Research Article

The Banach-Saks Properties in Orlicz-Lorentz Spaces

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Received 30 November 2013; Accepted 7 February 2014; Published 13 March 2014

Academic Editor: Mieczysław Mastyło

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The Banach-Saks index of an Orlicz-Lorentz space $\Lambda_{\varphi,w}(I)$ for both function and sequence case, is computed with respect to its Matuszewska-Orlicz indices of φ . It is also shown that an Orlicz-Lorentz function space has weak Banach-Saks (resp., Banach-Saks) property if and only if it is separable (resp., reflexive).

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space. A bounded sequence $\{x_n\} \subset X$ is called a *weak Banach-Saks sequence* whenever there exists a subsequence $\{y_n\} \subset \{x_n\}$ such that its Cesàro means converge in norm to zero; that is,

$$\frac{1}{n} \sum_{i=1}^n y_i \longrightarrow 0. \quad (1)$$

The space X is said to satisfy the *weak Banach-Saks property*, in short $X \in (wBS)$, if every weakly null sequence $\{x_n\}$ in X is a weak Banach-Saks sequence. Recall that a Banach space X satisfies the *Banach-Saks property* (BS) if for every bounded sequence $\{x_n\}$ in X , there is a subsequence $\{y_j\}$ such that its Cesàro means converge; that is, the sequence $\{(1/m) \sum_{j=1}^m y_j\}$ is convergent in norm. It is well known that a Banach space has the (BS)-property if and only if it is reflexive and it has the (wBS)-property [1].

Schreier constructed a separable Banach space without the weak Banach-Saks property [2] (which is called a Schreier space) and later it was shown by Baernstein [3] that there is a reflexive separable Banach space without the weak Banach-Saks property. Hence the space ℓ_∞ does not have the weak Banach-Saks property, since every separable space is embeddable in ℓ_∞ .

Following Johnson [4], given $1 \leq p \leq \infty$, a bounded sequence $\{x_n\} \subset X$ is called a *BS_p sequence* if there exists a subsequence $\{y_n\} \subset \{x_n\}$ such that

$$\sup_{m \in \mathbb{N}} m^{-1/p} \left\| \sum_{i=1}^m y_i \right\| < \infty. \quad (2)$$

Here $m^{1/\infty} = 1$ for all $m \in \mathbb{N}$. We then say that X has the *p-Banach-Saks property*, shortly $X \in (BS_p)$, if each weakly null sequence contains a *BS_p* subsequence. It is clear that every Banach space has a *(BS₁)* property. The *Banach-Saks index* is the number $\gamma(X) = \sup\{p : X \in (BS_p)\}$. The set $\Gamma(X) = \{p : X \in (BS_p)\}$ is always an interval $[1, \gamma(X))$ or $[1, \gamma(X)]$. The Bessaga-Pełczyński method [5, Proposition 1.a.12] shows that $\Gamma(\ell_p) = [1, p]$ and the Schur property of ℓ_1 shows that $\Gamma(\ell_1) = [1, \infty]$. The Banach-Saks index of Nakano sequence space was computed in [6].

The stronger property (S_p) was introduced by Knaust and Odell in [7]. It is said that X has property (S_p) , $1 < p \leq \infty$, if every weakly null sequence in X has a subsequence $\{x_k\}$ so that there is a constant $C > 0$ such that, for all $m \in \mathbb{N}$ and all real sequences $a = (a_n) \in \ell_p$,

$$\left\| \sum_{j=1}^m a_j x_j \right\| \leq C \|a\|_p, \quad (3)$$

where $\|a\|_p$ is the ℓ_p norm.

It is clear that $(S_p) \Rightarrow (BS_p) \Rightarrow (wBS)$ for all $1 < p \leq \infty$. The Elton c_0 -theorem [8, Theorem III 3.5] states that $(BS_\infty) \Leftrightarrow (S_\infty)$ and a Banach space X has (BS_∞) if and only if every normalized weakly null sequence contains a subsequence which is equivalent to the unit vector basis of c_0 , which is equivalent to the case in which every subspace of X has the Dunford-Pettis property (for details see [9]). In general, the two properties (BS_p) and (S_p) are not equivalent if $1 < p < \infty$ [10]; however, Rakov [11, Theorem 3] showed that if $1 < q < p < \infty$, then (BS_p) implies (S_q) . Various weak Banach-Saks properties and their relations were shown in [6] for Musielak-Orlicz sequence spaces.

In this paper we compute the Banach-Saks index of Orlicz-Lorentz space $\Lambda_{\varphi,w}(I)$ and show that Orlicz-Lorentz function space $\Lambda_{\varphi,w}(I)$ has the weak Banach-Saks property (resp., Banach-Saks property) if and only if it is separable (resp., reflexive). We start with several preliminary results about Orlicz-Lorentz spaces.

2. Preliminaries

Let $I = \mathbb{N}$ or $I = (0, a)$ with $0 < a \leq \infty$, equipped with the counting measure if $I = \mathbb{N}$ and the Lebesgue measure if $I = (0, a)$. If $A \subset I$ is a measurable set, we denote by $|A|$ its measure. By $L^0(I)$ we denote the collection of all real valued measurable functions on I . In the case when $I = \mathbb{N}$ the elements are sequences $x = (x(n))$, and in the other cases they are real valued Lebesgue measurable functions x .

