Research Article

Integrated Fractional Resolvent Operator Function and Fractional Abstract Cauchy Problem

Ya-Ning Li and Hong-Rui Sun

School of Mathematics and Statistics, Lanzhou University, Lanzhou, Gansu 730000, China

Correspondence should be addressed to Hong-Rui Sun; hrsun@lzu.edu.cn

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We firstly prove that β -times integrated α -resolvent operator function ((α, β) -ROF) satisfies a functional equation which extends that of β -times integrated semigroup and α -resolvent operator function. Secondly, for the inhomogeneous α -Cauchy problem ${}^c D_t^\alpha u(t) = Au(t) + f(t)$, $t \in (0, T)$, $u(0) = x_0$, $u'(0) = x_1$, if A is the generator of an (α, β) -ROF, we give the relation between the function $v(t) = S_{\alpha, \beta}(t)x_0 + (g_1 * S_{\alpha, \beta})(t)x_1 + (g_{\alpha-1} * S_{\alpha, \beta} * f)(t)$ and mild solution and classical solution of it. Finally, for the problem ${}^c D_t^\alpha v(t) = Av(t) + g_{\beta+1}(t)x$, $t > 0$, $v^{(k)}(0) = 0$, $k = 0, 1, \dots, N-1$, where A is a linear closed operator. We show that A generates an exponentially bounded (α, β) -ROF on a Banach space X if and only if the problem has a unique exponentially bounded classical solution v_x and $Av_x \in L^1_{\text{loc}}(\mathbb{R}^+, X)$. Our results extend and generalize some related results in the literature.

1. Introduction

This paper is concerned with the properties of β -integrated α -resolvent operator function ((α, β) -ROF) and two inhomogeneous fractional Cauchy problems.

Throughout this paper, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{N}$ denotes the set of natural numbers. $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let X, Y be Banach spaces, $B(X, Y)$ denote the space of all bounded linear operators from X to Y , $B(X) = B(X, X)$. If A is a closed linear operator, $\rho(A)$ denotes the resolvent set of A and $R(\lambda, A) = (\lambda I - A)^{-1}$ denotes the resolvent operator of A . $L^1(\mathbb{R}^+, X)$ denotes the space of X -valued Bochner integrable functions: $u : \mathbb{R}^+ \rightarrow X$ with the norm $\|u\|_{L^1(\mathbb{R}^+, X)} = \int_0^\infty \|u(t)\| dt$, it is a Banach space. By $*$ we denote the convolution of functions

$$(f * g)(t) = \int_0^t f(t-\tau)g(\tau)d\tau, \quad t \geq 0. \quad (1)$$

g_α denotes the function

$$g_\alpha(t) = \begin{cases} \frac{t^{\alpha-1}}{\Gamma(\alpha)}, & t > 0, \\ 0, & t \leq 0, \end{cases} \quad (2)$$

and $g_0(t) = \delta_0(t)$, the Dirac delta function.

In 1997, Mijatović et al. [1] introduced the concept of β -times integrated semigroup ($\beta \in \mathbb{R}^+$) which extends k -times integrated semigroup ($k \in \mathbb{N}_0$) [2], they showed an $R(\lambda)$ to be the pseudoresolvent of a β -times ($\beta > 0$) integrated semigroup $\{S(t)\}$ if and only if $\{S(t)\}$ satisfies the following functional equation:

$$\int_t^{t+s} (s+t-r)^{\beta-1} S(r) dr - \int_0^s (s+t-r)^{\beta-1} S(r) dr = \Gamma(\beta) S(t) S(s), \quad t, s \geq 0. \quad (3)$$

In the special case of $\beta = k \in \mathbb{N}$, the corresponding result is summarized in [2].

For the inhomogeneous Cauchy problem

$$u'(t) = Au(t) + f(t), \quad t \in [0, T], \quad u(0) = x, \quad (4)$$

where $T > 0$, $f \in L^1([0, T], X)$, $x \in X$, and A is the generator of a k -times integrated semigroup $\{S(t)\}$ on a Banach space X for some $k \in \mathbb{N}_0$. Let $v(t) = S(t)x + \int_0^t S(t-s)f(s)ds$, $t \in [0, T]$. Lemmas 3.2.9 and 3.2.10 of [2] show that if there is a mild(classical) solution u of (4), then $v \in C^k([0, T], X)$ ($C^{k+1}([0, T], X)$) and $u = v^{(k)}$. On the other

hand, if $v \in C^k([0, T], X) (C^{k+1}([0, T], X))$, then $v^{(k)}$ is also a mild (classical) solution of it.

Furthermore, if A generates an exponential bounded k -times integrated semigroup on a Banach space X , then, for any $x \in X$, $v(t) = \int_0^t S(s)x ds$ is the unique exponential bounded classical solution of the following problem:

$$u'(t) = Au(t) + g_{k+1}(t)x, \quad t \geq 0, \quad u(0) = 0. \quad (5)$$

In recent years, a considerable interest has been paid to fractional evolution equation due to its applications in different areas such as stochastic, finance, and physics; see [3–8]. One of the most important tools in the theory of fractional evolution equation is the solution operator (fractional resolvent family) [9–15]. The notion of solution operator was developed to study some abstract Volterra integral equations [16] and was first used by Bajlekova [17] to study a class of fractional order abstract Cauchy problem. In [9], Chen and Li introduced α -resolvent operator functions (α -ROF for short) defined by purely algebraic equation. They showed that a family $\{S_\alpha(t)\}_{t \geq 0} \subset B(X)$ is an α -ROF if and only if $\{S_\alpha(t)\}_{t \geq 0}$ is a solution of abstract fractional Cauchy problem

$$\begin{aligned} {}^c D_t^\alpha v(t) &= Av(t) + g_{\beta+1}(t)x, \quad t > 0, \\ v(0) &= x, \quad v^{(k)}(0) = 0, \quad k = 1, \dots, N-1. \end{aligned} \quad (6)$$

When $0 < \alpha < 1$, Peng and Li [18] proved that the solution operator $\{S_\alpha(t)\}_{t \geq 0}$ for (6) satisfies the following equality:

$$\begin{aligned} \int_t^{t+s} \frac{S_\alpha(\tau)}{(t+s-\tau)^\alpha} d\tau - \int_0^s \frac{S_\alpha(\tau)}{(t+s-\tau)^\alpha} d\tau \\ = \alpha \int_0^t \int_0^s \frac{S_\alpha(\tau_1) S_\alpha(\tau_2)}{(t+s-\tau_1-\tau_2)^{1+\alpha}} d\tau_1 d\tau_2, \quad t, s \geq 0. \end{aligned} \quad (7)$$

We refer to [5, 15, 16, 19] for further information concerning general resolvent operator functions. In addition, Chen and Li [9] also introduced the concept of integrated fractional resolvent operator function in an algebraic notion as follows.

