Research Article

Some Paranormed Double Difference Sequence Spaces for Orlicz Functions and Bounded-Regular Matrices

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The aim of this paper is to introduce some new double difference sequence spaces with the help of the Musielak-Orlicz function \( \mathcal{F} = (F_{j,k}) \) and four-dimensional bounded-regular (shortly, RH-regular) matrices \( A = (a_{nmjk}) \). We also make an effort to study some topological properties and inclusion relations between these double difference sequence spaces.

1. Introduction, Notations, and Preliminaries

In [1], Hardy introduced the concept of regular convergence for double sequences. Some important work on double sequences is also found by Bromwich [2]. Later on, it was studied by various authors, for example, Móricz [3], Móricz and Rhoades [4], Başar and Sönalcan [5], Mursaleen and Mohiuddine [6–8], and many others. Mursaleen [9] has defined and characterized the notion of almost strong regularity of four-dimensional matrices and applied these matrices to establish a core theorem (also see [10, 11]). Altay and Başar [12] have recently introduced the double sequence spaces \( BS, BS(t), CS_p, CS_{bp}, CS_r, \) and \( BF \) consisting of all double series whose sequence of partial sums are in the spaces \( BS, BS(t), CS_p, CS_{bp}, CS_r, \) and \( BF \), respectively. Başar and Sever [13] extended the well-known space \( \ell_p \), from single sequence to double sequences, denoted by \( \mathcal{L}_{pq} \), and established its interesting properties. The authors of [14] defined some convex and paranormed sequences spaces and presented some interesting characterization. Most recently, Mohiuddine and Alotaibi [15] introduced some new double sequences spaces for \( \sigma \)-convergence of double sequences and invariant mean and also determined some inclusion results for these spaces. For more details on these concepts, one can be referred to [16–18].

The notion of difference sequence spaces was introduced by Kızmaz [19], who studied the difference sequence spaces \( \ell_\infty(\Delta), c(\Delta), \) and \( c_0(\Delta) \). The notion was further generalized by Et and Çölak [20] by introducing the spaces \( \ell_\infty(\Delta^r), c(\Delta^r), \) and \( c_0(\Delta^r) \).

Let \( w \) be the space of all complex or real sequences \( x = (x_k) \) and let \( r \) and \( s \) be two nonnegative integers. Then for \( Z = \ell_\infty^r, c, c_0 \), we have the following sequence spaces:

\[
Z(\Delta^r) = \{ x = (x_k) \in w : (\Delta^r_0 x_k) \in Z \},
\]

where \( \Delta^r_0 x_k = (\Delta^r_0 x_k) = (\Delta^r_1 x_k) = (\Delta^r_{-1} x_k - \Delta^r_{-1} x_{k+1}) \) and \( \Delta^0_0 x_k = x_k \) for all \( k \in \mathbb{N} \), which is equivalent to the following binomial representation:

\[
\Delta^r_0 x_k = \sum_{\nu=0}^{r} \binom{r}{\nu} x_{k+s\nu}
\]

We remark that for \( s = 1 \) and \( r = s = 1 \), we obtain the sequence spaces which were introduced and studied by Et and Çölak [20] and Kızmaz [19], respectively. For more details about sequence spaces see [21–27] and references therein.
An Orlicz function \( F : [0, \infty) \to [0, \infty) \) is continuous, nondecreasing, and convex such that \( F(0) = 0 \), \( F(x) > 0 \) for \( x > 0 \) and \( F(x) \to \infty \) as \( x \to \infty \). If convexity of Orlicz function is replaced by \( F(x + y) \leq F(x) + F(y) \), then this function is called modulus function. Lindenstrauss and Tzafriri [28] used the idea of Orlicz function to define the following sequence space:

\[
\ell_p = \left\{ x = (x_k) \in \mathbb{W} : \sum_{k=1}^{\infty} F\left( \frac{|x_k|}{\rho} \right) < \infty, \, \rho > 0 \right\},
\]

which is known as an Orlicz sequence space. The space \( \ell_p \) is a Banach space with the norm

\[
\|x\| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} F\left( \frac{|x_k|}{\rho} \right) \leq 1 \right\}. \tag{4}
\]

Also it was shown in [28] that every Orlicz sequence space \( \ell_p \) contains a subspace isomorphic to \( \ell_n \) (\( p \geq 1 \)). An Orlicz function \( F \) can always be represented in the following integral form:

\[
F(x) = \int_0^x \eta(t) \, dt, \tag{5}
\]

where \( \eta \) is known as the kernel of \( F \), is a right differentiable for \( t \geq 0 \), \( \eta(0) = 0 \), \( \eta(t) > 0 \), \( \eta \) is nondecreasing, and \( \eta(t) \to \infty \) as \( t \to \infty \).

A sequence \( \mathcal{F} = (F_k) \) of Orlicz functions is said to be a Musielak-Orlicz function (see [29, 30]). A sequence \( \mathcal{A} = (N_k) \) is defined by

\[
N_k(v) = \sup \{|v| - F_k(u) : u \geq 0\}, \quad k = 1, 2, \ldots, \tag{6}
\]

which is called the complementary function of a Musielak-Orlicz function \( \mathcal{F} \). For a given Musielak-Orlicz function \( \mathcal{F} \), the Musielak-Orlicz sequence space \( t_{\mathcal{F}} \) and its subspace \( h_{\mathcal{F}} \) are defined as follows:

\[
t_{\mathcal{F}} = \{ x \in \mathbb{W} : I_{\mathcal{F}}(cx) < \infty \text{ for some } c > 0 \},
\]

\[
h_{\mathcal{F}} = \{ x \in \mathbb{W} : I_{\mathcal{F}}(cx) < \infty \forall c > 0 \},
\]

where \( I_{\mathcal{F}} \) is a convex modular defined by

\[
I_{\mathcal{F}}(x) = \sum_{k=1}^{\infty} F_k(x_k), \quad x = (x_k) \in t_{\mathcal{F}}. \tag{8}
\]