For a measurable function x , the decreasing rearrangement of x will be denoted by x^* , defined as $x^*(t) = \inf\{s > 0 : d_{|x|}(s) \leq t\}$, $t \geq 0$, where $d_x(s) = |\{t : x(t) \geq s\}|$, $s \geq 0$. For two measurable functions x and y , the relation $x \ll y$ means that, for all $t \geq 0$, we have

$$\int_0^t x^*(s) ds \leq \int_0^t y^*(s) ds. \quad (4)$$

For a sequence $x = (x(n))$, x^* will denote the decreasing rearrangement of x . That is, letting $\tilde{x}(t) = \sum_{n=1}^\infty x(n) \chi_{[n-1,1)}$, we have $x^*(n) = \tilde{x}^*(n-1)$ for all $n \in \mathbb{N}$. For sequences x and y , $x \ll y$ means that for all $n \in \mathbb{N}$, we have

$$\sum_{k=1}^n x^*(k) \leq \sum_{k=1}^n y^*(k). \quad (5)$$

Let $\varphi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an Orlicz function; that is, φ is convex, strictly increasing, and $\varphi(0) = 0$. Let the weight function $w \in L^0(I)$ be nonnegative and nonincreasing. For $I = (0, a)$, $0 < a \leq \infty$, the Orlicz-Lorentz space $\Lambda_{\varphi,w}(I) = \Lambda_{\varphi,w}(0, a)$ is the collection of all functions $f \in L^0(0, a)$ such that

$$\|f\| = \|f\|_{\varphi,w} = \inf \left\{ \epsilon > 0 : I \left(\frac{f}{\epsilon} \right) \leq 1 \right\} < \infty, \quad (6)$$

where $I(f) = I_{\varphi,w}(f) = \int_I \varphi(f^*)w$. Analogously in case when $I = \mathbb{N}$, the Orlicz-Lorentz sequence space $d(\varphi, w) = \Lambda_{\varphi,w}(\mathbb{N})$ contains all real sequences $x = (x(n))$ such that

$$\|x\| = \|x\|_{\varphi,w} = \inf \left\{ \epsilon > 0 : I \left(\frac{x}{\epsilon} \right) \leq 1 \right\} < \infty, \quad (7)$$

where $I(x) = I_{\varphi,w}(x) = \sum_{n=1}^\infty \varphi(x^*(n))w(n)$. It is well known that the space $(\Lambda_{\varphi,w}(I), \|\cdot\|)$ is a rearrangement invariant (r.i.) Banach function space satisfying the Fatou property [12–15]. The space $\Lambda_{\varphi,w}^0(0, a)$ (resp., $d^0(\varphi, w)$) is the order continuous part of $\Lambda_{\varphi,w}(0, a)$ (resp., $d(\varphi, w)$). In the sequence case $d^0(\varphi, w)$ is the closure of the linear span of the unit vectors (e_i) in $d(\varphi, w)$. If $1 \leq p < \infty$ and $\varphi(u) = u^p$, then $d(\varphi, w) = d(p, w)$ and $\Lambda_{\varphi,w}(0, a) = \Lambda_{p,w}(0, a)$ [16]. If $w \equiv 1$, then $\Lambda_{\varphi,w}(0, a) = L_\varphi(0, a)$ is an Orlicz function space, and $d(\varphi, w) = \ell_\varphi$ is an Orlicz sequence space. In this case also $\Lambda_{\varphi,w}^0(0, a) = L_\varphi^0(0, a)$ and $d^0(\varphi, w) = \ell_\varphi^0$ are subspaces of order continuous elements in Orlicz spaces.

We may assume that $\varphi(1) = 1$. In fact, the space $\Lambda_{\psi,w}(I)$, where $\psi(t) = \varphi(\alpha t)$ with $\varphi(\alpha) = 1$, is isometric to $\Lambda_{\varphi,w}(I)$; we have $\|f\|_{\psi,w} = \alpha \|f\|_{\varphi,w}$.

Definition 1. Given an Orlicz function φ , one defines the growth conditions connected to its lower and upper Matuszewska-Orlicz indices in three different categories, at zero, at infinity, and on $\mathbb{R}_+$.

We say that, for $1 \leq p < \infty$, $\varphi \in \Delta_0^{*p}$, $\varphi \in \Delta_\infty^{*p}$, and $\varphi \in \Delta^p$, whenever

$$\sup_{\substack{0 < a \leq 1 \\ 0 < t < 1}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty, \quad \sup_{\substack{0 < a \leq 1 \\ t > 1}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty, \quad (8)$$

$$\sup_{\substack{0 < a \leq 1 \\ t > 0}} \frac{\varphi(at)}{\varphi(t)a^p} < \infty,$$

respectively. Analogously we say that, given $1 \leq q < \infty$, $\varphi \in \Delta_0^q$, $\varphi \in \Delta_\infty^q$, and $\varphi \in \Delta^q$, whenever

$$\inf_{\substack{0 < a \leq 1 \\ 0 < t < 1}} \frac{\varphi(at)}{\varphi(t)a^q} > 0, \quad \inf_{\substack{0 < a \leq 1 \\ t > 1}} \frac{\varphi(at)}{\varphi(t)a^q} > 0, \quad (9)$$

$$\inf_{\substack{0 < a \leq 1 \\ t > 0}} \frac{\varphi(at)}{\varphi(t)a^q} > 0,$$

respectively.