Definition 1 (see [9, Definition 3.7]). Let $\alpha > 0$, $\beta \geq 0$. A function $S_{\alpha,\beta} : \mathbb{R}^+ \rightarrow B(X)$ is called a β -times integrated α -resolvent operator function or an (α, β) -resolvent operator function ((α, β) -ROF for short) if the following conditions hold:

- (a) $S_{\alpha,\beta}(\cdot)$ is strongly continuous on $\mathbb{R}^+$ and $S_{\alpha,\beta}(0) = g_{\beta+1}(0)I$;
- (b) $S_{\alpha,\beta}(s)S_{\alpha,\beta}(t) = S_{\alpha,\beta}(t)S_{\alpha,\beta}(s)$ for all $s, t \geq 0$;
- (c) the functional equation

$$\begin{aligned} S_{\alpha,\beta}(s) J_t^\alpha S_{\alpha,\beta}(t) - J_s^\alpha S_{\alpha,\beta}(s) S_{\alpha,\beta}(t) \\ = g_{\beta+1}(s) J_t^\alpha S_{\alpha,\beta}(t) - g_{\beta+1}(t) J_s^\alpha S_{\alpha,\beta}(s) \end{aligned} \quad (8)$$

holds for $s, t \geq 0$, where J_t^α is the Riemann-Liouville fractional integral of order α .

The generator A of $S_{\alpha,\beta}(t)$ is defined by

$$\begin{aligned} D(A) &:= \left\{ x \in X : \lim_{t \rightarrow 0^+} \frac{S_{\alpha,\beta}(t)x - g_{\beta+1}(t)x}{g_{\alpha+\beta+1}(t)} \text{ exists} \right\}, \\ Ax &:= \lim_{t \rightarrow 0^+} \frac{S_{\alpha,\beta}(t)x - g_{\beta+1}(t)x}{g_{\alpha+\beta+1}(t)}, \quad x \in D(A). \end{aligned} \quad (9)$$

Note that an $(\alpha, 0)$ -ROF is just an α -ROF.

In this paper, we firstly show that (α, β) -ROF satisfies an equality which extends (3) and (7) for β -integrated semigroup and α -ROF, respectively. Then, we consider the inhomogeneous fractional order abstract Cauchy problem

$$\begin{aligned} {}^c D_t^\alpha u(t) &= Au(t) + f(t), \quad t \in (0, T), \\ u(0) &= x_0, \quad u'(0) = x_1, \end{aligned} \quad (10)$$

where $1 < \alpha < 2$, $T > 0$, $f \in L^1((0, T), X)$, and A is assumed to be the generator of an (α, β) -ROF $S_{\alpha,\beta}(t)$ on X . We give the relation between the function $v(t) = S_{\alpha,\beta}(t)x_0 + (g_1 * S_{\alpha,\beta})(t)x_1 + (g_{\alpha-1} * S_{\alpha,\beta} * f)(t)$ and solution of (10). We also study the problem

$$\begin{aligned} {}^c D_t^\alpha v(t) &= Av(t) + g_{\beta+1}(t)x, \quad t > 0, \\ v^{(k)}(0) &= 0, \quad k = 0, 1, \dots, N-1, \end{aligned} \quad (11)$$

where $\alpha > 0$, $x \in X$, N is the smallest integer greater than or equal to α . We prove that if A generates an exponentially bounded (α, β) -ROF on X if and only if the problem (11) has a unique exponentially bounded classical solution v_x and $Av_x \in L_{\text{Loc}}^1(\mathbb{R}^+, X)$. If $\alpha \rightarrow 1^+$, $\beta = k \in \mathbb{N}$, our Theorem 13 reduces to Lemma 3.2.10 in [2]. When $\alpha = 1$, $\beta = k$, it is easy to see that our Theorem 15 extends and generalizes Theorem 3.2.13 in [2].

This paper is organized as follows. In Section 2, we provide some preliminaries of the fractional calculus and (α, β) -ROF. Section 3 is devoted to present an equality characteristic of the (α, β) -ROF. Finally, as an application of (α, β) -ROF, we discuss the solutions of fractional abstract Cauchy problem in Section 4.

2. Preliminary

Recall that the Riemann-Liouville fractional integral of order $\alpha > 0$ of f is defined by

$$J_t^\alpha f(t) = (g_\alpha * f)(t) = \int_0^t g_\alpha(t-s)f(s)ds, \quad (12)$$

and the Caputo fractional derivative of order $\alpha > 0$ of f can be written as

$${}^c D_t^\alpha f(t) = \frac{d^m}{dt^m} \left(g_{m-\alpha} * \left(f(t) - \sum_{k=0}^{m-1} f^{(k)}(0) g_{k+1}(t) \right) \right), \quad (13)$$

where m is the smallest integer greater than or equal to α . For more details in fractional calculus, we refer to [5, 20, 21].

The Mittag-Leffler function is defined by

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad E_{\alpha}(z) = E_{\alpha,1}(z), \quad (14)$$

$\operatorname{Re} \alpha > 0, \beta, z \in \mathbb{C}.$

And if $0 < \alpha < 2, \beta > 0$, then

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{(1-\beta)/\alpha} \exp(z^{1/\alpha}) + \varepsilon_{\alpha,\beta}(z),$$

$$|\arg z| \leq \frac{1}{2}\alpha\pi, \quad E_{\alpha,\beta}(z) = \varepsilon_{\alpha,\beta}(z), \quad (15)$$

$$|\arg(-z)| < \left(1 - \frac{1}{2}\alpha\right)\pi,$$

where

$$\varepsilon_{\alpha,\beta}(z) = -\sum_{n=1}^{N-1} \frac{z^{-n}}{\Gamma(\beta - \alpha n)} + O(|z|^{-N}) \quad \text{as } z \rightarrow \infty, \quad (16)$$

and the O -term is uniform in $\arg z$ if $|\arg(-z)| \leq (1 - (\alpha/2) - \epsilon)\pi$.

We now recall some properties of (α, β) -ROF.

Lemma 2 (see [9, Proposition 3.10]). *Let $S_{\alpha,\beta} : \mathbb{R}^+ \rightarrow B(X)$ be an (α, β) -ROF generated by A . The following assertions hold:*

- (a) $S_{\alpha,\beta}(t)D(A) \subset D(A)$ and $AS_{\alpha,\beta}(t)x = S_{\alpha,\beta}(t)Ax$ for $x \in D(A)$ and $t \geq 0$;
- (b) for all $x \in X$, $J_t^\alpha S_{\alpha,\beta}(t)x \in D(A)$ and $S_{\alpha,\beta}(t)x = g_{\beta+1}(t)x + A J_t^\alpha S_{\alpha,\beta}(t)x$, $t \geq 0$;
- (c) $x \in D(A)$ and $Ax = y$ if and only if $S_{\alpha,\beta}(t)x = g_{\beta+1}(t)x + J_t^\alpha S_{\alpha,\beta}(t)y$, $t \geq 0$;
- (d) A is closed.

