operator M_ω is defined by

$$M_\omega(f)(x) = \sup_{x \in Q} \frac{1}{\omega(Q)} \int_Q |f(y)| \omega(y) dy. \quad (8)$$

A variant of this maximal operator will become the main tool in our scheme; for $1 < r < \infty$,

$$M_{r,\omega}(f)(x) = M_\omega(|f|^r)^{1/r}(x). \quad (9)$$

For $0 < \alpha < n$, $r \geq 1$, the fractional maximal operator $M_{\alpha,r}$ is defined by

$$M_{\alpha,r}(f)(x) = \sup_{x \in Q} \left(\frac{1}{|Q|^{1-\alpha r/n}} \int_Q |f(y)|^r dy \right)^{1/r}, \quad (10)$$

and the fractional weighted maximal operator $M_{\alpha,r,\omega}$ is defined by

$$M_{\alpha,r,\omega}(f)(x) = \sup_{x \in Q} \left(\frac{1}{\omega(Q)^{1-\alpha r/n}} \int_Q |f(y)|^r \omega(y) dy \right)^{1/r}. \quad (11)$$

For any $f \in L^p(\mathbb{R}^n)$, $p \geq 1$, the sharp maximal function $M_L^\sharp f$ associated with the generalized approximations to the identity $\{e^{-tL}\}_{t>0}$ is given by

$$M_L^\sharp f(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y) - e^{-t_Q L} f(y)| dy, \quad (12)$$

where $t_Q = r_Q^2$ and r_Q is the radius of the ball Q .

In the above definitions, the supremum is taken over all balls Q containing x .

Definition 7. A weight function ω is said to be in the Muckenhoupt class A_p with $1 < p < \infty$ if, for every ball Q in $\mathbb{R}^n$, there exists a positive constant C which is independent of Q such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \left(\frac{1}{|Q|} \int_Q \omega(x)^{-1/(p-1)} dx \right)^{p-1} \leq C. \quad (13)$$

When $p = 1$, $\omega \in A_1$, if

$$\left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \leq C \operatorname{ess\,inf}_{x \in Q} \omega(x). \quad (14)$$

When $p = \infty$, $\omega \in A_\infty$, if there exist positive constants δ and C such that, given a ball Q and a measurable subset E of Q ,

$$\frac{\omega(E)}{\omega(Q)} \leq C \left(\frac{|E|}{|Q|} \right)^\delta. \quad (15)$$

Definition 8. A weight function ω belongs to $A_{p,q}$ for $1 < p < q < \infty$ if, for every ball Q in $\mathbb{R}^n$, there exists a positive constant C which is independent of Q such that

$$\left(\frac{1}{|Q|} \int_Q \omega(x)^q dx \right)^{1/q} \left(\frac{1}{|Q|} \int_Q \omega(x)^{-p'} dx \right)^{1/p'} \leq C, \quad (16)$$

where p' denotes the conjugate exponent of $p > 1$, that is, $1/p + 1/p' = 1$.

Definition 9. A weight function ω belongs to the reverse Hölder class RH_r if there exist two constants $r > 1$ and $C > 0$ such that the reverse Hölder inequality

$$\left(\frac{1}{|Q|} \int_Q \omega(x)^r dx \right)^{1/r} \leq C \left(\frac{1}{|Q|} \int_Q \omega(x) dx \right) \quad (17)$$

holds for every ball Q in $\mathbb{R}^n$.

It is well known that if $\omega \in A_p$ with $1 \leq p < \infty$, then there exists $r > 1$ such that $\omega \in \text{RH}_r$. It follows from Hölder's

inequality that $\omega \in \text{RH}_r$ implies $\omega \in \text{RH}_s$ for all $1 < s < r$. Moreover, if $\omega \in \text{RH}_r$, $r > 1$, then we have $\omega \in \text{RH}_{r+\epsilon}$ for some $\epsilon > 0$. We thus write $r_\omega = \sup\{r > 1 : \omega \in \text{RH}_r\}$ to denote the critical index of ω for the reverse Hölder condition.

We will make use of the following lemmas. We first provide a weighted version of the local good λ inequality for $M_L^\sharp$ which allows us to obtain an analog of the classical Fefferman-Stein (see [3, 12]) estimate on weighted Morrey spaces.