We consider \( t_{\mathcal{F}} \) equipped with the Luxemburg norm

\[
\|x\| = \inf \left\{ k > 0 : I_{\mathcal{F}}\left( \frac{x}{k} \right) \leq 1 \right\}, \tag{9}
\]

or equipped with the Orlicz norm

\[
\|x\|^0 = \inf \left\{ \frac{1}{k} (1 + I_{\mathcal{F}}(kx)) : k > 0 \right\}. \tag{10}
\]

A Musielak-Orlicz function \( \mathcal{F} = (F_k) \) is said to satisfy \( \Delta_2 \)-condition if there exist constants \( a, K > 0 \) and a sequence \( c = (c_k)_{k=1}^{\infty} \in l^1_+ \) (the positive cone of \( l^1 \)) such that the inequality

\[
F_k(2u) \leq KF_k(u) + c_k \tag{11}
\]

holds for all \( k \in \mathbb{N} \) and \( u \in \mathbb{R}^+ \), whenever \( F_k(u) \leq a \).

A double sequence \( x = (x_{jk}) \) is said to be bounded if \( \|x\|_{\mathcal{F}, \mathcal{A}} = \sup_{j,k} |x_{jk}| < \infty \). We denote by \( \ell_{\mathcal{F}, \mathcal{A}} \) the space of all bounded double sequences.

By the convergence of double sequence \( x = (x_{jk}) \) we mean the convergence in the Pringsheim sense; that is, a double sequence \( x = (x_{jk}) \) is said to converge to the limit \( L \) in Pringsheim sense (denoted by, \( P\)-lim \( x \to L \)) provided that given \( \epsilon > 0 \) there exists \( n \in \mathbb{N} \) such that \( |x_{jk} - L| < \epsilon \) whenever \( j, k > n \) (see [31]). We will write more briefly as \( P\)-convergent. If, in addition, \( x \in \ell_{\mathcal{F}, \mathcal{A}} \), then \( x \) is said to be \( P \)-convergent to \( L \). We will denote the space of all \( P \)-convergent double sequences (or \( P \)-convergent) by \( \ell_{\mathcal{F}, \mathcal{A}}^2 \).

Let \( S \subseteq \mathbb{N} \times \mathbb{N} \) and let \( \epsilon > 0 \) be given. By \( \mathcal{X}_\mathcal{S}(\epsilon) \), we denote the characteristic function of the set \( \mathcal{S}(x; \epsilon) = \{(j, k) \in \mathbb{N} \times \mathbb{N} : |x_{jk}| \geq \epsilon \} \).

Let \( \mathcal{A} = (a_{nmkj}) \) be a four-dimensional infinite matrix of scalars. For all \( m, n \in \mathbb{N}_0 \), where \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \), the sum

\[
y_{mn} = \sum_{j,k=0}^{\infty} a_{nmkj} x_{jk} \tag{12}
\]

is called the \( A \)-means of the double sequence \( (x_{jk}) \). A double sequence \( (x_{jk}) \) is said to be \( A \)-summable to the limit \( L \) if the \( A \)-means exist for all \( m, n \) in the sense of Pringsheim's convergence:

\[
P_{-} \lim_{p\to \infty} \sum_{j,k=0}^{p} a_{nmkj} x_{jk} = y_{mn}, \quad P_{-} \lim_{n,m \to \infty} y_{mn} = L. \tag{13}
\]

A four-dimensional matrix \( A \) is said to be \( bounded \)-regular (or \( RH \)-regular) if every bounded \( P \)-convergent sequence is \( A \)-summable to the same limit and the \( A \)-means are also bounded.

The following is a four-dimensional analogue of the well-known Silverman-Toeplitz theorem [32].

**Theorem 1** (Robison [33] and Hamilton [34]). The four-dimensional matrix \( A \) is \( RH \)-regular if and only if

- \( RH_1 \) \( P\)-lim \( a_{nmkj} = 0 \) for each \( j \) and \( k \),
- \( RH_2 \) \( P\)-lim \( \sum_{j,k=0}^{\infty} |a_{nmkj}| = 1 \),
- \( RH_3 \) \( P\)-lim \( \sum_{j=0}^{\infty} |a_{nmkj}| = 0 \) for each \( k \),
- \( RH_4 \) \( P\)-lim \( \sum_{k=0}^{\infty} |a_{nmkj}| = 0 \) for each \( j \),
- \( RH_5 \) \( \sum_{j,k=0}^{\infty} |a_{nmkj}| < \infty \) for all \( n, m \in \mathbb{N}_0 \).
2. The Double Difference Sequence Spaces

In this section, we define some new paranormed double difference sequence spaces with the help of Musielak-Orlicz functions and four-dimensional bounded-regular matrices. Before proceeding further, first we recall the notion of paranormed space as follows.

A linear topological space $X$ over the real field $\mathbb{R}$ (the set of real numbers) is said to be a paranormed space if there is a subadditive function $g : X \rightarrow \mathbb{R}$ such that $g(\theta) = 0$, $g(x) = g(-x)$, and scalar multiplication is continuous; that is, $|\alpha_n - \alpha| \rightarrow 0$ and $g(x_n - x) \rightarrow 0$ imply $g(\alpha_n x_n - \alpha x) \rightarrow 0$ for all $\alpha$'s in $\mathbb{R}$ and all $x$'s in $X$, where $\theta$ is the zero vector in the linear space $X$.

The linear spaces $l_\infty(p)$, $c(p)$, and $c_0(p)$ were defined by Maddox [35] (also, see Simons [36]).