Lemma 3 (see [9, Proposition 3.5, Theorem 3.11]). *Let $\alpha > 0$, $\beta \geq 0$. A generates an (α, β) -ROF $S_{\alpha,\beta}$ satisfying $\|S_{\alpha,\beta}(t)\| \leq Me^{\omega t}$, $t \geq 0$, for some constants $M > 0$ and $\omega \geq 0$, if and only if $(\omega^\alpha, \infty) \subset \rho(A)$ and there exists a strongly continuous function $S : \mathbb{R}^+ \rightarrow B(X)$ such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$ and $\int_0^\infty e^{-\lambda t} S(t)x dt = \lambda^{\alpha-\beta-1} R(\lambda^\alpha, A)x$, $\lambda > \omega$, for all $x \in X$. Furthermore, $S(t)$ is $S_{\alpha,\beta}(t)$.*

Lemma 4 (see [2, Proposition B.6]). *Let $U \subset \mathbb{C}$. If function $R : U \rightarrow B(X)$ satisfies $R(\lambda) - R(\mu) = (\mu - \lambda)R(\lambda)R(\mu)$, then there is an operator A on X such that $R(\lambda) = (\lambda I - A)^{-1}$ for all $\lambda \in U$ if and only if $\ker R(\lambda) = \{0\}$.*

3. An Novel Equality Characteristic for (α, β) -ROF

The following theorem shows that an (α, β) -ROF satisfies a functional equation and the treatment bases on the technique of Laplace transform. For convenience, we drop the subscript α, β from $\{S_{\alpha,\beta}\}_{t \geq 0}$ in this theorem.

Theorem 5. *Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0$, $\beta \in \mathbb{R}^+$ satisfy $\beta - \alpha > -1$. If $\{S(t)\}_{t \geq 0}$ is an (α, β) -ROF, then it satisfies the following equality:*

$$\int_t^{t+s} (s+t-r)^{\beta-\alpha} S(r) dr - \int_0^s (s+t-r)^{\beta-\alpha} S(r) dr$$

$$= \frac{\alpha\Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)} \int_0^t \int_0^s \frac{S(r_1)S(r_2)}{(t+s-r_1-r_2)^{1+\alpha}} dr_1 dr_2. \quad (17)$$

Proof. Denote by $L(t, s)$ and $R(t, s)$ the left and right sides of equality (17), respectively, and denote by $f_a(t)$ the truncation of $f(t)$ at a , that is, $f_a(t) = f(t)$ for $0 \leq t \leq a$ and $f_a(t) = 0$ otherwise.

We will show that the Laplace transform of $L_a(t, s)$ and $R_a(t, s)$ with respect to t and s is equivalent, and by the uniqueness of Laplace transform, we can get that $L_a(t, s) = R_a(t, s)$.

Taking Laplace transform of $L_a(t, s)$ with respect to s as follows

$$\begin{aligned} \widehat{L}_a(t, \lambda) &= \int_0^\infty e^{-\lambda s} \left[\int_t^{t+s} (s+t-r)^{\beta-\alpha} S_a(r) dr \right. \\ &\quad \left. - \int_0^s (s+t-r)^{\beta-\alpha} S_a(r) dr \right] ds \\ &= \int_t^\infty S_a(r) \int_{r-t}^\infty e^{-\lambda s} (s+t-r)^{\beta-\alpha} ds dr \\ &\quad - \int_0^\infty S_a(r) \int_r^\infty e^{-\lambda s} (s+t-r)^{\beta-\alpha} ds dr \\ &= \int_t^\infty S_a(r) e^{-\lambda(r-t)} \int_0^\infty e^{-\lambda \tau} \tau^{\beta-\alpha} d\tau dr \\ &\quad - \int_0^\infty S_a(r) e^{-\lambda(r-t)} \int_t^\infty e^{-\lambda \tau} \tau^{\beta-\alpha} d\tau dr \\ &= \frac{\Gamma(\beta - \alpha + 1)}{\lambda^{(\beta-\alpha+1)}} \int_t^\infty S_a(r) e^{-\lambda(r-t)} dr \\ &\quad - e^{\lambda t} \widehat{S}_a(\lambda) \int_t^\infty e^{-\lambda \tau} \tau^{\beta-\alpha} d\tau, \end{aligned} \quad (18)$$

then taking Laplace transform with respect to t , we have

$$\begin{aligned} \widehat{L}_a(\mu, \lambda) &= \int_0^\infty e^{-\mu t} \left[\frac{\Gamma(\beta - \alpha + 1)}{\lambda^{(\beta-\alpha+1)}} \int_t^\infty S_a(r) e^{-\lambda(r-t)} dr \right. \\ &\quad \left. - e^{\lambda t} \widehat{S}_a(\lambda) \int_t^\infty e^{-\lambda \tau} \tau^{\beta-\alpha} d\tau \right] dt \\ &= \frac{\Gamma(\beta - \alpha + 1)}{\lambda^{(\beta-\alpha+1)}} \int_0^\infty e^{-\mu t} \int_t^\infty S_a(r) e^{-\lambda(r-t)} dr dt \\ &\quad - \widehat{S}_a(\lambda) \int_0^\infty e^{(\lambda-\mu)t} \int_t^\infty e^{-\lambda \tau} \tau^{\beta-\alpha} d\tau dt \end{aligned}$$

$$\begin{aligned}
&= \frac{\Gamma(\beta - \alpha + 1)}{\lambda^{(\beta - \alpha + 1)}} \int_0^\infty e^{-\lambda r} S_a(r) \int_0^r e^{(\lambda - \mu)t} dt dr \\
&\quad - \int_0^\infty e^{-\lambda \tau} \tau^{\beta - \alpha} \int_0^\tau e^{(\lambda - \mu)t} dt d\tau \widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta - \alpha + 1)}{(\lambda - \mu) \lambda^{(\beta - \alpha + 1)}} \\
&\quad \times \left(\int_0^\infty e^{-\mu r} S_a(r) dr - \int_0^\infty e^{-\lambda r} S_a(r) dr \right) \\
&\quad - \frac{1}{\lambda - \mu} \\
&\quad \times \left(\int_0^\infty e^{-\mu \tau} \tau^{\beta - \alpha} d\tau - \int_0^\infty e^{-\lambda \tau} \tau^{\beta - \alpha} d\tau \right) \widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta - \alpha + 1)}{(\lambda - \mu) \lambda^{(\beta - \alpha + 1)}} (\widehat{S}_a(\mu) - \widehat{S}_a(\lambda)) \\
&\quad - \frac{1}{\lambda - \mu} \left(\frac{\Gamma(\beta - \alpha + 1)}{\mu^{(\beta - \alpha + 1)}} - \frac{\Gamma(\beta - \alpha + 1)}{\lambda^{(\beta - \alpha + 1)}} \right) \widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta - \alpha + 1)}{\lambda - \mu} (\lambda^{(\alpha - \beta - 1)} \widehat{S}_a(\mu) - \mu^{(\alpha - \beta - 1)} \widehat{S}_a(\lambda)) \\
&= \frac{\Gamma(\beta - \alpha + 1)}{(\lambda - \mu) (\lambda \mu)^{\beta - \alpha + 1}} \\
&\quad \times (\mu^{(\beta - \alpha + 1)} \widehat{S}_a(\mu) - \lambda^{(\beta - \alpha + 1)} \widehat{S}_a(\lambda)) \\
&= \frac{\Gamma(\beta - \alpha + 1)}{(\lambda - \mu) (\lambda \mu)^{\beta - \alpha + 1}} (R(\mu^\alpha, A) - R(\lambda^\alpha, A)),
\end{aligned} \tag{19}$$

where the last equality follows from Lemma 3.