The Matuszewska-Orlicz indices, lower α_φ^0 , α_φ^∞ , and α_φ and upper β_φ^0 , β_φ^∞ , and β_φ , are then defined as follows:

$$\begin{aligned} \alpha_\varphi^0 &= \sup \{p : \varphi \in \Delta_0^{*p}\}, & \alpha_\varphi^\infty &= \sup \{p : \varphi \in \Delta_\infty^{*p}\}, \\ \alpha_\varphi &= \sup \{p : \varphi \in \Delta^{*p}\}, \\ \beta_\varphi^0 &= \inf \{q : \varphi \in \Delta_0^q\}, & \beta_\varphi^\infty &= \inf \{q : \varphi \in \Delta_\infty^q\}, \\ \beta_\varphi &= \inf \{q : \varphi \in \Delta^q\}. \end{aligned} \quad (10)$$

It is clear that $\alpha_\varphi = \min(\alpha_\varphi^0, \alpha_\varphi^\infty)$ and $\beta_\varphi = \max(\beta_\varphi^0, \beta_\varphi^\infty)$. We say that φ satisfies conditions Δ_2^0 , Δ_2^∞ , and Δ_2 , whenever

$$\sup_{0 < t < 1} \frac{\varphi(2t)}{\varphi(t)} < \infty, \quad \sup_{t > 1} \frac{\varphi(2t)}{\varphi(t)} < \infty, \quad \sup_{t > 0} \frac{\varphi(2t)}{\varphi(t)} < \infty, \quad (11)$$

respectively. Let φ_1 and φ_2 be Orlicz functions. We call them *equivalent at zero*, *equivalent at infinity*, or just *equivalent*, if there is $D > 0$ such that $\varphi_1(D^{-1}t) \leq \varphi_2(t) \leq \varphi_1(Dt)$, for all $0 < t < 1$, $t > 1$, or for all $t > 0$, respectively. It is well known and easy to prove that Matuszewska-Orlicz indices of Orlicz functions are preserved by the corresponding equivalence relations (e.g., see [17]); that is, if φ_1 and φ_2 are equivalent (resp., equivalent at 0 and equivalent at ∞), then $\alpha_{\varphi_1} = \alpha_{\varphi_2}$ (resp., $\alpha_{\varphi_1}^0 = \alpha_{\varphi_2}^0$ and $\alpha_{\varphi_1}^\infty = \alpha_{\varphi_2}^\infty$). Similar equalities hold for upper indices. Recall also that any upper index of φ is finite if and only if φ satisfies the corresponding condition Δ_2 [17, 18].

We also have that if the Orlicz functions φ_1 and φ_2 are equivalent (resp., equivalent at infinity and equivalent at zero), then $\Lambda_{\varphi_1, w}(0, \infty) = \Lambda_{\varphi_2, w}(0, \infty)$ (resp., $\Lambda_{\varphi_1, w}(0, a) = \Lambda_{\varphi_2, w}(0, a)$, $a < \infty$; $d(\varphi_1, w) = d(\varphi_2, w)$) with equivalent norms.

The space $\Lambda_{\varphi, w}(0, \infty)$ (resp., $\Lambda_{\varphi, w}(0, a)$ ($a < \infty$); $d(\varphi, w)$) is separable if and only if $\varphi \in \Delta_2$ and $W(\infty) = \infty$ (resp., $\varphi \in \Delta_2^\infty$; $\varphi \in \Delta_2^0$ and $W(\infty) = \infty$), if $a = \infty$ (resp., $a < \infty$; $I = \mathbb{N}$). The separability is also equivalent to the case in which the space $\Lambda_{\varphi, w}(0, \infty)$ does not contain the isomorphic copy of ℓ_∞ [12, 15, 19].

Lemma 2. Let $1 \leq p, q < \infty$;

- (1) ([20, Lemma 6], [21, Lemma 3], and [22]) An Orlicz function $\varphi \in \Delta^{*p}$ (resp., $\varphi \in \Delta_\infty^{*p}$; $\varphi \in \Delta_0^{*p}$) if and only if there exists an Orlicz function ψ equivalent (resp., equivalent at ∞ ; equivalent at 0) to φ such that $\psi(t^{1/p})$ is convex;
- (2) ([20, Lemma 5], [21, Lemma 2], and [22]) An Orlicz function $\varphi \in \Delta^q$ (resp., $\varphi \in \Delta_\infty^q$; $\varphi \in \Delta_0^q$) if and only if there exists an Orlicz function ψ equivalent (resp., equivalent at ∞ and equivalent at 0) to φ such that $\psi(t^{1/q})$ is concave.

We will use the following functions:

$$\begin{aligned} W(t) &= \int_0^t w(s) ds \quad \text{for } t \in (0, a), \text{ or} \\ W(m) &= \sum_{i=1}^m w(i) \quad \text{for } m \in \mathbb{N}. \end{aligned} \quad (12)$$

Recall that the weight w is *regular* whenever there exists $C > 0$ such that $W(s) \leq Csw(s)$ for all $s \in I$. Since always $sw(s) \leq W(s)$, the weight w is regular if the functions $W(s)$ and $sw(s)$ are equivalent on I . Notice that for any regular weight w on $I = (0, \infty)$ or $I = \mathbb{N}$ it holds that $W(\infty) = \infty$.