Lemma 10 (see [13]). Assume that the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (3). Take $\lambda > 0$, $f \in L_0^1(\mathbb{R}^n)$, and a ball Q_0 such that there exists $x_0 \in Q_0$ with $Mf(x_0) \leq \lambda$. Then, for every $\omega \in A_\infty$, $0 < \eta < 1$, one can find $\gamma > 0$ (independent of λ , Q_0 , f , x_0) and constant C_ω , $r > 0$ (which only depend on ω), such that

$$\omega\{x \in Q_0 : Mf(x) > A\lambda, M_L^\sharp f(x) \leq \gamma\lambda\} \leq C_\omega \eta^r \omega(Q_0), \quad (18)$$

where $A > 1$ is a fixed constant which depends only on n .

As a consequence, by using the standard arguments, we have the following estimates.

For every $f \in L^{p,k}(\mu, \nu)$, with $1 < p < \infty$, if $\mu, \nu \in A_\infty$, $1 < p < \infty$, $0 \leq k < 1$, then

$$\|f\|_{L^{p,k}(\mu, \nu)} \leq \|Mf\|_{L^{p,k}(\mu, \nu)} \leq C \|M_L^\sharp f\|_{L^{p,k}(\mu, \nu)}. \quad (19)$$

In particular, when $\mu = \nu = \omega$ and $\omega \in A_\infty$, we have

$$\|f\|_{L^{p,k}(\omega)} \leq \|Mf\|_{L^{p,k}(\omega)} \leq C \|M_L^\sharp f\|_{L^{p,k}(\omega)}. \quad (20)$$

Lemma 11 (see [11]). Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $\omega^{q/p} \in A_1$. Then if $0 < k < p/q$ and $r_\omega > (1-k)/(p/q - k)$, one has

$$\|M_{\alpha,1} f\|_{L^{q,kq/p}(\omega^{q/p}, \omega)} \leq C \|f\|_{L^{p,k}(\omega)}. \quad (21)$$

The same conclusion still holds for I_α .

Lemma 12 (see [11]). Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $\omega^{q/p} \in A_1$. Then if $0 < k < p/q$, $1 < r < p$, and $r_\omega > (1-k)/(p/q - k)$, one has

$$\|M_{r,\omega} f\|_{L^{q,kq/p}(\omega^{q/p}, \omega)} \leq C \|f\|_{L^{p,k}(\omega)}. \quad (22)$$

Lemma 13 (see [11]). Consider $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < k < p/q$, and $\omega \in A_\infty$. For any $1 < r < p$, one has

$$\|M_{\alpha,r,\omega} f\|_{L^{q,kq/p}(\omega)} \leq C \|f\|_{L^{p,k}(\omega)}. \quad (23)$$

Remark 14. By checking the proof of Lemmas 11–13, we know that the three lemmas above still hold when $k = 0$.

Lemma 15. Let $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, and $\omega^{q/p} \in A_1$. Then if $0 \leq k < p/q$ and $r_\omega > (1-k)/(p/q - k)$, one has

$$\|L^{-\alpha/2} f\|_{L^{q,kq/p}(\omega^{q/p}, \omega)} \leq C \|f\|_{L^{p,k}(\omega)}. \quad (24)$$

Proof. Since the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (3), it is easy to check that $L^{-\alpha/2}(f)(x) \leq CI_\alpha(|f|)(x)$ for all $x \in \mathbb{R}^n$. Using the boundedness property of I_α on weighted Morrey space (see Lemma 11), we have

$$\|L^{-\alpha/2}f\|_{L^{q,kq/p}(\omega^{q/p}, \omega)} \leq \|I_\alpha f\|_{L^{q,kq/p}(\omega^{q/p}, \omega)} \leq C\|f\|_{L^{p,k}(\omega)}, \quad (25)$$

where $1 < p < n/\alpha$ and $1/q = 1/p - \alpha/n$. $\square$

Remark 16. Since I_α is of weak type $(1, n/(n - \alpha))$, from the proof of Lemma 15, we can get that $L^{-\alpha/2}$ is also of weak type $(1, n/(n - \alpha))$.

Lemma 17 (see [8, 14]). *Assume that the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (3). Then for $0 < \alpha < n$, the difference operator $L^{-\alpha/2} - e^{-tL}L^{-\alpha/2}$ has an associated kernel $K_{\alpha,t}(x, y)$ which satisfies*

$$|K_{\alpha,t}(x, y)| \leq \frac{C}{|x - y|^{n-\alpha}} \frac{t}{|x - y|^2}, \quad (26)$$

for some positive constant C .