On the other hand, observing that

$$R_a(t, s) = \frac{\alpha \Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)} \int_0^t \frac{S_a(r)}{(t + s - r)^{1 + \alpha}} dr * S_a(s), \tag{20}$$

Then taking Laplace transform with respect to t and s , respectively, we deduce

$$\begin{aligned}
\widehat{R}_a(t, \lambda) &= \frac{\alpha \Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)} \\
&\quad \times \int_0^\infty e^{-\lambda s} \int_0^t \frac{S_a(r)}{(t + s - r)^{1 + \alpha}} dr ds \widehat{S}_a(\lambda) \\
&= \frac{\alpha \Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)} \\
&\quad \times \int_0^\infty e^{-\lambda s} (t + s)^{-\alpha - 1} * S_a(t) ds \widehat{S}_a(\lambda),
\end{aligned}$$

$$\begin{aligned}
\widehat{R}_a(\mu, \lambda) &= \frac{\alpha \Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)} \\
&\quad \times \int_0^\infty e^{-\mu t} \int_0^\infty e^{-\lambda s} (t + s)^{-\alpha - 1} * S_a(t) ds dt \widehat{S}_a(\lambda) \\
&= \frac{\alpha \Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)} \\
&\quad \times \int_0^\infty e^{-\lambda s} \int_0^\infty e^{-\mu t} (t + s)^{-\alpha - 1} * S_a(t) ds dt \widehat{S}_a(\lambda) \\
&= \frac{\alpha \Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)} \\
&\quad \times \int_0^\infty e^{-\lambda s} \int_0^\infty e^{-\mu t} (t + s)^{-\alpha - 1} dt ds \widehat{S}_a(\mu) \widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)} \frac{\Gamma(1 - \alpha)}{\lambda - \mu} (\lambda^\alpha - \mu^\alpha) \widehat{S}_a(\mu) \widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta - \alpha + 1)}{\lambda - \mu} (\lambda^\alpha - \mu^\alpha) \widehat{S}_a(\mu) \widehat{S}_a(\lambda) \\
&= \frac{\Gamma(\beta - \alpha + 1)}{(\lambda - \mu)} (\lambda^\alpha - \mu^\alpha) \\
&\quad \times \frac{1}{(\lambda \mu)^{\beta - \alpha + 1}} R(\mu^\alpha, A) R(\lambda^\alpha, A) \\
&= \frac{\Gamma(\beta - \alpha + 1)}{(\lambda - \mu) (\lambda \mu)^{\beta - \alpha + 1}} (R(\mu^\alpha, A) - R(\lambda^\alpha, A)),
\end{aligned} \tag{21}$$

where the last equality follows from the resolvent identity. In view of (19), (21), and the uniqueness of Laplace transform, we obtain $L_a(t, s) = R_a(t, s)$, $t, s \geq 0$. The arbitrariness of a implies $L(t, s) = R(t, s)$ for $t, s \geq 0$. $\square$

Remark 6. (a) If $\beta = 0$, then $(\alpha, 0)$ -ROF $S_{\alpha, 0}(t)$ is an α -ROF and the equality (17) degenerates to be equality (7).

(b) If we assume that, for each $x \in X$, the map $t \rightarrow S_{\alpha, \beta}(t)x$ is continuously differentiable on $[0, \infty)$ and the limit of (α, β) -ROF $S_{\alpha, \beta}(t)$ exists as $\alpha \rightarrow 1^-$, then multiplying both sides of (17) with $1 - \alpha$ and integrating by parts to the right side of (17) and letting $\alpha \rightarrow 1^-$, we can get that (3) is just the limit state of (17).

By Lemma 3, (α, β) -ROF generated by operator A is exactly operator valued functions whose Laplace transforms are $\lambda^{\alpha - \beta - 1} R(\lambda, A)$. In the following theorem, we show that this property corresponds to the functional equation (17) for $S_{\alpha, \beta}(t)$. The proof of this theorem is proved by Ardent [2, proposition 3.2.4] for $\alpha \rightarrow 1^-$, $\beta = k \in \mathbb{N}$. Our proof is different since we could not use the binomial formula as in [2].

Theorem 7. Let $\tilde{S} : \mathbb{R}^+ \rightarrow B(X)$ be a strongly continuous function satisfying $\|\tilde{S}(t)\| \leq Me^{\omega t}$ ($t \geq 0$) for some M , $\omega \geq 0$. Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0$, $\beta \in \mathbb{R}^+$ satisfy that $\beta - \alpha > -1$, set

$$R(\lambda^\alpha) := \lambda^{-\alpha+\beta+1} \int_0^\infty e^{-\lambda t} \tilde{S}(t) dt. \quad (22)$$

Then the following assertions are equivalent.

- (i) There exists an operator A such that $(\omega^\alpha, \infty) \subset \rho(A)$ and $R(\lambda^\alpha) = (\lambda^\alpha I - A)^{-1}$ for $\lambda > \omega$.
- (ii) For $s, t \geq 0$, the equality

$$\begin{aligned} & \int_t^{t+s} (s+t-r)^{\beta-\alpha} \tilde{S}(r) dr - \int_0^s (s+t-r)^{\beta-\alpha} \tilde{S}(r) dr \\ &= \frac{\alpha \Gamma(\beta - \alpha + 1)}{\Gamma(1 - \alpha)} \int_0^s \int_0^t \frac{\tilde{S}(r_1) \tilde{S}(r_2)}{(t+s-r_1-r_2)^{1+\alpha}} dr_1 dr_2 \end{aligned} \quad (23)$$

holds and $\tilde{S}(t)x = 0$ for all $t \geq 0$ implies that $x = 0$.

Proof. Assume that (i) holds; then $(\omega^\alpha, \infty) \subset \rho(A)$, $(\lambda^\alpha I - A)^{-1} = \lambda^{-\alpha+\beta+1} \int_0^\infty e^{-\lambda t} \tilde{S}(t) dt$ for $\lambda > \omega$; from Lemma 3, we know that $\tilde{S}(t)$ is the (α, β) -ROF generated by A ; then Theorem 5 shows that equality (23) holds. It follows from $(\omega^\alpha, \infty) \subset \rho(A)$ and $R(\lambda^\alpha) = (\lambda^\alpha I - A)^{-1}$ for $\lambda > \omega$ that $R(\lambda^\alpha)$ is injective. If $\tilde{S}(t)x = 0$ for all $t \geq 0$, from $R(\lambda^\alpha) := \lambda^{-\alpha+\beta+1} \int_0^\infty e^{-\lambda t} \tilde{S}(t) dt$, we have $R(\lambda^\alpha)x = 0$; thus $x = 0$.