Let $\mathcal{F} = (F_{j,k})$ be a Musielak-Orlicz function; that is, $\mathcal{F}$ is a sequence of Orlicz functions and let $A = (a_{nmj,k})$ be a nonnegative four-dimensional bounded-regular matrix. Then, we define the following:

$$W_0^2(A, \mathcal{F}, u, \Delta'_s, p)$$

$$= \left\{ x = (x_{j,k}) : \right.$$  

$$P\lim_{n,m} \sum_{j,k=0}^{\infty} a_{nmj,k} \left[ F_{j,k} \left( u_{j,k} \Delta'_s x_{j,k} \right) \right]^{p_{j,k}} = 0 \right\},$$

$$W^2(A, \mathcal{F}, u, \Delta'_s, p)$$

$$= \left\{ x = (x_{j,k}) : \right.$$  

$$P\lim_{n,m} \sum_{j,k=0}^{\infty} a_{nmj,k} \left[ F_{j,k} \left( u_{j,k} \Delta'_s x_{j,k} - L \right) \right]^{p_{j,k}} = 0 \right\}$$  

(14)

where $p = (p_{j,k})$ is a double sequence of real numbers such that $p_{j,k} > 0$ for $j,k$, $\sup_{j,k} p_{j,k} = H < \infty$, and $u = (u_{j,k})$ is a double sequence of strictly positive real numbers.

Remark 2. If we take $\mathcal{F}(x) = x$ in $W_0^2(A, \mathcal{F}, u, \Delta'_s, p)$ and $W^2(A, \mathcal{F}, u, \Delta'_s, p)$, then we have the following spaces:

$$W_0^2(A, u, \Delta'_s, p)$$

$$= \left\{ x = (x_{j,k}) : \right.$$  

$$P\lim_{n,m} \sum_{j,k=0}^{\infty} a_{nmj,k} \left[ u_{j,k} \Delta'_s x_{j,k} \right]^{p_{j,k}} = 0 \right\},$$

$$W^2(A, u, \Delta'_s, p)$$

$$= \left\{ x = (x_{j,k}) : \right.$$  

$$P\lim_{n,m} \sum_{j,k=0}^{\infty} a_{nmj,k} \left[ u_{j,k} \Delta'_s x_{j,k} - L \right]^{p_{j,k}} = 0 \right\}$$  

(15)

Remark 3. Let $p = (p_{j,k}) = 1$ for all $j,k$. Then $W_0^2(A, \mathcal{F}, u, \Delta'_s, p)$ and $W^2(A, \mathcal{F}, u, \Delta'_s, p)$ are reduced to

$$W_0^2(A, \mathcal{F}, u, \Delta'_s)$$

$$= \left\{ x = (x_{j,k}) : \right.$$  

$$P\lim_{n,m} \sum_{j,k=0}^{\infty} a_{nmj,k} \left[ F_{j,k} \left( u_{j,k} \Delta'_s x_{j,k} \right) \right] = 0 \right\},$$

$$W^2(A, \mathcal{F}, u, \Delta'_s)$$

$$= \left\{ x = (x_{j,k}) : \right.$$  

$$P\lim_{n,m} \sum_{j,k=0}^{\infty} a_{nmj,k} \left[ F_{j,k} \left( u_{j,k} \Delta'_s x_{j,k} - L \right) \right] = 0 \right\}$$  

(16)

respectively.
Remark 4. Let \( u = (u_{jk}) = 1 \) for all \( j, k \). Then, the spaces \( W^2_0(A, \mathcal{F}, u, \Delta'_r, p) \) and \( W^2(A, \mathcal{F}, u, \Delta'_r, p) \) are reduced to

\[
W^2_0(A, \mathcal{F}, \Delta'_r, p) = \left\{ x = (x_{jk}) : \lim_{n,m}^{\infty, \infty} a_{nmjk} \left[ F_{jk}(|\Delta'_r x_{jk}|)^{p_n} \right] = 0 \right\},
\]

\[
W^2(A, \mathcal{F}, \Delta'_r, p) = \left\{ x = (x_{jk}) : \lim_{n,m}^{m-1,n-1} \sum_{j,k=0}^{\infty} a_{nmjk} \left[ F_{jk}(|\Delta'_r x_{jk} - L|)^{p_n} \right] = 0 \right\} = 0 \quad \text{for some } L \in \mathbb{C},
\]

(17)

respectively.

Remark 5. If we take \( A = (C, 1, 1) \) in \( W^2_0(A, \mathcal{F}, u, \Delta'_r, p) \) and \( W^2(A, \mathcal{F}, u, \Delta'_r, p) \), then we have the following spaces:

\[
W^2_0(\mathcal{F}, u, \Delta'_r, p) = \left\{ x = (x_{jk}) : \lim_{n,m}^{m-1,n-1} \sum_{j,k=0}^{\infty} a_{nmjk} \left[ F_{jk}(|\Delta'_r x_{jk}|)^{p_n} \right] = 0 \right\},
\]

\[
W^2(\mathcal{F}, u, \Delta'_r, p) = \left\{ x = (x_{jk}) : \lim_{n,m}^{m-1,n-1} \sum_{j,k=0}^{\infty} a_{nmjk} \left[ F_{jk}(|\Delta'_r x_{jk} - L|)^{p_n} \right] = 0 \right\} = 0 \quad \text{for some } L \in \mathbb{C},
\]

(18)

Remark 6. If we take \( A = (C, 1, 1) \) and \( \mathcal{F}(x) = x \) in \( W^2_0(A, \mathcal{F}, u, \Delta'_r, p) \) and \( W^2(A, \mathcal{F}, u, \Delta'_r, p) \), then we have the following spaces:

\[
W^2_0(u, \Delta'_r, p) = \left\{ x = (x_{jk}) : \lim_{n,m}^{m-1,n-1} \sum_{j,k=0}^{\infty} a_{nmjk} \left[ F_{jk}(|\Delta'_r x_{jk}|)^{p_n} \right] = 0 \right\},
\]

\[
W^2(u, \Delta'_r, p) = \left\{ x = (x_{jk}) : \lim_{n,m}^{m-1,n-1} \sum_{j,k=0}^{\infty} a_{nmjk} \left[ F_{jk}(|\Delta'_r x_{jk} - L|)^{p_n} \right] = 0 \right\} = 0 \quad \text{for some } L \in \mathbb{C},
\]

(19)

