Research Article

Arcwise Connected Domains, Quasiconformal Mappings, and Quasidisks

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We prove that a homeomorphism $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a quasiconformal mapping if and only if $f(D)$ is an arcwise connected domain for any arcwise connected domain $D \subseteq \mathbb{R}^2$, and D is a quasidisk if and only if both D and its exterior $D^* = \mathbb{R}^2 \setminus \bar{D}$ are arcwise connected domains.

1. Introduction

We shall assume throughout this paper that D is a Jordan proper subdomain of the complex plane $\mathbb{R}^2$ with a boundary containing at least three points. For convenience we shall adopt the notation and terminology as in paper [1]. For $x \in \mathbb{R}^2$ and $0 < r < \infty$, let $B^2(x, r) = \{z \in \mathbb{R}^2 : |z - x| < r\}$, $\bar{B}^2(x, r)$ be the closure of $B^2(x, r)$, $S^1(x, r) = \partial B^2(x, r)$, $B^2(r) = B^2(0, r)$, and $B^2 = B^2(1)$. Suppose that f is a homeomorphism in $\mathbb{R}^2$, $x \in \mathbb{R}^2$ and $0 < r < \infty$, let $L(x, f, r) = \max_{|y-x|=r} |f(y) - f(x)|$, and $l(x, f, r) = \min_{|y-x|=r} |f(y) - f(x)|$.

Suppose that $b \geq 1$ is a constant, we say that D is a b -arcwise connected domain if each pair of points x and y in D can be joined by an arc $\gamma \subseteq D$ with $\text{dia}(\gamma) \leq b|x - y|$. Here $\text{dia}(\gamma)$ is the Euclidean diameter of γ . And we call D an arcwise connected domain if D is a b -arcwise connected domain for some $b \geq 1$.

Arcwise connected domain is an important concept; it has been used extensively in the research fields of superspaces [2], multiobjective programmings [3], color images [4], continuous mappings [5], fixed points theory [6], continuity and differential problems [7], and topology groups theory [8].

D is called a quasidisk if there exists a K -quasiconformal mapping ($K \geq 1$) $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ such that D is the image of the unit disk B^2 under f .

It is well-known that quasidisks play a very important role in quasiconformal mappings, complex dynamics, Fuchsian groups, and Teichmüller space theory [9–12].

The purpose of this paper is to prove the following Theorems 1 and 2.

Theorem 1. *A homeomorphism $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a quasiconformal mapping if and only if $f(D)$ is a arcwise connected domain for any arcwise connected domain $D \subseteq \mathbb{R}^2$.*

Theorem 2. *D is a quasidisk if and only if both D and $D^* = \mathbb{R}^2 \setminus \bar{D}$ are arcwise connected domains.*

For convenience we shall introduce the following concepts and they will be used in the next section.

Let $c \geq 1$ be a constant. (1) If for any $x_0 \in \mathbb{R}^2$ and $0 < r < +\infty$, points in $D \cap \bar{B}(x_0, r)$ can be joined by an arc in $D \cap \bar{B}(x_0, cr)$; then we say that D is a c -inner linearly locally connected domain, denoted by $D \in c\text{-ILC}$; (2) If for any $x_0 \in \mathbb{R}^2$ and $0 < r < +\infty$, points in $D \setminus B^2(x_0, r/c)$ can be joined by an arc in $D \setminus B^2(x_0, r/c)$; then we say that D is a c -outer linearly locally connected domain, denoted by $D \in c\text{-OLC}$.

D is said to be a linearly locally connected domain if there exists $c \geq 1$ such that $D \in c\text{-ILC}$ and $D \in c\text{-OLC}$.

Let $b > 0$ be a constant; then D is said to be a b -cigar domain if each pair of points $x_1, x_2 \in D$ can be joined by an arc $\gamma \subseteq D$ for which

$$\min_{j=1,2} \text{dia}(\gamma(x_j, x)) \leq bd(x, \partial D) \quad \forall x \in \gamma, \quad (1)$$

where $\gamma(x_j, x)$ is the part of γ between x_j and x , and $d(x, \partial D)$ is the Euclidean distance from x to ∂D .

We say that D is a cigar domain if D is a b -cigar domain for some $b > 0$.