Lemma 18. *Assume that the semigroup e^{-tL} has a kernel $p_t(x, y)$ which satisfies the upper bound (3), and let $b \in BMO(\omega)$, $\omega \in A_1$. Then, for every function $f \in L^p(\mathbb{R}^n)$, $p > 1$, and for all $x \in \mathbb{R}^n$, one has*

$$\begin{aligned} & \sup_{x \in Q} \frac{1}{|Q|} \int_Q |e^{-t_Q L}(b(y) - b_Q)f(y)| dy \\ & \leq C\|b\|_{BMO(\omega)} \omega(x) M_{r,\omega}(f)(x), \end{aligned} \quad (27)$$

where $t_Q = r_Q^2$, r_Q being the radius of Q .

Proof. For any $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$ and $x \in Q$. We have

$$\begin{aligned} & \frac{1}{|Q|} \int_Q |e^{-t_Q L}((b(\cdot) - b_Q)f)(y)| dy \\ & \leq \frac{1}{|Q|} \int_Q \int_{\mathbb{R}^n} |p_{t_Q}(y, z)| |(b(z) - b_Q)f(z)| dz dy \\ & \leq \frac{1}{|Q|} \int_Q \int_{2Q} |p_{t_Q}(y, z)| |(b(z) - b_Q)f(z)| dz dy \\ & \quad + \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} |p_{t_Q}(y, z)| \\ & \quad \times |(b(z) - b_Q)f(z)| dz dy \\ & \doteq \mathbf{M} + \mathbf{N}. \end{aligned} \quad (28)$$

For any $y \in Q$ and $z \in 2Q$. We have

$$|p_{t_Q}(y, z)| \leq Ct_Q^{-n/2} \leq C \frac{1}{|2Q|}. \quad (29)$$

Thus,

$$\begin{aligned} \mathbf{M} & \leq C \frac{1}{|2Q|} \int_{2Q} |(b(z) - b_Q)f(z)| dz \\ & \leq C \frac{1}{|2Q|} \left(\int_{2Q} |b(z) - b_Q|^{r'} \omega(z)^{1-r'} dz \right)^{1/r'} \\ & \quad \times \left(\int_{2Q} |f(z)|^r \omega(z) dz \right)^{1/r} \\ & \leq C\|b\|_{BMO(\omega)} \frac{\omega(2Q)}{|2Q|} \left(\frac{1}{\omega(2Q)} \int_{2Q} |f(z)|^r \omega(z) dz \right)^{1/r} \\ & \leq C\|b\|_{BMO(\omega)} \omega(x) M_{r,\omega}f(x). \end{aligned} \quad (30)$$

Moreover, for any $y \in Q$ and $z \in 2^{k+1}Q \setminus 2^k Q$, we have $|y - z| \geq 2^{k-1}r_Q$ and $|p_{t_Q}| \leq C(e^{-c2^{2(k-1)}} 2^{(k+1)n}/|2^{k+1}Q|)$:

$\mathbf{N}$

$$\begin{aligned} & = \frac{1}{|Q|} \int_Q \sum_{k=1}^{\infty} \int_{2^{k+1}Q \setminus 2^k Q} |p_{t_Q}(y, z)| |(b(z) - b_Q)f(z)| dz dy \\ & \leq C \sum_{k=1}^{\infty} \frac{e^{-c2^{2(k-1)}} 2^{(k+1)n}}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(b(z) - b_Q)f(z)| dz \\ & \leq C \sum_{k=1}^{\infty} \frac{e^{-c2^{2(k-1)}} 2^{(k+1)n}}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(b(z) - b_{2^{k+1}Q})f(z)| dz \\ & \quad + C \sum_{k=1}^{\infty} \frac{e^{-c2^{2(k-1)}} 2^{(k+1)n}}{|2^{k+1}Q|} \int_{2^{k+1}Q} |(b_{2^{k+1}Q} - b_Q)f(z)| dz \\ & \doteq \mathbf{N}_1 + \mathbf{N}_2. \end{aligned} \quad (31)$$