Remark 7. Let \( p_{jk} = u_{jk} = 1 \) for all \( j, k \). If, in addition, \( \mathcal{F}(x) = F(x) \) and \( r = 0 \), then the spaces \( W^2_0(A, \mathcal{F}, u, \Delta'_r, p) \) and \( W^2(A, \mathcal{F}, u, \Delta'_r, p) \) are reduced to \( W^2_0(A, F) \) and \( W^2(A, F) \) which were introduced and studied by Yurdakadim and Tas [37] as below:

\[
W^2_0(A, F) = \left\{ x = (x_{jk}) : \lim_{n,m}^{m-1,n-1} \sum_{j,k=0}^{\infty} a_{nmjk} F_{jk}(|x_{jk}|) = 0 \right\},
\]

\[
W^2(A, F) = \left\{ x = (x_{jk}) : \lim_{n,m}^{m-1,n-1} \sum_{j,k=0}^{\infty} a_{nmjk} F_{jk}(|x_{jk} - L|) = 0 \right\} = 0 \quad \text{for some } L \in \mathbb{C},
\]

(20)

Throughout the paper, we will use the following inequality: let \( (a_{jk}) \) and \( (b_{jk}) \) be two double sequences. Then

\[
|a_{jk} + b_{jk}|^{p_{jk}} \leq K \left( |a_{jk}|^{p_{jk}} + |b_{jk}|^{p_{jk}} \right),
\]

(21)

where \( K = \max(1, 2^{H-1}) \) and \( \sup_{j,k} p_{jk} = H \) (see [15]). We will also assume throughout this paper that the symbol \( \mathcal{F} \) will denote the sublinear Musielak-Orlicz function.
3. Main Results

Theorem 8. Let $\mathcal{F} = (F_{jk})$ be a sublinear Musielak-Orlicz function, $A = (a_{mnj})$ a nonnegative four-dimensional RH-regular matrix, $p = (p_{jk})$ a bounded sequence of positive real numbers, and $u = (u_{jk})$ a sequence of strictly positive real numbers. Then $W^2_0(A, \mathcal{F}, u, \Delta'_s, p)$ and $W^2(A, \mathcal{F}, u, \Delta'_s, p)$ are linear spaces over the complex field $\mathbb{C}$.

Proof. Let $x = (x_{jk}), y = (y_{jk}) \in W^2_0(A, \mathcal{F}, u, \Delta'_s, p)$ and $\alpha, \beta \in \mathbb{C}$.

Since $\mathcal{F} = (F_{jk})$ is a nondecreasing function, so by inequality (21), we have

$$
\begin{align*}
\sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s (ax_{jk} + \beta y_{jk}) \right|) \right]^{p_{jk}} & \leq \sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| ax_{jk} + \beta y_{jk} \right|) \right]^{p_{jk}} \\
& \leq K \sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s x_{jk} \right|) \right]^{p_{jk}} \\
& + K \sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s y_{jk} \right|) \right]^{p_{jk}} \\
& \leq KM^2 \sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s x_{jk} \right|) \right]^{p_{jk}} \\
& + KN^2 \sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s y_{jk} \right|) \right]^{p_{jk}} \longrightarrow 0.
\end{align*}
$$

Thus $ax + \beta y \in W^2_0(A, \mathcal{F}, u, \Delta'_s, p)$. This proves that $W^2_0(A, \mathcal{F}, u, \Delta'_s, p)$ is a linear space. Similarly we can prove that $W^2(A, \mathcal{F}, u, \Delta'_s, p)$ is also a linear space. \(\square\)

Theorem 9. Let $\mathcal{F} = (F_{jk})$ be a sublinear Musielak-Orlicz function, $A = (a_{mnj})$ a nonnegative four-dimensional RH-regular matrix, $p = (p_{jk})$ a bounded sequence of positive real numbers, and $u = (u_{jk})$ a sequence of strictly positive real numbers. Then $W^2_0(A, \mathcal{F}, u, \Delta'_s, p)$ and $W^2(A, \mathcal{F}, u, \Delta'_s, p)$ are paranormed spaces with the paranorm

$$
g(x) = \sup_{n,m} \left\{ \sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s x_{jk} \right|) \right]^{p_{jk}} \right\}^{1/M}, \tag{23}
$$

where $0 < p_{jk} \leq \sup p_{jk} = H < \infty$ and $M = \max(1,H)$.

Proof. We will prove the result for $W^2_0(A, \mathcal{F}, u, \Delta'_s, p)$. Let $x = (x_{jk}) \in W^2_0(A, \mathcal{F}, u, \Delta'_s, p)$. Then for each $x = (x_{jk}) \in W^2_0(A, \mathcal{F}, u, \Delta'_s, p)$, $g(x)$ exists. Also it is clear that $g(0) = 0$, $g(-x) = g(x)$, and $g(x + y) \leq g(x) + g(y)$.

We now show that the scalar multiplication is continuous. First observe the following:

$$
g(\lambda x) = \sup_{n,m} \sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| \lambda \Delta'_s x_{jk} \right|) \right]^{p_{jk}} \leq (1 + [\lambda]) g(x), \tag{24}
$$

where $[\lambda]$ denotes the integer part of $|\lambda|$. It is also clear that if $x \to 0$ and $\lambda \to 0$ implies $g(\lambda x) \to 0$. For fixed $\lambda$, if $x \to 0$, then $g(\lambda x) \to 0$. We need to show that for fixed $x, \lambda \to 0$ implies $g(\lambda x) \to 0$. Let $x \in W^2(A, \mathcal{F}, u, \Delta'_s, p)$. Thus

$$
P \lim_{n,m} \sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s x_{jk} - L \right|) \right]^{p_{jk}} = 0. \tag{25}
$$

Then, for $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that

$$
\sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s x_{jk} - L \right|) \right]^{p_{jk}} < \frac{\epsilon}{4}, \tag{26}
$$

for $m, n > N$. Also, for each $m, n$ with $1 \leq m, n \leq N$, since

$$
\sum_{j,k=0}^{\infty} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s x_{jk} - L \right|) \right]^{p_{jk}} < \infty, \tag{27}
$$

there exists an integer $M_{m,n}$ such that

$$
\sum_{j,k>M_{m,n}} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s x_{jk} - L \right|) \right]^{p_{jk}} < \frac{\epsilon}{4}. \tag{28}
$$