Gehring and Osgood proved the following result in [13].

Theorem A. *D is a quasidisk if and only if D is a linearly locally connected domain.*

2. Proof of Theorems 1 and 2

To prove our Theorems 1 and 2, we first establish and introduce several lemmas.

Lemma 3. *If D is an arcwise connected domain, then $D^* = R^2 \setminus \overline{D}$ is a cigar domain.*

Proof. For any $z_1, z_2 \in D^*$, let γ be the hyperbolic geodesic joining z_1 and z_2 in D^* , for any $z \in \gamma \setminus \{z_1, z_2\}$; suppose that $f: B^2 \rightarrow D^*$ is a conformal mapping with $f(0) = z$ and

$$f^{-1}(\gamma) \subseteq R = \{z : z \in R^2, \text{Im } z = 0\}, \quad (2)$$

where f^{-1} is the inverse of f . Then there exist

$$x_1 \in \{z : z \in S^1, \text{Im } z > 0\}, \quad x_2 \in \{z : z \in S^1, \text{Im } z < 0\} \quad (3)$$

by [14, Corollary 10.3] such that $\alpha_j = f([0, x_j])$ is rectifiable with

$$l(\alpha_j) < a_0 d(z, \partial D^*), \quad j = 1, 2, \quad (4)$$

where a_0 is an absolute constant, and $[0, x_j]$ is the half open segment joining the origin O and x_j ; $j = 1, 2$.

Let $y_j = f(x_j)$, $\alpha = \alpha_1 \cup \alpha_2$, then $y_j \in \partial D$ and

$$l(\alpha) \leq l(\alpha_1) + l(\alpha_2) < 2a_0 d(z, \partial D^*). \quad (5)$$

For above $y_1, y_2 \in \partial D^* = \partial D$, there exist a constant $b \geq 1$ and a simple curve $\beta \subseteq D$ joining y_1 and y_2 such that

$$\text{dia}(\beta) \leq b|y_1 - y_2| \leq bl(\alpha) \quad (6)$$

by D is an arcwise connected domain.

If we denote by D_0 the bounded domain with boundary $\alpha \cup \beta$, then one of the points z_1 and z_2 must be in D_0 . Without loss of generality, we may assume that $z_1 \in D_0$; then it follows from (5) and (6) that

$$\begin{aligned} \text{dia}(\gamma(z_1, z)) &\leq \text{dia}(D_0) = \text{dia}(\partial D_0) \\ &\leq \text{dia}(\alpha) + \text{dia}(\beta) \\ &\leq l(\alpha) + \text{dia}(\beta) \\ &< 2a_0(b+1)d(z, \partial D^*). \end{aligned} \quad (7)$$

Inequality (7) leads to

$$\min_{j=1,2} \text{dia}(\gamma(z_j, z)) < 2a_0(b+1)d(z, \partial D^*). \quad (8)$$

Therefore, D^* is a cigar domain that follows from (8). $\square$

Lemma 4. *If D is a c -cigar domain, then $D \in (2c+2)$ -OLC.*

Proof. Let $b = 2c + 2$. If $D \notin b$ -OLC, then there exist $y_0 \in R^2$, $0 < r < \infty$ and $x_1, x_2 \in D \setminus B^2(y_0, r)$, such that x_1 and x_2 can not be joined by any arc in $D \setminus B^2(y_0, r/b)$.

Since D is a c -cigar domain, there exists an arc $\gamma \subset D$ such that γ joining x_1 and x_2 with

$$\min_{j=1,2} \text{dia}(\gamma(x_j, x)) \leq cd(x, \partial D) \quad (9)$$

for all $x \in \gamma$.