We estimate each term in turn. For $\mathbf{N}_1$, we apply Hölder's inequalities with exponent r . Then we have

$$\begin{aligned} \mathbf{N}_1 & \leq C \sum_{k=1}^{\infty} \frac{e^{-c2^{2(k-1)}} 2^{(k+1)n}}{|2^{k+1}Q|} \\ & \quad \times \left(\int_{2^{k+1}Q} |b(z) - b_{2^{k+1}Q}|^{r'} \omega(z)^{1-r'} dz \right)^{1/r'} \\ & \quad \times \left(\int_{2^{k+1}Q} |f(z)|^r \omega(z) dz \right)^{1/r} \\ & \leq C \sum_{k=1}^{\infty} 2^{(k+1)n} e^{-c2^{2(k-1)}} \|b\|_{BMO(\omega)} \frac{\omega(2^{k+1}Q)}{|2^{k+1}Q|} \\ & \quad \times \left(\frac{1}{\omega(2^{k+1}Q)} \int_{2^{k+1}Q} |f(z)|^r \omega(z) dz \right)^{1/r} \\ & \leq C\|b\|_{BMO(\omega)} \omega(x) M_{r,\omega}f(x). \end{aligned} \quad (32)$$

Since $\omega \in A_1$, then $|b_{2^{k+1}Q} - b_Q| \leq Ck\omega(x)\|b\|_{BMO(\omega)}$. This fact together with Hölder's inequality implies

$$\begin{aligned}
 N_2 &\leq C \sum_{k=1}^{\infty} 2^{(k+1)n} e^{-c2^{2(k-1)}} \frac{k}{|2^{k+1}Q|} \omega(x) \\
 &\quad \times \|b\|_{BMO(\omega)} \int_Q |f(z)| dz \\
 &\leq C \sum_{k=1}^{\infty} k 2^{(k+1)n} e^{-c2^{2(k-1)}} \omega(x) \\
 &\quad \times \|b\|_{BMO(\omega)} \left(\frac{1}{|2^{k+1}Q|} \int_{2^{k+1}Q} |f(z)|^r dz \right)^{1/r} \\
 &= C \sum_{k=1}^{\infty} k 2^{(k+1)n} e^{-c2^{2(k-1)}} \omega(x) \\
 &\quad \times \|b\|_{BMO(\omega)} \left(\frac{\omega(2^{k+1}Q)}{|2^{k+1}Q|} \frac{1}{\omega(2^{k+1}Q)} \int_{2^{k+1}Q} |f(z)|^r dz \right)^{1/r} \\
 &\leq C \sum_{k=1}^{\infty} k 2^{(k+1)n} e^{-c2^{2(k-1)}} \omega(x) \\
 &\quad \times \|b\|_{BMO(\omega)} \left(\frac{1}{\omega(2^{k+1}Q)} \int_{2^{k+1}Q} |f(z)|^r \omega(x) dz \right)^{1/r} \\
 &\leq C \|b\|_{BMO(\omega)} \omega(x) M_{r,\omega} f(x).
 \end{aligned} \tag{33}$$

Then Lemma 18 is proved. $\square$

Lemma 19. Let $0 < \alpha < n$, $\omega \in A_1$, and $b \in BMO(\omega)$. Then for all $r > 1$ and for all $x \in \mathbb{R}^n$, one has

$$\begin{aligned}
 M_L^\#([b, L^{-\alpha/2}] f)(x) \\
 \leq C \|b\|_{BMO(\omega)} \left(\omega(x) M_{r,\omega}(L^{-\alpha/2} f)(x) + \omega(x)^{1-\alpha/n} \right. \\
 \left. \times M_{\alpha,r,\omega}(f)(x) + \omega(x) M_{\alpha,1}(f)(x) \right).
 \end{aligned} \tag{34}$$

Proof. For any given $x \in \mathbb{R}^n$, fix a ball $Q = Q(x_0, r_Q)$ which contains x . We decompose $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}$. Observe that

$$\begin{aligned}
 [b, L^{-\alpha/2}] f(x) &= (b - b_Q) L^{-\alpha/2} f \\
 &\quad - L^{-\alpha/2} (b - b_Q) f_1 \\
 &\quad - L^{-\alpha/2} (b - b_Q) f_2, \\
 e^{-t_Q L}([b, L^{-\alpha/2}] f) &= e^{-t_Q L}[(b - b_Q) L^{-\alpha/2} f \\
 &\quad - L^{-\alpha/2} (b - b_Q) f_1 \\
 &\quad - L^{-\alpha/2} (b - b_Q) f_2].
 \end{aligned} \tag{35}$$