Let $M = \max_{1 \leq (m,n) \leq N \setminus (M_{m,n})} M_{m,n}$. We have for each $m, n$ with $1 \leq m, n \leq N$

$$
\sum_{j,k=M} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s x_{jk} - L \right|) \right]^{p_{jk}} < \frac{\epsilon}{4}. \tag{29}
$$

Also from (26), for $m, n > N$, we have

$$
\sum_{j,k>M} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s x_{jk} - L \right|) \right]^{p_{jk}} < \frac{\epsilon}{4}. \tag{30}
$$

Thus $M$ is an integer independent of $m, n$ such that

$$
\sum_{j,k>M} a_{mnj} \left[ F_{jk}(u_{jk} \left| \Delta'_s x_{jk} - L \right|) \right]^{p_{jk}} < \frac{\epsilon}{4}. \tag{31}
$$
Since $|\lambda|^{p_\lambda} \leq \max(1, |\lambda|^{H})$, therefore
\[
\sum_{j,k=0}^{\infty} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') \right]^{p_\lambda} \\
= \sum_{j,k=0}^{\infty} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda + \lambda L) \right]^{p_\lambda} \\
\leq \sum_{j,k>0} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} \\
+ \sum_{j,k>0} a_{nmjk} \left[ F(jk(u_{jk} | \lambda L) \right]^{p_\lambda} \\
\leq \sum_{j,k>0} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} \\
+ \sum_{j,k>0} a_{nmjk} \left[ F(jk(u_{jk} | \lambda L) \right]^{p_\lambda} \\
\leq \sum_{j,k>0} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} \\
+ \sum_{j,k>0} a_{nmjk} \left[ F(jk(u_{jk} | \lambda L) \right]^{p_\lambda}.
\]
For each $m, n$ and by the continuity of $F$ as $\lambda \to 0$, we have the following:
\[
\sum_{j,k} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} \\
+ \sum_{j,k=0}^{\infty} a_{nmjk} \left[ F(jk(u_{jk} | \lambda L) \right]^{p_\lambda} \to 0
\]
in Pringsheim’s sense. Now choose $\delta < 1$ such that $|\lambda| < \delta$ implies
\[
\sum_{j,k} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} \\
+ \sum_{j,k=0}^{\infty} a_{nmjk} \left[ F(jk(u_{jk} | \lambda L) \right]^{p_\lambda} < \varepsilon.
\]
In the same manner, we have
\[
\sum_{j,k} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} < \varepsilon/4, \quad (35)
\]
\[
\sum_{j,k} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} < \varepsilon/4, \quad (36)
\]
It follows from (31), (34), (35), and (36) that
\[
\sum_{j,k} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') \right]^{p_\lambda} < \varepsilon \quad \forall m,n.
\]
Thus $g(\lambda x) \to 0$ as $\lambda \to 0$. Therefore $W^2(\lambda A, F, u, \Delta^\nu, p)$ is a paranormed space. Similarly, we can prove that $W^2(A, \mathcal{F}, u, \Delta^\nu, p)$ is a paranormed space. This completes the proof.

**Theorem 10.** Let $\mathcal{F} = (F(jk))$ be a sublinear Musielak-Orlicz function, $A = (a_{nmjk})$ a nonnegative four-dimensional RH-regular matrix, $p = (p_{jk})$ a bounded sequence of positive real numbers, and $u = (u_{jk})$ a sequence of strictly positive real numbers. Then $W^2(\lambda A, \mathcal{F}, u, \Delta^\nu, p)$ and $W^2(A, \mathcal{F}, u, \Delta^\nu, p)$ are complete topological linear spaces.

**Proof.** Let $(x_{jk}^q)$ be a Cauchy sequence in $W^2(\lambda A, \mathcal{F}, u, \Delta^\nu, p)$; that is, $g(x_{jk}^q - x_{jk}^{q'}) \to 0$ as $q, t \to \infty$. Then, we have
\[
\sum_{j,k=0}^{\infty} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} \to 0.
\]
Thus for each fixed $j$ and $k$ as $q, t \to \infty$, since $A = (a_{nmjk})$ is nonnegative, we are granted that
\[
F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \to 0,
\]
and by continuity of $\mathcal{F} = (F(jk))$, $(x_{jk}^q)$ is a Cauchy sequence in $\mathcal{C}$ for each fixed $j$ and $k$.

Since $\mathcal{C}$ is complete as $t \to \infty$, we have $x_{jk}^q \to x_{jk}$ for each $(j,k)$. Now from (36), we have that, for $\varepsilon > 0$, there exists a natural number $N$ such that
\[
\sum_{j,k=0}^{\infty} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} < \varepsilon \quad \forall m,n.
\]
Since for any fixed natural number $M$, from (38) we have
\[
\sum_{j,k=0}^{\infty} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} < \varepsilon \quad \forall m,n.
\]
By letting $t \to \infty$ in the above expression we obtain
\[
\sum_{j,k=0}^{\infty} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} < \varepsilon.
\]
Since $M$ is arbitrary, by letting $M \to \infty$ we obtain
\[
\sum_{j,k=0}^{\infty} a_{nmjk} \left[ F(jk(u_{jk} | \Delta_{x_{jk}}^\nu') - \lambda L) \right]^{p_\lambda} < \varepsilon \quad \forall m,n.
\]
Thus $g(x_{jk}^q - x_{jk}^{q'}) \to 0$ as $q \to \infty$. This proves that $W^2(A, \mathcal{F}, u, \Delta^\nu, p)$ is a complete topological linear space.
Now we will show that $W^2(A, \mathcal{F}, u, \Delta'_s, p)$ is a complete topological linear space. For this, since $(\mathcal{X}_n^p)$ is also a sequence in $W^2(A, \mathcal{F}, u, \Delta'_s, p)$ by definition of $W^2(A, \mathcal{F}, u, \Delta'_s, p)$, for each $q$, there exists $L^3$ with
\[
\sum_{j,k=0}^{\infty, \infty} a_{nmj}(F_j u_j [\Delta'_s x_{jk} - L])^{p,n} \to 0
\]
(44)
as $m,n \to \infty$;
whence from the fact that $\sup_{r=0} a_{nmj} < \infty$ and from the definition of Musielak-Orlicz function, we have $F_j u_j [\Delta'_s x_{jk} - L] \to 0$ as $q \to \infty$ and so $L^3$ converges to $L$. Thus
\[
\sum_{j,k=0}^{\infty, \infty} a_{nmj}(F_j u_j [\Delta'_s x_{jk} - L])^{p,n} \to 0
\]
as $m,n \to \infty$.