It is obvious that $\gamma \cap S^1(y_0, r/b) \neq \emptyset$. Let $y \in \gamma \cap S^1(y_0, r/b)$; then (9) implies

$$\begin{aligned} d(y, \partial D) &\geq \frac{1}{c} \min_{j=1,2} \text{dia}(\gamma(x_j, y)) \geq \frac{1}{c} \min_{j=1,2} |x_j - y| \\ &\geq \frac{1}{c} \left(1 - \frac{1}{b}\right) r. \end{aligned} \quad (10)$$

But

$$d(y_0, \partial D) \leq \frac{r}{b}. \quad (11)$$

Inequalities (10) and (11) together with the triangular inequality yield

$$\begin{aligned} \frac{1}{c} \left(1 - \frac{1}{b}\right) r &\leq d(y, \partial D) \leq |y - y_0| + d(y_0, \partial D) \leq \frac{2}{b} r, \\ b &\leq 2c + 1. \end{aligned} \quad (12)$$

Inequality (12) contradicts with $b = 2c + 2$. Hence $D \in (2c+2)$ -OLC. $\square$

Lemma 5. *If D^* is a c_0 -cigar domain, then $D \in (16c_0 + 21)$ -ILC.*

Proof. Let $\delta = 8c_0 + 10$. For any $u \in R^2$, $s > 0$, and $z_1, z_2 \in D \cap \overline{B}^2(u, s)$, $z_1 \neq z_2$. Denote $z = (1/2)(z_1 + z_2)$ and $r = |z_1 - z_2|$. We first prove that z_1, z_2 must be in the same component of $\overline{B}^2(z, (1/2)\delta r) \setminus D^*$.

If z_1, z_2 belong to different components of $\overline{B}^2(z, (1/2)\delta r) \setminus D^*$, then z_1, z_2 must be in the different components of $\overline{B}^2(z, (1/2)r) \setminus D^*$. Let β be the line segment which joins z_1 and z_2 ; then β contains a subcurve $\alpha \subseteq D^*$ such that α divides D^* into D_1 and D_2 , and $\text{dia}(D_j) \geq (1/2)r(\delta - 1)$, $j = 1, 2$. This yields

$$\min_{j=1,2} \text{dia}(D_j) \geq \frac{1}{2} r (\delta - 1). \quad (13)$$

For any $x \in \alpha$, if $D_1 \not\subseteq B^2(x, (2c_0 + 2)\text{dia}(\alpha))$ and $D_2 \not\subseteq B^2(x, (2c_0 + 2)\text{dia}(\alpha))$, then take

$$x_j \in D_j \setminus \bar{B}^2(x, (2c_0 + 2)\text{dia}(\alpha)), \quad j = 1, 2. \quad (14)$$

Since D^* is a c_0 -cigar domain, there exists an arc $\gamma \subseteq D^*$ joining x_1 and x_2 with

$$\min_{j=1,2} \text{dia}(\gamma(x_j, w)) \leq c_0 d(w, \partial D^*) \quad (15)$$

for all $w \in \gamma$.

Let $y \in \gamma \cap S^1(x, \text{dia}(\alpha))$; then

$$\begin{aligned} |y - x_j| &\geq (2c_0 + 1)\text{dia}(\alpha), \\ \min_{j=1,2} \text{dia}(\gamma(x_j, y)) &\leq c_0 d(y, \partial D^*), \end{aligned} \quad (16)$$

$j = 1, 2.$

This implies

$$d(y, \partial D^*) \geq \frac{2c_0 + 1}{c_0} \text{dia}(\alpha). \quad (17)$$

But

$$d(x, \partial D^*) \leq \text{dia}(\alpha). \quad (18)$$

Inequalities (17) and (18) together with the triangular inequality yield

$$\begin{aligned} \frac{2c_0 + 1}{c_0} \text{dia}(\alpha) &\leq d(y, \partial D^*) \leq |y - x| + d(x, \partial D^*) \\ &\leq 2\text{dia}(\alpha), \end{aligned} \quad (19)$$

so

$$\text{dia}(\alpha) \leq 0. \quad (20)$$

This is obviously impossible. Hence $D_1 \subseteq \bar{B}^2(x, (2c_0 + 2)\text{dia}(\alpha))$ or $D_2 \subseteq \bar{B}^2(x, (2c_0 + 2)\text{dia}(\alpha))$, and we obtain

$$\min_{j=1,2} \text{dia}(D_j) \leq 2(2c_0 + 2)\text{dia}(\alpha). \quad (21)$$

Inequalities (13) and (21) together with $\text{dia}(\alpha) \leq r$ imply

$$\delta \leq 8c_0 + 9. \quad (22)$$

This contradicts with $\delta = 8c_0 + 10$. Hence z_1, z_2 must be in the same component of $\bar{B}^2(z, (1/2)\delta r) \setminus D^*$, and there exists an arc $\gamma \subseteq D$ joining z_1 and z_2 with

$$\text{dia}(\gamma) \leq \delta r = \delta |z_1 - z_2| \leq 2\delta s. \quad (23)$$