A Banach lattice is said to be *p-convex* (resp., *p-concave*) for some $1 \leq p < \infty$ if there is a constant $K > 0$ such that

$$\begin{aligned} \left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\| &\leq K \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \\ \left(\text{resp. } \left(\sum_{i=1}^n \|x_i\|^p \right)^{1/p} \right) &\leq K \left\| \left(\sum_{i=1}^n |x_i|^p \right)^{1/p} \right\| \end{aligned} \quad (13)$$

for every choice of vectors $x_1, \dots, x_n$ in X . A Banach lattice is said to satisfy *upper p-estimate* (resp., *lower p-estimate*) if the definition of *p-convexity* (resp., *p-concavity*) holds true for any choice of disjointly supported elements $x_1, \dots, x_n$ in X . It is known that, given $1 \leq p < \infty$, if X is *p-convex* (resp., *p-concave*), then X is *r-convex* (resp., *r-concave*) for $1 \leq r \leq p$ (resp., $r \geq p$) [23, 24].

Theorem 3 (see [25]). The space $\Lambda_{\varphi, w}(0, \infty)$ (resp., $\Lambda_{\varphi, w}(0, a)$ ($a < \infty$); $d(\varphi, w)$) has finite concavity if and only if φ satisfies the Δ_2 (resp., Δ_2^∞ ; Δ_2^0) condition and the weight w is regular.

Recall that a Banach lattice has the finite concavity if and only if it is of finite cotype [26].

Theorem 4 (see [19]). Let $1 \leq p \leq \infty$;

- (1) [19, Theorem 7.18] For $I = (0, \infty)$, the following conditions are equivalent:

- (i) ℓ^p (replaced by c_0 when $p = \infty$) is order isomorphic to a sublattice of $\Lambda_{\varphi, w}^0(0, \infty)$,
- (ii) either $p \in [\alpha_\varphi^0, \beta_\varphi^0] \cup [\alpha_\varphi^\infty, \beta_\varphi^\infty] = [\alpha_\varphi, \beta_\varphi]$ or, for some $c > 0$, $\int_0^\infty \varphi(ct^{-1/p})w(t)dt < \infty$;

- (2) [19, Theorem 6.13] For $I = (0, a)$, $a < \infty$, the following assertions are equivalent:

- (i) ℓ^p (replaced by c_0 when $p = \infty$) is order isomorphic to a sublattice of $\Lambda_{\varphi, w}^0(0, a)$,
- (ii) $p \in [\alpha_\varphi^\infty, \beta_\varphi^\infty]$,
- (iii) ℓ^p (replaced by c_0 when $p = \infty$) is order isomorphic to a sublattice of $L_\varphi^0(0, a)$;

- (3) [19, Theorem 7.8] For $I = \mathbb{N}$ the following statements are equivalent (where c_0 is meant in place of ℓ^p if $p = \infty$):

- (i) ℓ^p is order isomorphic to a sublattice of $d_0(w, \varphi)$,
- (ii) ℓ^p is isomorphic to a subspace of $d_0(w, \varphi)$,
- (iii) $p \in [\alpha_\varphi^0, \beta_\varphi^0]$,
- (iv) ℓ^p is isomorphic to a subspace of h_φ .

3. Main Results

Theorem 5. If $1 < p < \infty$ and $\varphi \in \Delta^{*p}$ (resp., $\varphi \in \Delta_\infty^{*p}$ and $\varphi \in \Delta_0^{*p}$), then $\Lambda_{\varphi, w}(0, \infty)$ (resp., $\Lambda_{\varphi, w}(0, a)$, $a < \infty$; $d(\varphi, w)$) is *p-convex*. If $\Lambda_{\varphi, w}(0, \infty)$ (resp., $\Lambda_{\varphi, w}(0, a)$, $a < \infty$; $d(\varphi, w)$) is *p-convex* for some $1 < p < \infty$, then $p \leq \alpha_\varphi$ (resp., $p \leq \alpha_\varphi^\infty$; $p \leq \alpha_\varphi^0$).

Proof. (1) By Lemma 2 we assume that $\psi(t) = \varphi(t^{1/p})$ is convex. Then we have the equality $\|f\|_{\Lambda_{\psi,w}^{(p)}(I)} = \|f\|_{\varphi,w}$, where $\Lambda_{\psi,w}^{(p)}(I)$ is a p -convexification of $\Lambda_{\psi,w}(I)$. Indeed,

$$\begin{aligned} \|f\|_{\Lambda_{\psi,w}^{(p)}(I)} &= \| |f|^p \|_{\psi,w}^{1/p} \\ &= \inf \left\{ \epsilon^{1/p} > 0 : \int_I \psi \left(\frac{(|f|^p)^*}{\epsilon} \right) \leq 1 \right\} \\ &= \inf \left\{ \epsilon^{1/p} > 0 : \int_I \varphi \left(\frac{f^*}{\epsilon^{1/p}} \right) \leq 1 \right\} = \|f\|_{\varphi,w}. \end{aligned} \quad (14)$$

Since ψ is a convex Orlicz function, the space $\Lambda_{\psi,w}(I)$ is a Banach space, so its p -convexification $\Lambda_{\psi,w}^{(p)}(I) = \Lambda_{\varphi,w}(I)$ is p -convex [26].