If (ii) is satisfied, similar as the calculations of (19) and (21), we can get that the Laplace transform of the left side and the right side of (17) are

$$\begin{aligned} & \frac{\Gamma(\beta - \alpha + 1)}{(\lambda - \mu)(\lambda\mu)^{\beta-\alpha+1}} (R(\mu^\alpha) - R(\lambda^\alpha)), \\ & \frac{\Gamma(\beta - \alpha + 1)}{\lambda - \mu} \frac{\lambda^\alpha - \mu^\alpha}{(\lambda\mu)^{\beta-\alpha+1}} R(\mu^\alpha) R(\lambda^\alpha), \end{aligned} \quad (24)$$

respectively. So,

$$R(\mu^\alpha) - R(\lambda^\alpha) = (\lambda^\alpha - \mu^\alpha) R(\mu^\alpha) R(\lambda^\alpha). \quad (25)$$

On the other hand, if $R(\lambda^\alpha)x = 0$, by $R(\lambda^\alpha) = \lambda^{-\alpha+\beta+1} \int_0^\infty e^{-\lambda t} \tilde{S}(t) dt$ and uniqueness of Laplace transform, we have $\tilde{S}(t)x = 0$ for all $t \geq 0$, then from (ii) we know $x = 0$, so, $\text{Ker } R(\lambda^\alpha) = 0$, by (25) and Lemma 4, we get the conclusion. $\square$

4. Fractional Abstract Cauchy Problems

In this section, we study the following inhomogeneous fractional abstract Cauchy problem:

$$\begin{aligned} {}^c D_t^\alpha u(t) &= Au(t) + f(t), \quad t \in (0, T), \\ u(0) &= x_0, \quad u'(0) = x_1, \end{aligned} \quad (26)$$

where $1 < \alpha < 2$, $T > 0$, $f \in L^1((0, T), X)$, $x_0, x_1 \in X$, A is a linear closed operator.

First, we give the definitions of solutions to (26).

Definition 8. A function $u \in C([0, T]; X)$ is called a mild solution of (26), if $(g_\alpha * u)(t) \in D(A)$ and $u(t) = x_0 + tx_1 + A(g_\alpha * u)(t) + (g_\alpha * f)(t)$, $t \in [0, T]$.

Definition 9. A function $u \in C([0, T]; X)$ is called a classical solution of (26) if u satisfies the following.

- (a) $u \in C([0, T]; D(A)) \cap C^1([0, T]; X)$.
- (b) $g_{2-\alpha} * (u - x_0 - tx_1) \in C^2([0, T]; X)$.
- (c) u satisfies (26).

From the above definitions, it is clear that a classical solution of (26) is a mild solution of it. The following assertion shows that a mild solution of the problem (26) with suitable regularity is also a classical solution.

Theorem 10. Let u be a mild solution of (26) and $f \in C([0, T]; X)$, if $g_{2-\alpha} * (u - x_0 - tx_1) \in C^2([0, T]; X)$, and for any $t \in (0, T)$, $g_\alpha * u \in L^1((0, t), D(A))$; then u is also a classical solution of (26).

Proof. Since u is a mild solution of (26), we have

$$\begin{aligned} (g_\alpha * u)(t) &\in D(A), \\ u(t) &= x_0 + tx_1 + A(g_\alpha * u)(t) + (g_\alpha * f)(t), \end{aligned} \quad (27)$$

$t \in [0, T]$.

If we denote $w(t) := u(t) - x_0 - tx_1$, then it follows from (27) that

$$\begin{aligned} (g_{2-\alpha} * w)(t) &= g_{2-\alpha} * (A(g_\alpha * u)(t) + (g_\alpha * f)(t)) \\ &= A(g_2 * u)(t) + (g_2 * f)(t). \end{aligned} \quad (28)$$

Since $g_{2-\alpha} * w \in C^2([0, T]; X)$, then ${}^c D_t^\alpha u(t) = (d^2/dt^2)(g_{2-\alpha} * w)(t)$ is well defined, and by (28), we have

$$\begin{aligned} {}^c D_t^\alpha u(t) &= \frac{d^2}{dt^2} (g_{2-\alpha} * w)(t) \\ &= \lim_{h \rightarrow 0} \frac{1}{h^2} [(g_{2-\alpha} * w)(t) - 2(g_{2-\alpha} * w)(t-h) \\ &\quad + (g_{2-\alpha} * w)(t-2h)] \\ &= \lim_{h \rightarrow 0} \frac{1}{h^2} [A(g_2 * u)(t) - 2A(g_2 * u)(t-h) \\ &\quad + A(g_2 * u)(t-2h)] \\ &\quad + \lim_{h \rightarrow 0} \frac{1}{h^2} [(g_2 * f)(t) - 2(g_2 * f)(t-h) \\ &\quad + (g_2 * f)(t-2h)] \\ &= \lim_{h \rightarrow 0} \frac{1}{h^2} [A(g_2 * u)(t) - 2A(g_2 * u)(t-h) \\ &\quad + A(g_2 * u)(t-2h)] + f(t). \end{aligned} \quad (29)$$

Thus,

$$\lim_{h \rightarrow 0} \frac{1}{h^2} [A(g_2 * u)(t) - 2A(g_2 * u)(t-h) + A(g_2 * u)(t-2h)] = {}^c D_t^\alpha u(t) - f(t). \quad (30)$$

On the other hand, from the closeness of A and $g_\alpha * u \in L^1((0, t), D(A))$ for $t \in [0, T]$, by Proposition 1.1.7 in [2], we have

$$(g_2 * u)(t) = (g_{2-\alpha} * (g_\alpha * u))(t) \in D(A), \quad (31)$$

Then from (30) and the closeness of A , we obtain

$$\begin{aligned} u(t) &= \lim_{h \rightarrow 0} \frac{1}{h^2} [(g_2 * u)(t) - 2(g_2 * u)(t-h) \\ &\quad + (g_2 * u)(t-2h)] \in D(A), \quad (32) \\ {}^c D_t^\alpha u(t) &= Au(t) + f(t), \quad t \in [0, T]. \end{aligned}$$

It is clear that $u(0) = x_0$, $u'(0) = x_1$. Thus, u is a classical solution of (26). $\square$

Lemma 11. Let $1 < \alpha < 2$, $f \in L^1((0, T), X)$. Suppose A is the generator of an (α, β) -ROF $S_{\alpha, \beta}(t)$ on X for some $\beta \in \mathbb{R}^+$. Then, for every $t \in [0, T]$, $(g_{\alpha-1} * S_{\alpha, \beta} * f)(t)$ exists, and $(g_{\alpha-1} * S_{\alpha, \beta} * f) \in C([0, T], X)$.

Proof. For every $t \in [0, T]$, since $g_{\alpha-1} \in L^1((0, t), \mathbb{R}^+)$, $f \in L^1((0, t), X)$, we get $g_{\alpha-1} * f \in L^1((0, t), X)$, hence, from

$$\begin{aligned} (g_{\alpha-1} * S_{\alpha, \beta} * f)(t) &= (S_{\alpha, \beta} * g_{\alpha-1} * f)(t) \\ &= \int_0^t S_{\alpha, \beta}(t-s) (g_{\alpha-1} * f)(s) ds, \end{aligned} \quad (33)$$

we obtain that $(g_{\alpha-1} * S_{\alpha, \beta} * f)(t)$ exists.