It follows from (23) that

$$\begin{aligned} \gamma &\subseteq D \cap \bar{B}^2(u, s + \text{dia}(\gamma)) \subseteq D \cap \bar{B}^2(u, (2\delta + 1)s) \\ &= D \cap \bar{B}^2(u, (16c_0 + 21)s). \end{aligned} \quad (24)$$

Hence $D \in (16c_0 + 21)\text{-ILC}$; this completes the proof of Lemma 5. $\square$

Lemma 6. *If D is a b -cigar domain, then D^* is a $(4b + 3)$ -arcwise connected domain.*

Proof. For any $z_1, z_2 \in D^*$, $z_1 \neq z_2$, let $z_0 = (z_1 + z_2)/2$, $r = |z_1 - z_2|/2$, and $d = 4b + 3$. We shall prove that z_1 and z_2 can be joined by an arc in $D^* \cap \bar{B}^2(z_0, dr)$.

If z_1 and z_2 can not be joined by any arc in $D^* \cap \bar{B}^2(z_0, dr)$, then z_1 and z_2 must be in the different components of $D^* \cap \bar{B}^2(z_0, dr)$, and hence z_1 and z_2 must be in the different components of $D^* \cap \bar{B}^2(z_0, (d/2)r)$. There exist points $y_1, y_2 \in \partial D^* \setminus \bar{B}^2(z_0, (d/2)r)$ such that $\beta \cap B^2(z_0, r) \neq \emptyset$ for any arc $\beta \subseteq D$ joining y_1 and y_2 .

Since D is a b -cigar domain, hence there exists an arc $\gamma \subseteq D$ joining y_1 and y_2 such that

$$\min_{j=1,2} \text{dia}(\gamma(y_j, y)) \leq bd(y, \partial D) \quad (25)$$

for all $y \in \gamma$.

Let $y_0 \in \gamma \cap B^2(z_0, r)$; then

$$\min_{j=1,2} \text{dia}(\gamma(y_j, y_0)) \leq bd(y_0, \partial D), \quad (26)$$

$$d(y_0, \partial D) \leq |y_0 - z_j| \leq 2r,$$

$$\begin{aligned} \min_{j=1,2} \text{dia}(\gamma(y_j, y_0)) &\geq \min_{j=1,2} |y_j - y_0| \\ &\geq \min_{j=1,2} (|y_j - z_0| - |y_0 - z_0|) \\ &\geq \left(\frac{d}{2} - 1\right)r. \end{aligned} \quad (27)$$

It follows from (26) and (27) that

$$d \leq 4b + 2, \quad (28)$$

which contradicts with $d = 4b + 3$. Hence z_1 and z_2 can be joined by an arc $\alpha \subseteq D^* \cap \bar{B}^2(z_0, dr)$ with

$$\text{dia}(\alpha) \leq 2dr = (4b + 3)|z_1 - z_2|. \quad (29)$$

This shows that D^* is a $(4b + 3)$ -arcwise connected domain. $\square$

Lemma 7. *If $f : R^2 \rightarrow R^2$ is a K -quasiconformal mapping, and $c \geq 1$ is a constant, then for any $x \in R^2$ and $0 < r < +\infty$, we have*

$$L(x, f, cL(x', f^{-1}, r)) \leq ar, \quad (30)$$

where $x' = f(x)$, f^{-1} is the inverse of f , and $a = a(K, c)$ is a constant which depends only on K and c .

Proof. Let Γ be the curve family which joins $S^1(x, cL(x', f^{-1}, r))$ and $\partial(f^{-1}(B^2(x', r)))$ in $B^2(x, cL(x', f^{-1}, r)) \setminus f^{-1}(B^2(x', r))$; $M(\Gamma)$ denotes the modulus of Γ , $\Gamma' = f(\Gamma)$.