We only show the second part in the case of $I = (0, \infty)$. If $p > \alpha_\varphi = \min(\alpha_\varphi^0, \alpha_\varphi^\infty)$, then, by Theorem 4(1), there is $\alpha_\varphi \leq r < p$ such that ℓ^r is an order copy in $\Lambda_{\varphi,w}(0, \infty)$, and thus the space cannot be p -convex. $\square$

Example 6. In general, the parallel statement on concavity in Theorem 5 does not hold. As shown by Pisier in [26, Example 1.f.19], the Lorentz space $d(p, w)$, where $w_n = n^{1/p} - (n-1)^{1/p}$ and $1 < p < 2$, is not 2-concave even though the function $\varphi(t) = t^p$ satisfies condition Δ_2 and w is regular.

Proposition 7. Let $1 < p \leq 2$. If $\varphi \in \Delta^{*p}$ and $\varphi \in \Delta_2$ if $a = \infty$ (resp., $\varphi \in \Delta_\infty^{*p}$ and $\varphi \in \Delta_2^\infty$ in the case when $a < \infty$) and w is regular, then $\Lambda_{\varphi,w}(0, a)$, $0 < a \leq \infty$, has type p .

Proof. By Theorem 5 the space is p -convex, and thus in view of Theorem 3 and by applying the well-known fact [26, Proposition 1.f.3], the space has type p . $\square$

Rakov [27, Theorem 1] showed that if a Banach space has type $1 < p \leq 2$, then it has p -Banach-Saks property. Hence we get the following.

Theorem 8. If a Banach space X has type p for some $1 < p \leq 2$, then $X \in (BS_p)$. Consequently $p \leq \gamma(X)$.

Notice that for every separable r.i. space E on $[0, 1]$, it is shown in [28] that $\gamma(E) \leq 2$. Because every r.i. function space X on $[0, \infty)$ contains the r.i. space $\tilde{X} = \{f\chi_{[0,1]} : f \in X\}$ isometrically, we get the following.

Proposition 9 (see [29]). If X is a separable r.i. space on $(0, a)$, $0 < a \leq \infty$, then $\gamma(X) \leq 2$.

Recall that a Banach space X is said to be p -smooth ($1 \leq p \leq 2$) if its modulus of smoothness $\rho_X(\epsilon)$ is majorized by the function ϵ^p . That is, there is a constant $A > 0$ such that, for all $\epsilon > 0$,

$$\rho_X(\epsilon) := \inf_{\{\|x\|=\|y\|=1\}} \frac{\|x + \epsilon y\| + \|x - \epsilon y\| - 2}{2} \leq A\epsilon^p. \quad (15)$$

Rakov [27, Theorem 1] showed that if X is a p -smooth space, then it has the (S_p) property. It is well known [26] that a Banach lattice is isomorphic to a p -smooth Banach space for some $1 < p \leq 2$ if and only if it has the type p and satisfies a lower q -estimate for some $1 \leq q < \infty$. Notice also that the finite concavity is equivalent to the lower q -estimate for some $q < \infty$ [26]. Then we obtain immediately the following corollary as a consequence of definition of lower indices and by Propositions 7 and 9 and Theorem 8.

Corollary 10. Let $1 < p \leq 2$. If $\varphi \in \Delta^{*p}$ and $\varphi \in \Delta_2$ if $a = \infty$ (resp., $\varphi \in \Delta_\infty^{*p}$ and $\varphi \in \Delta_2^\infty$ in the case when $a < \infty$) and w is regular, then $\Lambda_{\varphi,w}(0, a)$, $0 < a \leq \infty$, has the S_p -property.

Consequently, if $\varphi \in \Delta_2$ when $a = \infty$ (resp., $\varphi \in \Delta_2^\infty$ in the case when $a < \infty$) and w is regular, then

$$\begin{aligned} \min(2, \alpha_\varphi) &\leq \gamma(\Lambda_{\varphi,w}(0, \infty)), \\ \min(2, \alpha_\varphi^\infty) &\leq \gamma(\Lambda_{\varphi,w}(0, a)), \\ a &< \infty. \end{aligned} \quad (16)$$

The next corollary is a result of the facts that $\gamma(\ell^\infty) = 1$ and that if φ does not satisfy appropriate condition Δ_2 , then $\Lambda_{\varphi,w}(I)$ contains an isomorphic copy of ℓ^∞ .

Corollary 11. If $\varphi \notin \Delta_2$ when $a = \infty$, $\varphi \notin \Delta_2^\infty$ in the case when $a < \infty$, and $\varphi \notin \Delta_2^0$ when $I = \mathbb{N}$, then

$$\begin{aligned} \gamma(\Lambda_{\varphi,w}(0, \infty)) &= 1, \quad \gamma(\Lambda_{\varphi,w}(0, a)) = 1, \\ a &< \infty, \quad \gamma(d(\varphi, w)) = 1. \end{aligned} \quad (17)$$

Corollary 12. An Orlicz-Lorentz space $\Lambda_{\varphi,w}^0(0, \infty)$ (resp., $\Lambda_{\varphi,w}^0(0, a)$, $a < \infty$; $d^0(w, \varphi)$) does not contain an isomorphic copy of ℓ^1 if and only if $\alpha_\varphi > 1$ (resp., $\alpha_\varphi^\infty > 1$; $\alpha_\varphi^0 > 1$).

Proof. This hypothesis for sequence spaces is an immediate consequence of Theorem 4(3).