Hence $x \in W^2(A, \mathcal{F}, u, \Delta'_s, p)$ and this completes the proof. \hfill \Box

Theorem 11. Let $\mathcal{F} = (F_j)$ be a sublinear Musielak-Orlicz function which satisfies the $\Delta_2$-condition. Then $W^2(A, u, \Delta'_s, p) \subseteq W^2(A, \mathcal{F}, u, \Delta'_s, p)$.

Proof. Let $x = (x_j) \in W^2(A, u, \Delta'_s, p)$; that is,
\[
\lim_{n \to \infty} \sum_{j,k} a_{nmj}(F_j u_j [\Delta'_s x_{jk} - L])^{p,n} = 0.
\]
(46)

Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $F_j(t) < \epsilon$ for $0 \leq t \leq \delta$. Write $y_{jk} = (u_j [\Delta'_s x_{jk} - L])$ and consider
\[
\sum_{j,k} a_{nmj}(F_j(y_j))^{p,n} = \sum_{j,k : |y_{jk}| < \delta} a_{nmj}(F_j(y_j))^{p,n} \\
+ \sum_{j,k : |y_{jk}| \geq \delta} a_{nmj}(F_j(y_j))^{p,n} = \epsilon \sum_{j,k : |y_{jk}| < \delta} a_{nmj}(F_j(y_j))^{p,n} \\
+ \sum_{j,k : |y_{jk}| \geq \delta} a_{nmj}(F_j(y_j))^{p,n}.
\]
(47)

For $y_{jk} > \delta$, we use the fact that $y_{jk} < y_{jk}/\delta < 1 + y_{jk}/\delta$. Hence
\[
F_j(y_{jk}) < F_j \left(1 + \frac{y_{jk}}{\delta}\right) < F_j(2) + \frac{1}{\delta} y_{jk}(2) \frac{y_{jk}}{\delta}.
\]
(48)

Since $\mathcal{F}$ satisfies the $\Delta_2$-condition, we have
\[
F_j(y_{jk}) < K \frac{y_{jk}}{\delta} F_j(2) + K \frac{y_{jk}}{\delta} F_j(2) = K \frac{y_{jk}}{\delta} F_j(2),
\]
(49)

and hence
\[
\sum_{j,k : |y_{jk}| \geq \delta} a_{nmj}(F_j(y_{jk}))^{p,n} \leq K \frac{F_j(2)}{\delta} \sum_{j,k} a_{nmj}(u_j [\Delta'_s x_{jk} - L])^{p,n}.
\]
(50)

Since $A$ is RH-regular and $x \in W^2(A, u, \Delta'_s, p)$, we get $x \in W^2(A, \mathcal{F}, u, \Delta'_s, p)$. \Box

Theorem 12. Let $\mathcal{F} = (F_j)$ be a sublinear Musielak-Orlicz function and let $A = (a_{nmj})$ be a nonnegative four-dimensional RH-regular matrix. Suppose that $\beta = \lim_{t \to \infty} (F_j(t)/t) < \infty$. Then
\[
W^2(A, u, \Delta'_s, p) = W^2(A, \mathcal{F}, u, \Delta'_s, p).
\]
(51)

Proof. In order to prove that $W^2(A, u, \Delta'_s, p) = W^2(A, \mathcal{F}, u, \Delta'_s, p)$, it is sufficient to show that $W^2(A, \mathcal{F}, u, \Delta'_s, p) \subseteq W^2(A, u, \Delta'_s, p)$. Now, let $\beta > 0$. By definition of $\beta$, we have $F_j(t) \geq \beta t$ for all $t \geq 0$. Since $\beta > 0$, we have $t \leq (1/\beta) F_j(t)$ for all $t \geq 0$. Let $x = (x_{jk}) \in W^2(A, \mathcal{F}, u, \Delta'_s, p)$. Thus, we have
\[
\sum_{j,k=0}^{\infty, \infty} a_{nmj}(F_j u_j [\Delta'_s x_{jk} - L])^{p,n} \leq \frac{1}{\beta} \sum_{j,k=0}^{\infty, \infty} a_{nmj}(F_j u_j [\Delta'_s x_{jk} - L])^{p,n},
\]
(52)

which implies that $x = (x_{jk}) \in W^2(A, u, \Delta'_s, p)$. This completes the proof. \Box

Theorem 13. (i) Let $0 < \inf p_j < p_j \leq 1$. Then
\[
W^2(A, \mathcal{F}, u, \Delta'_s, p) \subseteq W^2(A, \mathcal{F}, u, \Delta'_s).
\]
(53)

(ii) Let $1 \leq p_j \leq \sup p_j < \infty$. Then
\[
W^2(A, \mathcal{F}, u, \Delta'_s) \subseteq W^2(A, \mathcal{F}, u, \Delta'_s, p).
\]
(54)

Proof. (i) Let $x = (x_{jk}) \in W^2(A, \mathcal{F}, u, \Delta'_s, p)$. Then since $0 < \inf p_j < p_j \leq 1$, we obtain the following:
\[
\sum_{j,k=0}^{\infty, \infty} a_{nmj}(F_j u_j [\Delta'_s x_{jk} - L])^{p,n} \leq \sum_{j,k=0}^{\infty, \infty} a_{nmj}(F_j u_j [\Delta'_s x_{jk} - L])^{p,n},
\]
(55)

Thus $x = (x_{jk}) \in W^2(A, \mathcal{F}, u, \Delta'_s)$.