For $h \in \mathbb{R}$, $|h| \ll 1$ and $t+h \in [0, T]$, we have

$$\begin{aligned} &(g_{\alpha-1} * S_{\alpha, \beta} * f)(t+h) - (g_{\alpha-1} * S_{\alpha, \beta} * f)(t) \\ &= \int_0^{t+h} S_{\alpha, \beta}(t+h-s) (g_{\alpha-1} * f)(s) ds \\ &\quad - \int_0^t S_{\alpha, \beta}(t-s) (g_{\alpha-1} * f)(s) ds \\ &= \int_0^{t+h} (S_{\alpha, \beta}(t+h-s) - S_{\alpha, \beta}(t-s)) (g_{\alpha-1} * f)(s) ds \\ &\quad - \int_t^{t+h} S_{\alpha, \beta}(t-s) (g_{\alpha-1} * f)(s) ds. \end{aligned} \quad (34)$$

From the dominated convergence theorem and absolute continuity of integral, we deduce

$$\lim_{h \rightarrow 0} ((g_{\alpha-1} * S_{\alpha, \beta} * f)(t+h) - (g_{\alpha-1} * S_{\alpha, \beta} * f)(t)) = 0. \quad (35)$$

So, $(g_{\alpha-1} * S_{\alpha, \beta} * f) \in C([0, T], X)$. $\square$

Let

$$v(t) = S_{\alpha, \beta}(t) x_0 + (g_1 * S_{\alpha, \beta})(t) x_1 + (g_{\alpha-1} * S_{\alpha, \beta} * f)(t). \quad (36)$$

From Lemma 11, we know that v is well defined, and $v \in C([0, T], X)$.

The following theorem is proved by Arendt [2, Lemma 3.2.9] for $\alpha = 1$, $\beta = l \in \mathbb{N}$. Our proof is different because we could not use the formula of integration by parts as [2, Lemma 3.2.9].

Theorem 12. Suppose that A is the generator of an (α, β) -ROF $S_{\alpha, \beta}(t)$ on X for some $\beta \in \mathbb{R}^+$. Let v be defined by (36). Then one has the following results.

- (a) If (26) has a mild solution u , then $g_{m-\beta} * (v - \sum_{k=0}^{m-1} v^{(k)}(0) g_{k+1}(t)) \in C^m([0, T]; X)$ and $u(t) = {}^c D_t^\beta v(t)$.
- (b) If there is a classical solution u of (26), then $g_{2-\alpha} * ({}^c D_t^\beta v(t) - x_0 - tx_1) \in C^2([0, T]; X)$ and $u(t) = {}^c D_t^\beta v(t)$.

Proof. If u is a mild solution of (26), then $(g_\alpha * u)(t) \in D(A)$ and

$$u(t) = x_0 + tx_1 + A(g_\alpha * u)(t) + (g_\alpha * f)(t), \quad t \in [0, T]. \quad (37)$$

Using Lemma 2(b) and the closeness of A , we have

$$\begin{aligned} (g_{\beta+1} * u)(t) &= (S_{\alpha, \beta} - A(g_\alpha * S_{\alpha, \beta}))(t) * u(t) \\ &= (S_{\alpha, \beta} * u)(t) - (A(g_\alpha * S_{\alpha, \beta}) * u)(t) \\ &= (S_{\alpha, \beta} * u)(t) - S_{\alpha, \beta} * A(g_\alpha * u)(t) \\ &= (S_{\alpha, \beta} * u)(t) \\ &\quad - S_{\alpha, \beta} * (u - x_0 - tx_1 - (g_\alpha * f)(t)) \\ &= S_{\alpha, \beta} * x_0 + S_{\alpha, \beta} * tx_1 + (S_{\alpha, \beta} * g_\alpha * f)(t); \end{aligned} \quad (38)$$

that is, $(g_{\beta+1} * u)(t) = (1 * S_{\alpha, \beta})(t) x_0 + (g_2 * S_{\alpha, \beta})(t) x_1 + (g_\alpha * S * f)(t)$. So

$$\begin{aligned} J_t^\beta u(t) &= (g_\beta * u)(t) = \frac{d}{dt} (g_{\beta+1} * u)(t) \\ &= S_{\alpha, \beta}(t) x_0 + (g_1 * S_{\alpha, \beta})(t) x_1 \\ &\quad + (g_{\alpha-1} * S_{\alpha, \beta} * f)(t) = v(t). \end{aligned} \quad (39)$$

Thus, it follows from $u \in C([0, T], X)$ that $g_{m-\beta} * (v - \sum_{k=0}^{m-1} v^{(k)}(0) g_{k+1}(t)) \in C^m([0, T]; X)$ and $u(t) = {}^c D_t^\beta v(t)$. Hence (a) holds. If u is a classical solution of (26), then u is a mild solution of (26). So, assertion (b) follows immediately from (a). $\square$

Theorem 13. Let v be defined by (36). Assume that $v \in C^{m-1}([0, T]; X)$, $v^{(k)}(0) = 0$ for $k = 0, 1, \dots, m-1$, and $g_{m-\beta} * v \in C^m([0, T]; X)$; then ${}^c D_t^\beta v(t)$ is a mild solution of the problem (26). Moreover, if $g_{2-\alpha} * ({}^c D_t^\beta v(t) - x_0 - tx_1) \in C^2([0, T]; X)$, and for any $t \in (0, T)$, $g_\alpha * {}^c D_t^\beta v(t) \in L^1((0, t), D(A))$, then ${}^c D_t^\beta v(t)$ is also a classical solution of (26).

Proof. Consider the following steps.

Step 1. We first claim that $J_t^\alpha v(t) \in D(A)$ and

$${}^c D_t^\beta A J_t^\alpha v(t) = {}^c D_t^\beta v(t) - x_0 - tx_1 - g_\alpha * f. \quad (40)$$

In view of definition of $v(t)$, we have

$$\begin{aligned} J_t^\alpha v(t) &= (g_\alpha * S_{\alpha,\beta})(t) x_0 + (g_\alpha * g_1 * S_{\alpha,\beta})(t) x_1 \\ &\quad + (g_\alpha * g_{\alpha-1} * S_{\alpha,\beta} * f)(t) \\ &= (g_\alpha * S_{\alpha,\beta})(t) x_0 + \int_0^t (g_\alpha * S_{\alpha,\beta})(\tau) x_1 d\tau \\ &\quad + \int_0^t (g_\alpha * S_{\alpha,\beta})(t - \tau) (g_{\alpha-1} * f)(\tau) d\tau, \end{aligned} \quad (41)$$

for $t \in [0, T]$.

From Lemma 2(b), for $0 < \tau < t$, we have

$$\begin{aligned} (g_\alpha * S_{\alpha,\beta})(t) x_0 &\in D(A), \quad (g_\alpha * S_{\alpha,\beta})(\tau) x_1 \in D(A), \\ (g_\alpha * S_{\alpha,\beta})(t - \tau) (g_{\alpha-1} * f)(\tau) &\in D(A), \\ A(g_\alpha * S_{\alpha,\beta})(\tau) x_1 &= S_{\alpha,\beta}(\tau) x_1 - g_{\beta+1}(\tau) x_1 \in L^1(0, t), \\ A(g_\alpha * S_{\alpha,\beta})(t - \tau) (g_{\alpha-1} * f)(\tau) &= S_{\alpha,\beta}(t - \tau) (g_{\alpha-1} * f)(\tau) \\ &\quad - g_{\beta+1}(t - \tau) (g_{\alpha-1} * f)(\tau) \in L^1(0, t), \end{aligned} \quad (42)$$

combining with the closeness of A , one has

$$\begin{aligned} \int_0^t (g_\alpha * S_{\alpha,\beta})(\tau) x_1 d\tau &\in D(A), \\ \int_0^t (g_\alpha * S_{\alpha,\beta})(t - \tau) (g_{\alpha-1} * f)(\tau) d\tau &\in D(A). \end{aligned} \quad (43)$$