Let now $a < \infty$. From [19, Corollary 3.5(2), (3)], the space $\Lambda_{\varphi,w}^0(0, a)$ contains an isomorphic copy of ℓ^1 if and only if either ℓ^1 is an order isomorphic copy, that is, in view of Theorem 4(2), $\alpha_\varphi^\infty = 1$, or the sequence of functions $\min(t^{-1}, n)$ is bounded in $L_\varphi(0, a)$. However $\alpha_\varphi^\infty > 1$ implies that there exists $p > 1$ such that $\varphi(\beta u) \geq K\beta^p \varphi(u)$ for all $u, \beta \geq 1$ and some $K > 0$. Hence for any constant $c > 0$ and large enough $n \in \mathbb{N}$,

$$\begin{aligned} \int_0^a \varphi(c \min(t^{-1}, n)) dt &\geq \int_0^{1/n} \varphi(cn) dt \\ &\geq \int_0^{1/n} Kc^p n^p \varphi(1) dt \\ &= K\varphi(c) n^{p-1} \rightarrow \infty. \end{aligned} \quad (18)$$

Thus whenever $\alpha_\varphi^\infty > 1$ the sequence $\min(t^{-1}, n)$ is not bounded, and the proof is done in case of finite interval.

Assume that $a = \infty$. Again applying Corollary 3.5(2) in [19], we get dichotomy that ℓ^1 is an isomorphic copy in $\Lambda_{\varphi,w}^0(0, \infty)$ if and only if either ℓ^1 is an order isomorphic copy in $\Lambda_{\varphi,w}^0(0, \infty)$ or the sequence $\min(t^{-1}\chi_{(0,1)}(t), n)$ is bounded in $L_{\varphi}(0, \infty)$. Assuming now that $\alpha_{\varphi} > 1$, we can show similarly as above that the second alternative cannot occur. Recall then that, by Theorem 4(1), ℓ^1 is an order isomorphic copy in the space if and only if either $\alpha_{\varphi} = 1$ or, for some $c > 0$, $\int_0^{\infty} \varphi(ct^{-1})w(t)dt < \infty$. We will exclude again the second possibility. In fact if $\alpha_{\varphi} > 1$, then for some $p > 1$ and $K > 0$ and for all $\beta \geq 1$ and $u \geq 0$, $\varphi(\beta u) \geq K\beta^p\varphi(u)$. Therefore for every $c > 0$ and some $t_0 < 1$ such that $w(t_0) > 0$, we have

$$\begin{aligned} \int_0^{\infty} \varphi(ct^{-1})w(t)dt &\geq \int_0^{t_0} \varphi(ct^{-1})w(t_0)dt \\ &\geq K\varphi(c)w(t_0) \int_0^{t_0} t^{-p}dt = \infty, \end{aligned} \quad (19)$$

and this completes the proof. $\square$

Corollary 13. Assume that $\Lambda_{\varphi,w}(0, a)$, $0 < a \leq \infty$, does not contain an isomorphic copy of ℓ^1 , equivalently that $\alpha_{\varphi} > 1$ in case $a = \infty$ and $\alpha_{\varphi}^{\infty} > 1$ in case $a < \infty$. Then

$$\begin{aligned} \gamma(\Lambda_{\varphi,w}(0, \infty)) &\leq \min(2, \alpha_{\varphi}), \\ \gamma(\Lambda_{\varphi,w}(0, a)) &\leq \min(2, \alpha_{\varphi}^{\infty}), \\ a &< \infty. \end{aligned} \quad (20)$$

Proof. By Proposition 9 the Banach-Saks index of any r.i. space over the interval $(0, a)$ cannot be bigger than 2. On the other hand, recall that $\Gamma(\ell_p) = [1, p]$ for $1 < p < \infty$ and $\Gamma(c_0) = \Gamma(\ell_1) = [1, \infty]$. The space ℓ_{∞} does not have wBS property, so $\Gamma(\ell_{\infty}) = 1$. Hence Theorem 4 shows that

$$\begin{aligned} \gamma(\Lambda_{\varphi,w}(0, a)) &\leq A \\ &:= \inf \{p > 1 : \ell^p \text{ is an isomorphic copy in } \Lambda_{\varphi,w}(0, a)\}, \end{aligned} \quad (21)$$

where $A \leq \alpha_{\varphi}$ when $a = \infty$ and $A \leq \alpha_{\varphi}^{\infty}$ if $a < \infty$. $\square$

Finally by Corollaries 10 and 11 we get the result on Banach-Saks index in the Orlicz-Lorentz spaces over the interval $(0, a)$, $a \leq \infty$.

Theorem 14. Assume that $\Lambda_{\varphi,w}(0, a)$, $0 < a \leq \infty$, does not contain an isomorphic copy of ℓ^1 and equivalently that $\alpha_{\varphi} > 1$ in case $a = \infty$ and $\alpha_{\varphi}^{\infty} > 1$ in case $a < \infty$. Assume also that

$\varphi \in \Delta_2$ when $a = \infty$, $\varphi \in \Delta_2^{\infty}$ when $a < \infty$, and w is regular. Then

$$\begin{aligned} \min(2, \alpha_{\varphi}) &= \gamma(\Lambda_{\varphi,w}(0, \infty)), \\ \min(2, \alpha_{\varphi}^{\infty}) &= \gamma(\Lambda_{\varphi,w}(0, a)), \\ a &< \infty. \end{aligned} \quad (22)$$

Using the Bessaga-Pełczyński method, we get the following.