Then from the comparison principle of modulus and the result given in [1, 7.5] we get

$$M(\Gamma) \geq \frac{2\pi}{\log(cL(x', f^{-1}, r)/l(x', f^{-1}, r))}, \quad (31)$$

$$M(\Gamma') \leq \frac{2\pi}{\log(l(x, f, cL(x', f^{-1}, r))/r)}.$$

According to the properties of K -quasiconformal mapping in [1], we have

$$KM(\Gamma') \geq M(\Gamma) \geq \frac{M(\Gamma')}{K} \quad (32)$$

$$\frac{L(x', f^{-1}, r)}{l(x', f^{-1}, r)} \leq c' = c'(K), \quad (33)$$

where $c'(K)$ is a constant which depends only on K .

It follows from (31)–(33) that

$$\frac{l(x, f, cL(x', f^{-1}, r))}{r} \leq (cc')^K. \quad (34)$$

The same reason to get (33) implies

$$\frac{L(x, f, cL(x', f^{-1}, r))}{l(x, f, cL(x', f^{-1}, r))} \leq c'. \quad (35)$$

It follows from (34) and (35) that

$$\frac{L(x, f, cL(x', f^{-1}, r))}{r} \leq c'(cc')^K = a. \quad (36)$$

□

Lemma 8. Suppose that $f : R^2 \rightarrow R^2$ is a K -quasiconformal mapping. If D is a b -cigar domain for some $b > 0$, then $D' = f(D)$ is a b' -cigar domain. In here $b' = b'(b, K)$ is a constant which depends only on b and K .

Proof. For any points $y_1, y_2 \in D'$, let $x_1 = f^{-1}(y_1)$, $x_2 = f^{-1}(y_2)$. Since D is a b -cigar domain, hence there exists an arc $\gamma \subseteq D$ joining x_1 and x_2 such that $\min_{j=1,2} \text{dia}(\gamma(x_j, x)) \leq bd(x, \partial D)$ for all $x \in \gamma$. Let $\gamma' = f(\gamma)$; then γ' is an arc joining y_1 and y_2 in D' . For any $y \in \gamma'$, let $x = f^{-1}(y)$; then $x \in \gamma$ with $\min_{j=1,2} \text{dia}(\gamma(x_j, x)) \leq bd(x, \partial D)$.

Without loss of generality, we assume that $\text{dia}(\gamma(x_1, x)) \leq bd(x, \partial D)$; then $\gamma(x_1, x) \subseteq \bar{B}^2(x, bd(x, \partial D))$ and $\gamma'(y_1, y) = f(\gamma(x_1, x)) \subseteq \bar{B}^2(y, L(x, f, bd(x, \partial D)))$. From Lemma 7 and the fact $L(x, f, bd(x, \partial D)) \leq L(x, f, bL(y, f^{-1}, d(y, \partial D)))$ we know that there exists constant $b' = b'(b, K)$ which depends only on b and K such that $L(x, f, bd(x, \partial D)) \leq (b'/2)d(y, \partial D')$. This implies $\gamma'(y_1, y) \subseteq \bar{B}^2(y, (b'/2)d(y, \partial D'))$. Therefore,

$$\text{dia}(\gamma'(y_1, y)) \leq b'd(y, \partial D'), \quad (37)$$

$$\min_{j=1,2} \text{dia}(\gamma(y_j, y)) \leq b'd(y, \partial D').$$

This shows that D' is a b' -cigar domain. □

Lemma 9 (see [15]). Let $f : R^2 \rightarrow R^2$ be a homeomorphism. If there exists a constant $c > 0$ such that

$$[\text{dia}(f(B^2(x, r)))^2] \leq c \cdot m[f(B^2(x, r))] \quad (38)$$

for all $x \in R^2$ and $0 < r < \infty$; then f is a quasiconformal mapping, where $m[f(B^2(x, r))]$ denotes the 2-dimensional Lebesgue measure of $f(B^2(x, r))$.

Lemma 10. Let $f : R^2 \rightarrow R^2$ be a homeomorphism. If f maps any cigar domain D onto a cigar domain $D' = f(D)$, then f is a quasiconformal mapping.