Then

$$\begin{aligned}
 &\frac{1}{|Q|} \int_Q |[b, L^{-\alpha/2}] f(y) - e^{-t_Q L}[b, L^{-\alpha/2}] f(y)| dy \\
 &\leq \frac{1}{|Q|} \int_Q |(b(y) - b_Q) L^{-\alpha/2} f(y)| dy \\
 &\quad + \frac{1}{|Q|} \int_Q |L^{-\alpha/2} (b - b_Q) f_1(y)| dy \\
 &\quad + \frac{1}{|Q|} \int_Q |e^{-t_Q L}((b - b_Q) L^{-\alpha/2} f)(y)| dy \\
 &\quad + \frac{1}{|Q|} \int_Q |e^{-t_Q L} L^{-\alpha/2} ((b - b_Q) f_1)(y)| dy \\
 &\quad + \frac{1}{|Q|} \int_Q |(L^{-\alpha/2} - e^{-t_Q L} L^{-\alpha/2})((b - b_Q) f_2)(y)| dy \\
 &\doteq \text{I} + \text{II} + \text{III} + \text{IV} + \text{V}.
 \end{aligned} \tag{36}$$

We estimate each term separately.

Since $\omega \in A_1$, then it follows from Hölder's inequality that

$$\begin{aligned}
 \text{I} &\leq \frac{1}{|Q|} \int_Q |(b(y) - b_Q) L^{-\alpha/2} f(y)| dy \\
 &\leq \frac{1}{|Q|} \left(\int_Q |b(y) - b_Q|^{r'} \omega(y)^{1-r'} dy \right)^{1/r'} \\
 &\quad \times \left(\int_Q |L^{-\alpha/2} f(y)|^r \omega(y) dy \right)^{1/r} \\
 &\leq C \|b\|_{BMO(\omega)} \frac{\omega(Q)}{|Q|} \left(\frac{1}{\omega(Q)} \int_Q |L^{-\alpha/2} f(y)|^r \omega(y) dy \right)^{1/r} \\
 &\leq C \|b\|_{BMO(\omega)} \omega(x) M_{r,\omega}(L^{-\alpha/2} f)(x).
 \end{aligned} \tag{37}$$

Applying Kolmogorov's inequality (see [15, page 485]), Hölder's inequality, and the continuity of $L^{-\alpha/2}$, we thus have

$$\begin{aligned}
 \text{II} &= \frac{1}{|Q|} \int_Q |L^{-\alpha/2} (b - b_Q) f_1(y)| dy \\
 &\leq C \frac{1}{|Q|^{1-\alpha/n}} \|L^{-\alpha/2} (b - b_Q) f_1\|_{L^{n/(n-\alpha), \infty}} \\
 &\leq C \frac{1}{|Q|^{1-\alpha/n}} \int_Q (b - b_Q) f_1(y) dy \\
 &\leq C \frac{1}{|Q|^{1-\alpha/n}} \left(\int_Q |b(y) - b_Q|^{r'} \omega(y)^{1-r'} dy \right)^{1/r'}
 \end{aligned}$$

$$\begin{aligned}
& \times \left(\int_Q |f(y)|^r \omega(y) dy \right)^{1/r} \\
& \leq C \|b\|_{\text{BMO}(\omega)} \frac{\omega(Q)^{1-\alpha/n}}{|Q|^{1-\alpha/n}} \\
& \quad \times \left(\frac{1}{\omega(Q)^{1-\alpha/n}} \int_Q |f(y)|^r \omega(y) dy \right)^{1/r} \\
& \leq C \|b\|_{\text{BMO}(\omega)} \omega(x)^{1-\alpha/n} M_{\alpha,r,\omega}(f)(x).
\end{aligned} \tag{38}$$

By Lemma 18, we have

$$\text{III} \leq C \|b\|_{\text{BMO}(\omega)} \omega(x) M_{r,\omega}(L^{-\alpha/2} f)(x). \tag{39}$$

For IV, using the estimate obtained in II, we get

$$\begin{aligned}
\text{IV} & \leq \frac{1}{|Q|} \int_Q \int_{2Q} |p_{t_Q}(y, z)| |b(z) - b_Q| |f(z)| dz dy \\
& \leq \frac{1}{|2Q|} \int_{2Q} |L^{-\alpha/2}((b(z) - b_Q)) f(z)| dz \\
& \leq C \|b\|_{\text{BMO}(\omega)} \omega(x)^{1-\alpha/n} M_{\alpha,r,\omega}(f)(x).
\end{aligned} \tag{40}$$