Theorem 15. Let $1 < p < \infty$. If an order continuous sequence space X is p -convex, then X has the (S_p) property. Consequently $p \leq \gamma(X)$.

Proof. Since X is order continuous, the standard unit vectors $\{e_n\}$ form a basis of X . Suppose that $\{x_n\}$ is a normalized weakly null sequence in X . By the Bessaga-Pełczyński method [5, Proposition 1.a.12], we may assume that $x_n = y_n + z_n$ for all n , where $\{y_n\}$ is a block basic sequence of $\{e_n\}$ and $\|z_n\| \leq 1/2^n$ for all n . Then for real numbers $a_1, \dots, a_n$ with $\sum_{j=1}^n |a_j|^p \leq 1$, the p -convexity of X and disjointness of the supports of y_n 's show that

$$\begin{aligned} \left\| \sum_{j=1}^n a_j x_j \right\| &\leq \left\| \sum_{j=1}^n a_j y_j \right\| + \sum_{j=1}^n |a_j| \|z_j\| \\ &\leq \left\| \left(\sum_{j=1}^n |a_j y_j|^p \right)^{1/p} \right\| + \left(\sum_{j=1}^n |a_j|^p \right)^{1/p} \left(\sum_{j=1}^n \frac{1}{2^{jq}} \right)^{1/q} \\ &\leq 2C \left(\sum_{j=1}^n |a_j|^p \right)^{1/p} + \left(\sum_{j=1}^n |a_j|^p \right)^{1/p} \left(\sum_{j=1}^n \frac{1}{2^{jq}} \right)^{1/q} \\ &\leq (2C + 1) \left(\sum_{j=1}^n |a_j|^p \right)^{1/p}, \end{aligned} \quad (23)$$

where C is the constant from the p -convexity of X and q is the conjugate of p satisfying $1/p + 1/q = 1$. The proof is done. $\square$

As a consequence of the above Theorems 15 and 5 about p -convexity and Theorem 4 about ℓ^p -copies, we get the Banach-Saks index in sequence spaces as follows.

Theorem 16. Assume that $d(\varphi, w)$ does not contain an isomorphic copy of ℓ^1 and equivalently that $\alpha_{\varphi}^0 > 1$. Assume also that $\varphi \in \Delta_2^0$ and $W(\infty) = \infty$. Then

$$\gamma(d(\varphi, w)) = \alpha_{\varphi}^0. \quad (24)$$

Proof. By Theorems 5 and 15, $\alpha_{\varphi}^0 \leq \gamma(d(\varphi, w))$ and

$$\gamma(d(\varphi, w)) \leq \inf \{p > 1 : \ell^p \subset d(\varphi, w)\}. \quad (25)$$

Since $\gamma(\ell_p) = p$ ($1 < p < \infty$), Theorem 4 completes the proof. $\square$

We finish with a result on the weak Banach-Saks property.

Theorem 17. A separable Orlicz-Lorentz space $\Lambda_{\varphi,w}(0, a)$, $a \leq \infty$, has the weak Banach-Saks property.

Proof. Let $\{x_n\} \subset \Lambda_{\varphi,w}(0, a)$ be weakly null and assume without loss of generality that $\|x_n\| = 1$, $n \in \mathbb{N}$. Then by [29, Proposition 3.2] there exists a subsequence $\{x'_n\}$ splitted into three subsequences

$$x'_n = y_n + z_n + d_n, \quad (26)$$

such that $\{y_n\}, \{z_n\}, \{d_n\} \subset \Lambda_{\varphi,w}(0, a)$ are bounded and

- (a) $y_1^* = y_n^*$, $\|y_n\| \leq 1$, and $y_n \rightarrow 0$ weakly;
- (b) $z_n z_m = 0$ for $n \neq m$, $\|z_n\| \leq 2$, $z_n \rightarrow 0$ in measure, and $z_n \rightarrow 0$ weakly;
- (c) $\|d_n\| \rightarrow 0$.

We need to show that there is a subsequence of $\{x'_n\}$ whose Cesàro means approach zero in norm. It is obvious that we can do it for the sequence $\{d_n\}$ satisfying (c). We will show this now for the sequence $\{y_n\}$. By [29, Proposition 4.3] there exists a subsequence $\{y_{n(k)}\} \subset \{y_n\}$, $x \in L_1(0, a) + L_\infty(0, a)$, such that

$$\frac{1}{N} \sum_{k=1}^N y_{m(k)} \rightarrow x \quad (27)$$

in $L_1(0, a) + L_\infty(0, a)$ for all further subsequences $\{y_{m(k)}\} \subset \{y_{n(k)}\}$ (we can assume that $y_{m(1)} = y_1$). Hence

$$\frac{1}{N} \sum_{k=1}^N y_{m(k)} \rightarrow x \quad \text{in measure on } (0, a). \quad (28)$$

(Recall that if $x_n \rightarrow 0$ in any r.i. space X , then $x_n \rightarrow 0$ in measure.) Since $y_{n(k)} \rightarrow 0$ weakly in $\Lambda_{\varphi,w}(0, a)$, we have, for any subsequence $\{m(k)\}$ of $\{n(k)\}$,

$$\frac{1}{N} \sum_{k=1}^N y_{m(k)} \rightarrow 0 \quad \text{weakly in } L_1(0, a) + L_\infty(0, a). \quad (29)$$