Proof. For any $x \in R^2$ and $0 < r < \infty$, choose $y \in B^2(x, r)$ such that

$$\text{dia}(f(B^2(x, r))) \leq 3|f(y) - f(x)|. \quad (39)$$

It is easy to see that $B^2(x, r)$ is a 1-cigar domain if we take $\gamma = [x_1, x] \cup [x_2, x]$ for any $x_1, x_2 \in B^2(x, r)$, where $[x_j, x]$ denotes the closed line segment joining x_j and x , $j = 1, 2$. By the assumption of Lemma 10 we know that there exists a constant $b > 0$ such that $f(B^2(x, r))$ is a b -cigar domain; hence there exists an arc $\gamma \subseteq f(B^2(x, r))$ joining $f(x)$ and $f(y)$ with

$$\begin{aligned} \min\{\text{dia}(\gamma(f(x), z)), \text{dia}(\gamma(f(y), z))\} \\ \leq bd(z, \partial f(B^2(x, r))) \end{aligned} \quad (40)$$

for all $z \in \gamma$. If we choose $z_0 \in \gamma$ such that $\text{dia}(\gamma(f(x), z_0)) = \text{dia}(\gamma(f(y), z_0))$, then (39) and (40) imply

$$\begin{aligned} \text{dia}(z_0, \partial f(B^2(x, r))) &\geq \frac{\text{dia}(\gamma(f(x), z_0))}{b} \\ &\geq \frac{\text{dia}(\gamma(f(x), f(y)))}{2b} \\ &\geq \frac{|f(x) - f(y)|}{2b} \\ &\geq \frac{\text{dia}(f(B^2(x, r)))}{6b}. \end{aligned} \quad (41)$$

This yields

$$\begin{aligned} B^2\left(z_0, \frac{\text{dia}(f(B^2(x, r)))}{6b}\right) &\subseteq f(B^2(x, r)), \\ \pi\left(\frac{\text{dia}[f(B^2(x, r))]}{6b}\right)^2 &\leq m(f(B^2(x, r))), \end{aligned} \quad (42)$$

$$[\text{dia}(f(B^2(x, r)))]^2 \leq \frac{36b^2}{\pi} m(f(B^2(x, r))).$$

From the above argument and Lemma 9 we know that f is a quasiconformal mapping. □

Proof of Theorem 1. Consider the following.

Necessity. For any quasiconformal mapping $f: R^2 \rightarrow R^2$ and any arcwise connected domain $D \subseteq R^2$, we know that D^* is a cigar domain by Lemma 3. This and Lemma 8 imply that $f(D^*)$ is a cigar domain, and then $D' = f(D) = (f(D^*))^*$ is an arcwise connected domain by Lemma 6.

Sufficiency. To prove a homeomorphism $f: R^2 \rightarrow R^2$ is a quasiconformal mapping, making use of Lemma 10, we need only to prove that $D' = f(D)$ is a cigar domain for any cigar domain D . In fact, for any cigar domain D , Lemma 6 implies D^* is an arcwise connected domain. From this and the condition of Theorem 1 we know that $f(D^*)$ is an arcwise connected domain; then from Lemma 3 we know that $(f(D^*))^* = f(D) = D'$ is a cigar domain. $\square$

Proof of Theorem 2. Consider the following.

Necessity. Let D be a quasidisk; then there exists a quasiconformal mapping $f: R^2 \rightarrow R^2$ such that $D = f(B^2)$.

- (1) It is obvious that B^2 is a 1-arcwise connected domain; then from Theorem 1 we know that $D = f(B^2)$ is an arcwise connected domain.
- (2) Since B^2 is a 1-cigar domain, hence $(B^2)^*$ is an arcwise connected domain; then Theorem 1 implies that $f((B^2)^*) = D^*$ is an arcwise connected domain.

Sufficiency. Suppose that both D and D^* are arcwise connected domains.

- (1) Since D is an arcwise connected domain, hence D^* is a cigar domain by Lemma 3; then Lemma 5 implies that $D \in \text{ILC}$.
- (2) Since D^* is an arcwise connected domain, hence D is a cigar domain by Lemma 3; then Lemma 4 implies that $D \in \text{OLC}$.

From the above (1) and (2) together with Theorem A we know that D is a quasidisk. $\square$

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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