(ii) Let $p_j \geq 1$ for each $j$ and $k$ and $\sup p_j < \infty$. Let $x = (x_{jk}) \in W^2(A, \mathcal{F}, u, \Delta'_s)$. Then for each $0 < \epsilon < 1$ there exists a positive integer $N$ such that
\[
\sum_{j,k=0}^{\infty, \infty} a_{nmj}(F_j u_j [\Delta'_s x_{jk} - L])^{p,n} \leq \epsilon < 1 \quad \forall m,n \geq N.
\]
(56)
This implies that
\[
\sum_{j,k=0}^{\infty} a_{nmjk} \left[ F_{jk}(u_{jk} | \Delta^r x_{jk} - L) \right] \leq \sum_{j,k=0}^{\infty} a_{nmjk} \left[ F_{jk}(u_{jk} | \Delta^r y_{jk}) \right].
\]

Therefore \( x = (x_{jk}) \in W^2(A, F, u, \Delta_s, p) \). This completes the proof. \( \Box \)

**Lemma 14.** Let \( S = (F, K) \) be a sublinear Musielak-Orlicz function which satisfies the \( \Delta_2 \)-condition and let \( A = (a_{nmjk}) \) be a nonnegative four-dimensional RH-regular matrix. Then \( W^2_0(A, F, u, \Delta_s, p) \cap l^2_{\infty} \) is an ideal in \( l^2_{\infty} \).

**Proof.** Let \( x \in W^2_0(A, F, u, \Delta_s, p) \cap l^2_{\infty} \) and \( y \in l^2_{\infty} \). We need to show that \( xy \in W^2_0(A, F, u, \Delta_s, p) \cap l^2_{\infty} \). Since \( y \in l^2_{\infty} \), there exists \( T_1 > 1 \) such that \( \|y\| < T_1 \). In this case \( \|x_{jk} y_{jk}\| < T_1 \|x_{jk}\| \) for all \( j, k \). Since \( S \) is nondecreasing and satisfies the \( \Delta_2 \)-condition, we have
\[
\left[ F_{jk}(u_{jk} | \Delta^r x_{jk} y_{jk}) \right] \leq T(T_1) \left[ F_{jk}(u_{jk} | \Delta^r x_{jk}) \right],
\]
and for all \( j, k \) and \( T > 0 \). Therefore \( \lim_{n,m} \sum_{j,k} a_{nmjk} \left[ F_{jk}(u_{jk} | \Delta^r (x_{jk} y_{jk})) \right] = 0 \). Thus \( xy \in W^2_0(A, F, u, \Delta_s, p) \cap l^2_{\infty} \). This completes the proof. \( \Box \)

**Lemma 15.** Let \( G \) be an ideal in \( l^2_{\infty} \) and let \( x = (x_{jk}) \in l^2_{\infty} \). Then \( x \) is in the closure of \( G \) in \( l^2_{\infty} \) if and only if \( \|x_{jk}\| \leq K |x_{jk}| \) for all \( j, k \).

**Proof.** Let \( x \) be in the closure of \( G \) and let \( \epsilon > 0 \) be given. Suppose that \( z = (z_{jk}) \in G \) such that \( \|x - z\| < \epsilon/2 \) and observe that \( S(x; \epsilon) \subseteq S(z; \epsilon/2) \). Define a double sequence \( y = (y_{jk}) \in l^2_{\infty} \) by
\[
y_{jk} = \begin{cases} \frac{1}{z_{jk}}, & \text{if } |z_{jk}| \geq \frac{\epsilon}{2} \\ 0, & \text{otherwise}. \end{cases}
\]
Clearly \( yz = x_{\|x_{jk}\|/2} \in G \). Since \( S(x; \epsilon) \subseteq S(z; \epsilon/2) \) and \( x_{\|x_{jk}\|} \in l^2_{\infty} \), hence \( x_{\|x_{jk}\|} = x_{\|x_{jk}\|/2} \in G \).

Conversely, if \( x \in l^2_{\infty} \) then \( \|x - x_{\|x_{jk}\|/2}\| < \epsilon \). It follows that \( x_{\|x_{jk}\|/2} \in G \) for all \( \epsilon > 0 \); then \( x \) is in the closure of \( G \). \( \Box \)

**Lemma 16.** If \( A \) is a nonnegative four-dimensional RH-regular matrix, then \( W^2_0(A, u, \Delta_s, p) \cap l^2_{\infty} \) is a closed ideal in \( l^2_{\infty} \).

**Proof.** We have \( W^2_0(A, S, u, \Delta_s, p) \cap l^2_{\infty} \subset l^2_{\infty} \) and it is clear that \( W^2_0(A, S, u, \Delta_s, p) \cap l^2_{\infty} \neq \emptyset \). For \( x, y \in W^2_0(A, S, u, \Delta_s, p) \cap l^2_{\infty} \), we get \( |x_{jk} + y_{jk}| < |x_{jk}| + |y_{jk}| \). Now, we have
\[
\left[ F_{jk}(u_{jk} | \Delta^r x_{jk} y_{jk}) \right] \leq \left[ F_{jk}(u_{jk} | \Delta^r x_{jk}) \right] + \left[ F_{jk}(u_{jk} | \Delta^r y_{jk}) \right]
\]
where \( K = \max\{|K_1, K_2|\} \), so \( x + y, x - y \in W^2_0(A, S, u, \Delta_s, p) \cap l^2_{\infty} \).