Applying now [29, Lemma 4.4] we get that

$$\frac{1}{N} \sum_{k=1}^N y_{m(k)} \rightarrow 0 \quad \text{in } L_1(0, a) + L_\infty(0, a) \quad (30)$$

for all subsequences $\{y_{m(k)}\} \subset \{y_{n(k)}\}$. Let now $\{y_{m(k)}\}$ be an arbitrary subsequence of $\{y_{n(k)}\}$. Set

$$a_N = \frac{1}{N} \sum_{k=1}^N y_{m(k)}. \quad (31)$$

By $y_{m(k)}^* = y_1^*$ and by subadditivity of the operator $t \mapsto \int_0^t f^*$, $f \in L^0(I)$, we get, for $t \geq 0$,

$$\int_0^t \left(\frac{1}{N} \sum_{k=1}^N y_{m(k)} \right)^* \leq \frac{1}{N} \sum_{k=1}^N \int_0^t y_{m(k)}^* = \int_0^t y_1^*; \quad (32)$$

that is,

$$a_N \prec\prec y_1. \quad (33)$$

Since $a_N \rightarrow 0$ in measure and $a_N \prec\prec y_1$ for every $N \in \mathbb{N}$, by [29, Proposition 4.2],

$$\frac{1}{N} \sum_{k=1}^N y_{m(k)} \rightarrow 0 \quad (34)$$

in $\Lambda_{\varphi,w}(0, a)$, which finishes the case (a).

In the case (b), by [19, Theorem 5.5], there exists a subsequence $\{z_{n(k)}\} \subset \{z_n\}$ such that any further subsequence $\{z_{m(k)}\}$ of $\{z_{n(k)}\}$ is equivalent to the unit vector basis in some Orlicz sequence space ℓ_ψ^0 . Thus $z_{n(k)} \rightarrow 0$ weakly in ℓ_ψ^0 , and by wBS -property of ℓ_ψ^0 (see, e.g., [27]), there is a subsequence $\{z'_k\} \subset \{z_{n(k)}\}$ such that $(1/n) \sum_{k=1}^N z'_k \rightarrow 0$ in ℓ_ψ^0 and thus in $\Lambda_{\varphi,w}(0, a)$. This completes case (b) and proves the theorem. $\square$

A Banach space X is said to have the (*resp.*, *weakly*) *alternative signs Banach-Saks property* (ABS) (*resp.*, (wABS)) if, for every bounded (*resp.*, weakly null) sequence $(x_n)_{n=1}^\infty$, there is a subsequence (y_n) and a sequence (ϵ_n) of signs such that the Cesàro averages are norm convergent to zero. That is,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left\| \sum_{i=1}^n \epsilon_i y_i \right\| = 0. \quad (35)$$

It is shown by Rosenthal [30, 31] that if X has the (ABS), then X has the weak Banach-Saks property. In case that X does not contain ℓ_1 , then (ABS) is equivalent to the weak Banach-Saks property [31]. It is clear that ℓ_1 does not have the (ABS) and has the weak Banach-Saks property. Hence a Banach space X has the (ABS) if and only if X has the weak Banach-Saks property and does not contain ℓ_1 . The notion of (wABS) was introduced in [32] and it was shown that (wABS) is equivalent to (wBS).

Corollary 18. Let $I = (0, a)$ for some $0 < a \leq \infty$. Then the following are equivalent:

- (1) Orlicz-Lorentz space $\Lambda_{\varphi,w}(I)$ has the Banach-Saks property (BS);
- (2) Orlicz-Lorentz space $\Lambda_{\varphi,w}(I)$ is reflexive;
- (3) Orlicz-Lorentz space $\Lambda_{\varphi,w}(I)$ does not contain isomorphic copies of c_0 or ℓ_1 ;
- (4) Orlicz-Lorentz space $\Lambda_{\varphi,w}(I)$ is separable and does not contain an isomorphic copy of ℓ_1 ;
- (5) Orlicz-Lorentz space $\Lambda_{\varphi,w}(I)$ has the weak Banach-Saks property (wBS) and does not contain an isomorphic copy of ℓ_1 ;
- (6) Orlicz-Lorentz space $\Lambda_{\varphi,w}(I)$ has the alternative signs Banach-Saks property (ABS).

Proof. Recall that a Banach lattice is reflexive if and only if it does not contain isomorphic copies of c_0 or ℓ_1

[26, Theorem 1.c.5]. Hence we get $(2) \Leftrightarrow (3)$. Recall also that a Banach lattice X is a KB space if and only if X does not contain an isomorphic copy of c_0 [33, Theorem 14.12]. That is, if X is a Köthe function space, then it is equivalent to the case in which X is order continuous and satisfies the Fatou property. Since an Orlicz-Lorentz space satisfies the Fatou property and the order continuity is equivalent to the separability of the Orlicz-Lorentz space, $(3) \Leftrightarrow (4)$ follows.

Since a nonseparable Orlicz-Lorentz space contains ℓ_∞ and it does not have the weak Banach-Saks property, Theorem 17 shows that $(4) \Leftrightarrow (5)$. The preceding argument of the corollary shows that $(5) \Leftrightarrow (6)$. Since (6) implies both (5) and (2) , (6) implies (1) . The proof is done. $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgment

Han Ju Lee was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A1006869).

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