By virtue of Lemma 17, we have

$$\begin{aligned}
\text{V} & \leq \frac{1}{|Q|} \int_Q \int_{(2Q)^c} |K_{\alpha,t_Q}(y, z)| |b(z) - b_Q| |f(z)| dz dy \\
& \leq C \sum_{k=1}^{\infty} \int_{2^k r_Q \leq |x_0 - z| < 2^{k+1} r_Q} \frac{1}{|x_0 - z|^{n-\alpha}} \frac{t_Q}{|x_0 - z|^2} \\
& \quad \times |(b(z) - b_Q) f(z)| dz \\
& \leq C \sum_{k=1}^{\infty} 2^{-2k} \frac{1}{|2^{k+1} Q|^{1-\alpha/n}} \int_{2^{k+1} Q} |(b(z) - b_Q) f(z)| dz \\
& \leq C \sum_{k=1}^{\infty} 2^{-2k} \frac{1}{|2^{k+1} Q|^{1-\alpha/n}} \int_{2^{k+1} Q} |(b(z) - b_{2^{k+1} Q}) f(z)| dz \\
& \quad + C \sum_{k=1}^{\infty} 2^{-2k} (b_{2^{k+1} Q} - b_Q) \frac{1}{|2^{k+1} Q|^{1-\alpha/n}} \int_{2^{k+1} Q} |f(z)| dz \\
& \doteq \text{VI} + \text{VII}.
\end{aligned} \tag{41}$$

For VI, applying the same arguments as in II, we get

$$\begin{aligned}
\text{VI} & \leq C \|b\|_{\text{BMO}(\omega)} \sum_{k=1}^{\infty} 2^{-2k} \omega(x)^{1-\alpha/n} M_{\alpha,r,\omega}(f)(x) \\
& \leq C \|b\|_{\text{BMO}(\omega)} \omega(x)^{1-\alpha/n} M_{\alpha,r,\omega}(f)(x).
\end{aligned} \tag{42}$$

Since $\omega \in A_1$, then $|b_{2^{k+1} Q} - b_Q| \leq C k \omega(x) \|b\|_{\text{BMO}(\omega)}$. Thus,

$$\begin{aligned}
\text{VII} & \leq C \|b\|_{\text{BMO}(\omega)} \sum_{k=1}^{\infty} 2^{-2k} k \omega(x) M_{\alpha,1}(f)(x) \\
& \leq C \|b\|_{\text{BMO}(\omega)} \omega(x) M_{\alpha,1}(f)(x).
\end{aligned} \tag{43}$$

Then

$$\begin{aligned}
\text{V} & \leq C \|b\|_{\text{BMO}(\omega)} \\
& \quad \times \left(\omega(x)^{1-\alpha/n} M_{\alpha,r,\omega}(f)(x) + \omega(x) M_{\alpha,1}(f)(x) \right).
\end{aligned} \tag{44}$$

Combining the above estimates I–V, we get (34). The proof of Lemma 19 is complete. $\square$

3. Proofs of the Main Results

In this section we prove our main results. We start with the proof of Theorem 1.

Proof. (a) $\Rightarrow$ (b): Applying Lemmas 10 and 19, we get

$$\begin{aligned}
& \left\| [b, L^{-\alpha/2}] f \right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)} \\
& \leq \left\| M_L^{\sharp}([b, L^{-\alpha/2}] f) \right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)} \\
& \leq C \|b\|_{\text{BMO}(\omega)} \\
& \quad \times \left(\left\| \omega M_{r,\omega}(L^{-\alpha/2} f) \right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)} \right. \\
& \quad \left. + \left\| \omega^{1-\alpha/n} M_{\alpha,r,\omega} f \right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)} \right. \\
& \quad \left. + \left\| \omega M_{\alpha,1} f \right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)} \right) \\
& \leq C \|b\|_{\text{BMO}(\omega)} \\
& \quad \times \left(\left\| M_{r,\omega}(L^{-\alpha/2} f) \right\|_{L^{q,kq/p}(\omega^{q/p}, \omega)} \right. \\
& \quad \left. + \left\| M_{\alpha,r,\omega} f \right\|_{L^{q,kq/p}(\omega)} \right. \\
& \quad \left. + \left\| M_{\alpha,1} f \right\|_{L^{q,kq/p}(\omega^{q/p}, \omega)} \right).
\end{aligned} \tag{45}$$