Thus $J_t^\alpha v(t) \in D(A)$, and

$$\begin{aligned} A J_t^\alpha v(t) &= A(g_\alpha * S_{\alpha,\beta})(t) x_0 + g_1 * A(g_\alpha * S_{\alpha,\beta})(t) x_1 \\ &\quad + g_{\alpha-1} * A(g_\alpha * S_{\alpha,\beta} * f)(t) \end{aligned}$$

$$\begin{aligned} &= S_{\alpha,\beta}(t) x_0 - g_{\beta+1}(t) x_0 + (g_1 * S_{\alpha,\beta})(t) x_1 \\ &\quad - (g_1 * g_{\beta+1})(t) x_1 \\ &\quad + g_{\alpha-1} * (S_{\alpha,\beta} * f - g_{\beta+1} * f)(t) \\ &= S_{\alpha,\beta}(t) x_0 + (g_1 * S_{\alpha,\beta})(t) x_1 \\ &\quad + (g_{\alpha-1} * S_{\alpha,\beta} * f)(t) - g_{\beta+1}(t) x_0 \\ &\quad - (g_1 * g_{\beta+1})(t) x_1 - (g_{\alpha+\beta} * f)(t) \\ &= v(t) - g_{\beta+1}(t) x_0 - (g_1 * g_{\beta+1})(t) x_1 \\ &\quad - (g_{\alpha+\beta} * f)(t). \end{aligned} \quad (44)$$

So

$$\begin{aligned} A J_t^\alpha v(t) &= v(t) - g_{\beta+1}(t) x_0 - (g_1 * g_{\beta+1})(t) x_1 \\ &\quad - (g_{\alpha+\beta} * f)(t), \end{aligned} \quad (45)$$

$${}^c D_t^\beta A J_t^\alpha v(t) = {}^c D_t^\beta v(t) - x_0 - tx_1 - g_\alpha * f.$$

Step 2. We prove ${}^c D_t^\beta J_t^\alpha v(t) \in D(A)$ and $A {}^c D_t^\beta J_t^\alpha v(t) = {}^c D_t^\beta A J_t^\alpha v(t)$.

Since $v \in C^k([0, T]; X)$, $v^{(k)}(0) = 0$ for $k = 0, 1, \dots, m-1$, we have

$$\frac{d^k}{dt^k} (g_\alpha * v)(t)|_{t=0} = (g_\alpha * v^{(k)})(t)|_{t=0} = 0. \quad (46)$$

So

$$\begin{aligned} {}^c D_t^\beta J_t^\alpha v(t) &= \frac{d^m}{dt^m} (g_{m-\beta} * g_\alpha * v)(t) \\ &= \lim_{h \rightarrow 0} \frac{1}{h^m} \sum_{r=0}^m C_m^r (g_{m-\beta} * g_\alpha * v)(t - rh), \end{aligned} \quad (47)$$

where $C_m^r = (m(m-1) \cdots (m-r+1))/r!$. From (45), we know that $A J_t^\alpha v \in L^1((0, t), X)$, and the closeness of A implies that $(g_{m-\beta} * g_\alpha * v)(t) \in D(A)$, and $A(g_{m-\beta} * g_\alpha * v)(t) = g_{m-\beta} * A(g_\alpha * v)(t)$, by Step 1, ${}^c D_t^\beta A J_t^\alpha v(t)$ exists, then ${}^c D_t^\beta J_t^\alpha v(t) \in D(A)$, and

$$A {}^c D_t^\beta J_t^\alpha v(t) = {}^c D_t^\beta A J_t^\alpha v(t). \quad (48)$$

Step 3. We show that $v^{(k)}(0) = 0$ for $k = 0, 1, \dots, m-1$ implies

$${}^c D_t^\beta J_t^\alpha v(t) = J_t^\alpha {}^c D_t^\beta v(t). \quad (49)$$

In fact, if $\alpha \geq \beta$, we have ${}^c D_t^\beta J_t^\alpha v(t) = J_t^{\alpha-\beta} {}^c D_t^\beta v(t)$, and

$$\begin{aligned} J_t^\alpha {}^c D_t^\beta v(t) &= J_t^{\alpha-\beta} J_t^\beta {}^c D_t^\beta v(t) \\ &= J_t^{\alpha-\beta} \left(v(t) - \sum_{k=0}^{m-1} v^{(k)}(0) g_{k+1}(t) \right). \end{aligned} \quad (50)$$

If $\alpha < \beta$, we have ${}^c D_t^\beta J_t^\alpha v(t) = {}^c D_t^{\beta-\alpha} v(t)$, and

$$\begin{aligned} J_t^\alpha {}^c D_t^\beta v(t) &= {}^c D_t^{\beta-\alpha} J_t^\beta {}^c D_t^\beta v(t) \\ &= {}^c D_t^{\beta-\alpha} \left(v(t) - \sum_{k=0}^{m-1} v^{(k)}(0) g_{k+1}(t) \right). \end{aligned} \quad (51)$$

From the above discussion and $v^{(k)}(0) = 0$ for $k = 0, 1, \dots, m-1$, we conclude that (49) holds.

Finally, in view of (40), (48), and (49), we have

$$\begin{aligned} A J_t^\alpha {}^c D_t^\beta v(t) &= A {}^c D_t^\beta J_t^\alpha v(t) = {}^c D_t^\beta A J_t^\alpha v(t) \\ &= {}^c D_t^\beta v(t) - x_0 - t x_1 - (g_\alpha * f)(t). \end{aligned} \quad (52)$$

Therefore, ${}^c D_t^\beta v(t)$ is a mild solution of (26).

Moreover, if $g_{2-\alpha} * ({}^c D_t^\beta v(t) - x_0 - t x_1) \in C^2([0, T]; X)$, and for any $t \in (0, T)$, $g_\alpha * {}^c D_t^\beta v(t) \in L^1((0, t), D(A))$, applying Theorem 10, we have that ${}^c D_t^\beta v(t)$ is a classical solution of (26). $\square$

Remark 14. If $\alpha \rightarrow 1^+$, $\beta = k$, then (26) becomes (4). Theorem 13 degenerated to Lemma 3.2.10 in [2]. Note that the condition $v^{(j)}(0) = 0$ for $j = 0, 1, \dots, k-1$ is not necessary in Lemma 3.2.10 of [2], since from its proof, it is easy to see that $v(0) = 0$ implies that $v^{(j)}(0) = 0$ for $j = 1, \dots, k-1$.

Now, we turn our attention to the problem

$$\begin{aligned} {}^c D_t^\alpha v(t) &= A v(t) + g_{\beta+1} t(x), \quad t > 0, \\ v^{(k)}(0) &= 0, \quad k = 0, 1, \dots, N-1, \end{aligned} \quad (53)$$

where $\alpha > 0$, $x \in X$, A is a linear closed operator on X and N is the smallest integer greater than or equal to α .

Theorem 15. Let A be a closed operator on X and $\beta > 0$; the following two assertions are equivalent.