Let \( x \in W^2_0(A, S, u, \Delta_s, p) \cap l^2_{\infty} \) and \( y \in l^2_{\infty} \). Thus, there exists a positive integer \( K \), so that, for every \( j, k \), we have \( |x_{jk} y_{jk}| \leq K |x_{jk}| \). Therefore
\[
\left[ F_{jk}(u_{jk} | \Delta^r x_{jk} y_{jk}) \right] \leq \left[ F_{jk}(u_{jk} K | \Delta^r x_{jk}) \right] \leq T \left[ F_{jk}(u_{jk} | \Delta^r x_{jk}) \right],
\]
and so
\[
\sum_{j,k} a_{nmjk} \left[ F_{jk}(u_{jk} | \Delta^r x_{jk} y_{jk}) \right] \leq T \sum_{j,k} a_{nmjk} \left[ F_{jk}(u_{jk} | \Delta^r x_{jk}) \right]
\]
Hence \( xy \in W^2_0(A, S, u, \Delta_s, p) \cap l^2_{\infty} \). So \( W^2_0(A, S, u, \Delta_s, p) \cap l^2_{\infty} \) is an ideal in \( l^2_{\infty} \) for a Musielak-Orlicz function which satisfies the \( \Delta_2 \)-condition.

Now, we have to show that \( W^2_0(A, S, u, \Delta_s, p) \cap l^2_{\infty} \) is closed. Let \( x \in W^2_0(A, S, u, \Delta_s, p) \cap l^2_{\infty} \), there exists \( x_{\epsilon} = x_{\|x_{jk}\|/2} \in W^2_0(A, S, u, \Delta_s, p) \cap l^2_{\infty} \) such that \( x_{\epsilon} \rightarrow x \in l^2_{\infty} \).
For every $\varepsilon > 0$ there exists $N_1(\varepsilon) \in \mathbb{N}$ such that, for all $c, d > N_1(\varepsilon)$,
$$|\Delta r^s x_{c,d} - \Delta r^s x_{j,k}| < \varepsilon.$$ Now, for $\varepsilon > 0$, we have
$$\sum_{j,k} a_{n,mj,k} \left[ F_{jk}(u_{jk} | \Delta r^s x_{j,k}) \right]^{p,n}$$
$$= \sum_{j,k} a_{n,mj,k} \left[ F_{jk}(u_{jk} | \Delta r^s x_{j,k} - \Delta r^s x_{j,k} + \Delta r^s x_{j,k}) \right]^{p,n}$$
$$\leq \sum_{j,k} a_{n,mj,k} \left[ F_{jk}(u_{jk} | \Delta r^s x_{j,k} - \Delta r^s x_{j,k} + \Delta r^s x_{j,k}) \right]^{p,n} \right]^{p,n}$$
$$\leq \frac{1}{2} \sum_{j,k} a_{n,mj,k} \left[ F_{jk}(u_{jk} | \Delta r^s x_{j,k} - \Delta r^s x_{j,k}) \right]^{p,n} \right]^{p,n}$$
$$+ \frac{1}{2} \sum_{j,k} a_{n,mj,k} \left[ F_{jk}(u_{jk} | \Delta r^s x_{j,k} - \Delta r^s x_{j,k}) \right]^{p,n} \right]^{p,n}$$
$$\leq \frac{1}{2} K F_{jk}(\varepsilon) \sum_{j,k} a_{n,mj,k} + \frac{1}{2} k \sum_{j,k} a_{n,mj,k} \left[ F_{jk}(u_{jk} | \Delta r^s x_{j,k}) \right]^{p,n}$$
$$\leq \frac{1}{2} K F_{jk}(\varepsilon) \sum_{j,k} a_{n,mj,k} + \frac{1}{2} k \sum_{j,k} a_{n,mj,k} \left[ F_{jk}(u_{jk} | \Delta r^s x_{j,k}) \right]^{p,n}.$$

Since $x^{c,d} \in W_0^2(A, \mathcal{F}, u, \Delta^s, p) \cap l^2_{\infty}$ and $A$ is RH-regular, we get
$$\lim_{n \to \infty} \sum_{j,k} a_{n,mj,k} \left[ F_{jk}(u_{jk} | \Delta r^s x_{j,k}) \right]^{p,n} = 0;$$
so $x \in W_0^2(A, \mathcal{F}, u, \Delta^s, p) \cap l^2_{\infty}$. This completes the proof. $\Box$

**Theorem 17.** Let $x = (x_{j,k})$ be a bounded sequence, $\mathcal{F} = (F_{jk})$ a sublinear Musielak-Orlicz function which satisfies the $\Delta_2$-condition, and $A$ a nonnegative four-dimensional RH-regular matrix. Then $W^2(A, \mathcal{F}, u, \Delta^s, p) \cap l^2_{\infty} = W^2(A, u, \Delta^s, p) \cap l^2_{\infty}$.

Proof. Without loss of generality we may take $L = 0$ and establish
$$W^2_0(A, \mathcal{F}, u, \Delta^s, p) \cap l^2_{\infty} = W^2_0(A, u, \Delta^s, p) \cap l^2_{\infty}. \quad (66)$$
Since $W^2_0(A, u, \Delta^s, p) \subseteq W^2_0(A, \mathcal{F}, u, \Delta^s, p)$, therefore $W^2_0(A, u, \Delta^s, p) \cap l^2_{\infty} \subseteq W^2_0(A, \mathcal{F}, u, \Delta^s, p) \cap l^2_{\infty}$. We need to show that $W^2_0(A, \mathcal{F}, u, \Delta^s, p) \cap l^2_{\infty} \subseteq W^2_0(A, u, \Delta^s, p) \cap l^2_{\infty}$. Notice that if $S \subset \mathbb{N} \times \mathbb{N}$, then
$$\sum_{j,k} a_{n,mj,k} \left[ F_{jk}(\Delta S_{j,k}) \right]^{p,n} = F_{jk}(1) \sum_{j,k} a_{n,mj,k} \left[ \Delta S_{j,k} \right]^{p,n},$$
for all $n, m$. Observe that $\Delta S_{j,k} \in W^2_0(A, u, \Delta^s, p) \cap l^2_{\infty}$, whenever $x \in W^2_0(A, \mathcal{F}, u, \Delta^s, p) \cap l^2_{\infty}$ by Lemmas 14 and 15, so
$$W^2_0(A, \mathcal{F}, u, \Delta^s, p) \cap l^2_{\infty} \subseteq W^2_0(A, u, \Delta^s, p) \cap l^2_{\infty}. \quad (68)$$
The proof is complete. $\Box$

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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