Since $0 \leq k < p/q$, $\omega^{q/p} \in A_1$, and $r_\omega > (1-k)/(p/q-k)$, by making use of Lemmas 11–13, then we obtain

$$\begin{aligned}
& \left\| [b, L^{-\alpha/2}] f \right\|_{L^{q,kq/p}(\omega^{1-(1-\alpha/n)q}, \omega)} \\
& \leq C \|b\|_{\text{BMO}(\omega)} \left(\left\| L^{-\alpha/2} f \right\|_{L^{q,kq/p}(\omega^{q/p}, \omega)} + \|f\|_{L^{p,k}(\omega)} \right) \\
& \leq C \|b\|_{\text{BMO}(\omega)} \|f\|_{L^{p,k}(\omega)}.
\end{aligned} \tag{46}$$

The last inequality follows from Lemma 15. This completes the proof of (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a): Let $L = -\Delta$ be the Laplacian on $\mathbb{R}^n$; then $L^{-\alpha/2}$ is the classical fractional integral I_α . Choose $Z_0 \in \mathbb{R}^n$ so that $|Z_0| = 3$. For $x \in Q(Z_0, 2)$, $|x|^{-\alpha+n}$ can be written as the absolutely convergent Fourier series $|x|^{-\alpha+n} = \sum_{m \in \mathbb{Z}_n} a_m e^{i\langle \gamma_m, x \rangle}$

with $\sum_m |a_m| < \infty$ since $|x|^{-\alpha+n} \in C^\infty(Q(Z_0, 2))$. For any $x_0 \in \mathbb{R}^n$ and $\rho > 0$, let $Q = Q(x_0, \rho)$ and $Q_{Z_0} = Q(x_0 + Z_0\rho, \rho)$,

$$\begin{aligned} & \int_Q |b(x) - b_{Q_{Z_0}}| dx \\ &= \frac{1}{|Q_{Z_0}|} \int_Q \left| \int_{Q_{Z_0}} (b(x) - b(y)) dy \right| dx \\ &= \frac{1}{\rho^n} \int_Q s(x) \left(\int_{Q_{Z_0}} (b(x) - b(y)) |x - y|^{\alpha-n} |x - y|^{n-\alpha} dy \right) dx, \end{aligned} \quad (47)$$

where $s(x) = \text{sgn}(\int_{Q_{Z_0}} (b(x) - b(y)) dy)$. Fix $x \in Q$ and $y \in Q_{Z_0}$ and we have $(y - x)/\rho \in Q(Z_0, 2)$; hence, we have

$$\begin{aligned} & \frac{\rho^{-\alpha+n}}{\rho^n} \int_Q s(x) \left(\int_{Q_{Z_0}} (b(x) - b(y)) |x - y|^{\alpha-n} \left(\frac{|x - y|}{\rho} \right)^{n-\alpha} dy \right) dx \\ &= \rho^{-\alpha} \sum_{m \in \mathbb{Z}^n} a_m \\ & \quad \times \int_Q s(x) \left(\int_{Q_{Z_0}} (b(x) - b(y)) \right. \\ & \quad \left. \times |x - y|^{\alpha-n} e^{i\langle y_m, y/\rho \rangle} dy \right) e^{-i\langle y_m, x/\rho \rangle} dx \\ &\leq \rho^{-\alpha} \left| \sum_{m \in \mathbb{Z}^n} |a_m| \int_Q s(x) [b, L^{-\alpha/2}] \right. \\ & \quad \left. \times (\chi_{Q_{Z_0}} e^{i\langle y_m, \cdot/\rho \rangle}) \chi_Q(x) e^{-i\langle y_m, x/\rho \rangle} dx \right| \\ &\leq \rho^{-\alpha} \sum_{m \in \mathbb{Z}^n} |a_m| \| [b, L^{-\alpha/2}] (\chi_{Q_{Z_0}} e^{i\langle y_m, \cdot/\rho \rangle}) \|_{L^{q,0}(\omega^{1-(1-(\alpha/n)q})})} \\ & \quad \times \left(\int_Q \omega(x)^{q'[(1-(\alpha/n)-1/q)]} dx \right)^{1/q'} \\ &\leq C \rho^{-\alpha} \sum_{m \in \mathbb{Z}^n} |a_m| \| \chi_{Q_{Z_0}} \|_{L^{p,0}(\omega)} \left(\int_Q \omega(x)^{q'(1/q' - \alpha/n)} dx \right)^{1/q'} \\ &\leq C \omega(Q)^{1/p+1/q' - \alpha/n} \\ &= C \omega(Q). \end{aligned} \quad (48)$$

This implies $b \in \text{BMO}(\omega)$. Thus Theorem 1 is proved. $\square$

Similarly, to prove Theorem 3, we need the following lemmas.