- (i) A generates an exponentially bounded (α, β) -ROF $S_{\alpha, \beta}$ on X .
- (ii) For every $x \in X$, there exists a unique classical solution v_x of (53) which is exponentially bounded and $A v_x \in L_{\text{loc}}^1(\mathbb{R}^+; X)$.

Proof. If (i) is satisfied, for every $x \in X$, define $v_x : \mathbb{R}^+ \rightarrow X$ by $v_x(t) = (g_\alpha * S_{\alpha, \beta})(t)x$, then $v_x^{(k)}(0) = 0$ for $k = 0, 1, \dots, N-1$. By Lemma 2(b), we have $v_x(t) = (g_\alpha * S_{\alpha, \beta})(t)x \in D(A)$, and

$$\begin{aligned} {}^c D_t^\alpha v_x(t) &= S_{\alpha, \beta}(t)x = A(g_\alpha * S_{\alpha, \beta})(t)x + g_{\beta+1}(t)x \\ &= A v_x(t) + g_{\beta+1}(t)x, \quad t > 0. \end{aligned} \quad (54)$$

Thus, v_x is a classical solution of (53); it is unique by Theorem 12. Since $S_{\alpha, \beta}$ is exponentially bounded, we have that v_x is exponentially bounded. From

$$A v_x(t) = A(g_\alpha * S_{\alpha, \beta})(t)x = S_{\alpha, \beta}(t)x - g_{\beta+1}(t)x, \quad (55)$$

we know that $A v_x(t) \in L_{\text{loc}}^1(\mathbb{R}^+; X)$. So (ii) is true.

Assume that (ii) holds. From linearity of (53) and the uniqueness of its solution, we get that v_x is linear in x . So, for each $t \geq 0$, there exists a linear mapping $V(t) : X \rightarrow D(A)$ such that $V(t)x = v_x(t)$ for any $x \in X$.

Next, we show that, for each $t \geq 0$, $V(t) \in B(X, D(A))$.

We consider the mapping $\Phi : X \rightarrow C(\mathbb{R}^+, D(A))$ by $\Phi(x) = v_x(\cdot) = V(\cdot)x$. Then, Φ is a linear operator defined on X . Now we show that Φ is closed, if $x_n \rightarrow x$ in X and $\Phi(x_n) \rightarrow u$ in $C(\mathbb{R}^+, D(A))$. For $t > 0$, by the dominated convergence theorem, we have that $J_t^\alpha v_{x_n}(t)$ converges to $J_t^\alpha u(t)$, since $v_{x_n}(\cdot) = g_{\alpha+\beta+1}(t)x_n + J_t^\alpha A v_{x_n}(t)$, from the closeness of A , it follows that as $n \rightarrow \infty$, $u(t) = g_{\alpha+\beta+1}(t)x + J_t^\alpha A u(t)$, which implies that $u = \Phi(x)$ and Φ is closed. Therefore, by the closed graph theorem, Φ is bounded. So, for each $t \geq 0$, $V(t) \in B(X, D(A))$. Then, the exponential boundedness of $V(t)x$ and Lemma 3.2.14 in [2], imply that $\|V(t)\| \leq M e^{\omega t}$ ($t \geq 0$) for some constants $M, \omega \geq 0$. So $Q(\lambda)x = \lambda^{\beta+1} \int_0^\infty e^{-\lambda t} V(t)x dt$ is well defined for $\lambda > \omega$, $(\omega^\alpha, \infty) \subset \rho(A)$.

Since $AV(t)x \in L^1(\mathbb{R}^+, X)$, then the Laplace transform of $AV(t)x$ is well defined, and from the closeness of A , for $\lambda > \omega$, we have

$$\begin{aligned} (\lambda^\alpha - A)Q(\lambda)x &= \lambda^{\alpha+\beta+1} \int_0^\infty e^{-\lambda t} V(t)x dt - \lambda^{\alpha+\beta+1} \\ &\quad \times \int_0^\infty e^{-\lambda t} AV(t)x dt \\ &= \lambda^{\alpha+\beta+1} \int_0^\infty e^{-\lambda t} V(t)x dt - \lambda^{\beta+1} \\ &\quad \times \int_0^\infty e^{-\lambda t} {}^c D_t^\alpha V(t)x dt + \lambda^{\beta+1} \\ &\quad \times \int_0^\infty e^{-\lambda t} g_{\beta+1}(t)x dt \\ &= \lambda^{\alpha+\beta+1} \widehat{V}(\lambda)x - \lambda^{\beta+1} \lambda^\alpha \widehat{V}(\lambda)x \\ &\quad + \lambda^{\beta+1} \lambda^{-(\beta+1)} x = x. \end{aligned} \quad (56)$$

Now, we show that $(\lambda^\alpha - A)$ is injective for $\lambda > \omega$. Assume that $(\lambda^\alpha - A)x = 0$ for some $x \in D(A)$ and $\lambda > \omega$. Then, by the method of Laplace transform, we have that the solution of (53) is $t^{\alpha+\beta} E_{\alpha, \alpha+\beta+1}(\lambda^\alpha t^\alpha)$. Since $\|v_x(t)\| \leq M e^{\omega t}$, for all $t \geq 0$, combine with (15), it follows that $x = 0$. Hence $(\lambda^\alpha - A)^{-1} = Q(\lambda)$ for $\lambda > \omega$ and $V(t)$ is an $(\alpha, \alpha + \beta)$ -ROF. Let

$$S_{\alpha, \beta}(t)x := {}^c D_t^\alpha V(t)x = AV(t)x + g_{\beta+1}(t)x; \quad (57)$$

then $S_{\alpha, \beta}(t)x$ exists and $V(t)x = J_t^\alpha S_{\alpha, \beta}(t)x$ for all $t \geq 0$ and all $x \in X$. So

$$S_{\alpha, \beta}(t)x = A J_t^\alpha S_{\alpha, \beta}(t)x + g_{\beta+1}(t)x, \quad (58)$$

and taking the Laplace transform, we have

$$\widehat{S}_{\alpha, \beta}(\lambda)x = A \lambda^{-\alpha} \widehat{S}_{\alpha, \beta}(\lambda)x + \lambda^{-\beta-1} x, \quad \lambda > \omega; \quad (59)$$

that is,

$$\widehat{S}_{\alpha,\beta}(\lambda)x = \lambda^{\alpha-\beta-1}(\lambda^\alpha - A)^{-1}x, \quad \lambda > \omega. \quad (60)$$

From Lemma 3, we know that $S_{\alpha,\beta}$ is the (α, β) -ROF generated by A . $\square$

Remark 16. Theorem 15 extends and generalizes Theorem 3.2.13 in [2]. In fact, when $\alpha = 1$ and $\beta = k$, (53) becomes (5), $S_{\alpha,\beta}(t)$ is a k -times integrated semigroup. For problem (5), the condition $Av_x \in L^1_{\text{loc}}(\mathbb{R}^+; X)$ in (ii) is not necessary. Since from the proof of Theorem 3.2.13 in [2], it is easy to see that the assumption that exponentially boundedness of the unique classical solution to the problem (5) imply that $Av_x \in L^1_{\text{loc}}(\mathbb{R}^+; X)$.

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