Lemma 20. Let $0 < \alpha < n$, $1 < r, s < \infty$ such that $rs < p < n/\alpha$ and $b \in \text{BMO}$. Then for all $r > 1$ and for all $x \in \mathbb{R}^n$, one has

$$\begin{aligned} & M_L^\sharp([b, L^{-\alpha/2}] f)(x) \\ &\leq C \|b\|_{\text{BMO}} (M_r(L^{-\alpha/2} f)(x) + M_{\alpha,rs}(f)(x)), \end{aligned} \quad (49)$$

where $M_r(f)(x) = M(|f|^r)^{1/r}(x)$.

Proof. The case $0 < \alpha < 1$ was proved by Duong and Yan (see [8] for details). The general case $0 < \alpha < n$ follows by repeating the same steps as in Lemma 19. Since the main steps and the ideas are almost the same, here we omit the proof. $\square$

Lemma 21 (see [5]). If $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < k < p/q$, and $\omega \in A_{p,q}$, then the fractional maximal operator $M_{\alpha,1}$ is bounded from $L^{p,k}(\omega^p, \omega^q)$ to $L^{q,kq/p}(\omega^q)$.

Lemma 22 (see [5]). If $0 < \alpha < n$, $1 < p < n/\alpha$, $1/q = 1/p - \alpha/n$, $0 < k < p/q$, and $\omega \in A_{p,q}$, then the fractional maximal operator I_α is bounded from $L^{p,k}(\omega^p, \omega^q)$ to $L^{q,kq/p}(\omega^q)$.

Lemma 23 (see [5]). If $1 < p < \infty$, $0 < k < 1$, and $\omega \in A_p$, then M is bounded on $L^{p,k}(\omega)$.

Remark 24. By applying the same argument as in Lemma 15, we know that the conclusion in Lemma 22 still holds for $L^{-\alpha/2}$. We omit the proof here.

Remark 25. By checking the proof of Lemmas 21–23, we know that the three lemmas above still hold when $k = 0$.

Now we prove Theorem 3.

Proof. (a) $\Rightarrow$ (b): Since $\omega^{rs} \in A_{p/rs,q/rs}$, then we get $\omega^q \in A_{q/rs}$ and $\omega^p \in A_{p/rs}$. Applying Lemmas 10 and 20–23, we get

$$\begin{aligned} & \| [b, L^{-\alpha/2}] f \|_{L^{q,kq/p}(\omega^q)} \\ &\leq \| M_L^\sharp([b, L^{-\alpha/2}] f) \|_{L^{q,kq/p}(\omega^q)} \\ &\leq C \|b\|_{\text{BMO}} \left(\| M_r(L^{-\alpha/2} f) \|_{L^{q,kq/p}(\omega^q)} + \| M_{\alpha,rs}(f) \|_{L^{q,kq/p}(\omega^q)} \right) \\ &\leq C \|b\|_{\text{BMO}} \left(\| L^{-\alpha/2} f \|_{L^{q,kq/p}(\omega^q)} + \| f \|_{L^{p,k}(\omega^p, \omega^q)} \right) \\ &\leq C \|b\|_{\text{BMO}} \| f \|_{L^{p,k}(\omega^p, \omega^q)}. \end{aligned} \quad (50)$$

In the last inequality, we used the fact that $L^{-\alpha/2}$ is bounded from $L^{p,k}(\omega^p, \omega^q)$ to $L^{q,kq/p}(\omega^q)$ (see Remark 24).

(b) $\Rightarrow$ (a): Let $L = -\Delta$ be the Laplacian on $\mathbb{R}^n$; then $L^{-\alpha/2}$ is the classical fractional integral I_α . Let $k = 0$ and weight $\omega \equiv 1$, and then $L^{p,k}(\omega^p, \omega^q) = L^p$ and $L^{q,kq/p}(\omega^q) = L^q$. From [10] we deduce that the (L^p, L^q) boundedness of $[b, I_\alpha]$ implies $b \in \text{BMO}$. Thus Theorem 3 is proved. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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