Research Article

Hybrid Extragradient Method with Regularization for Convex Minimization, Generalized Mixed Equilibrium, Variational Inequality and Fixed Point Problems

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We introduce two iterative algorithms by the hybrid extragradient method with regularization for finding a common element of the set of solutions of the minimization problem for a convex and continuously Fréchet differentiable functional, the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inequalities for inverse strong monotone mappings and the set of fixed points of an asymptotically κ-strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. We prove some strong and weak convergence theorems for the proposed iterative algorithms under mild conditions.

1. Introduction

Throughout this paper, we assume that H is a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$; let C be a nonempty closed convex subset of H and let P_C be the metric projection of H onto C. Let $S : C \to C$ be a self-mapping on C. We denote by Fix(S) the set of fixed points of S and by $\mathbf{R}$ the set of all real numbers.

Let $\varphi : C \to \mathbf{R}$ be a real-valued function, let $A : C \to H$ be a nonlinear mapping, and let $F : C \times C \to \mathbf{R}$ be a bifunction. In 2008, Peng and Yao [1] introduced the following generalized mixed equilibrium problem (GMEP) of finding $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x \rangle \geq 0, \quad \forall y \in C. \quad (1)$$

We denote the set of solutions of GMEP (1) by GMEP(F, φ, A). The GMEP (1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, and Nash equilibrium problems in noncooperative games. The GMEP is further considered and studied in [2–5].

Let $f : C \to \mathbf{R}$ be a convex and continuously Fréchet differentiable functional. Consider the convex minimization problem (CMP) of minimizing f over the constraint set C

$$\min_{x \in C} f(x) \quad (2)$$

(assuming the existence of minimizers). We denote by Γ the set of minimizers of CMP (2). The gradient-projection algorithm (GPA) generates a sequence $\{x_n\}$ determined by the gradient ∇f and the metric projection P_C as follows:

$$x_{n+1} := P_C(x_n - \lambda \nabla f(x_n)), \quad \forall n \geq 0, \quad (3)$$

or more generally,

$$x_{n+1} := P_C(x_n - \lambda_n \nabla f(x_n)), \quad \forall n \geq 0, \quad (3)'$$

where, in both (3) and (3)′, the initial guess x_0 is taken from C arbitrarily and the parameters λ or λ_n are positive real numbers. The convergence of algorithms (3) and (3)′ depends on the behavior of the gradient ∇f.

Since the Lipschitz continuity of the gradient ∇f implies that it is actually $(1/L)$-inverse strongly monotone (ism) [6],

its complement can be an averaged mapping (i.e., it can be expressed as a proper convex combination of the identity mapping and a nonexpansive mapping). Consequently, the GPA can be rewritten as the composite of a projection and an averaged mapping, which is again an averaged mapping. This shows that averaged mappings play an important role in the GPA. Recently, Xu [7] used averaged mappings to study the convergence analysis of the GPA, which is hence an operator-oriented approach.

Assume that the CMP (2) is consistent and the gradient ∇f is L-Lipschitz continuous with $L > 0$. Let $Q : C \to C$ be a ρ-contraction with $\rho \in [0, 1)$. Xu [7] introduced the following hybrid GPA:

$$x_{n+1} = \theta_n Q x_n + (1 - \theta_n) P_C (x_n - \lambda_n \nabla f (x_n)), \quad \forall n \geq 0, \tag{4}$$

where $\{\theta_n\} \subset [0, 1]$ and $0 < \liminf_{n \to \infty} \lambda_n \leq \limsup_{n \to \infty} \lambda_n < 2/L$. It was proven that under appropriate conditions the sequence $\{x_n\}$ converges in norm to a minimizer of CMP (2); see [7, Theorem 5.2].

It is worth emphasizing that the regularization, in particular the traditional Tikhonov regularization, is usually used to solve ill-posed optimization problems. Consider the regularized minimization problem

$$\min_{x \in C} f_\alpha (x) := f (x) + \frac{\alpha}{2} \|x\|^2, \tag{5}$$

where $\alpha > 0$ is the regularization parameter and again f is convex with L-Lipschitz continuous gradient ∇f. In [7], Xu introduced another hybrid GPA with regularization

$$\begin{aligned} x_{n+1} &= P_C \left(I - \lambda_n \nabla f_{\alpha_n} \right) x_n \\ &= P_C \left(I - \lambda_n \left(\nabla f + \alpha_n I \right) \right) x_n, \quad \forall n \geq 0, \end{aligned} \tag{6}$$

where (i) $0 < \lambda_n \leq \alpha_n/(L + \alpha_n)^2$ for all $n \geq 0$; (ii) $\alpha_n \to 0$ (and $\lambda_n \to 0$) as $n \to \infty$; (iii) $\sum_{n=0}^{\infty} \alpha_n \lambda_n = \infty$; and (iv) $(|\lambda_n - \lambda_{n-1}| + |\alpha_n \lambda_n - \alpha_{n-1} \lambda_{n-1}|)/(\alpha_n^2 \lambda_n^2) \to 0$ as $n \to \infty$. It was proven that $\{x_n\}$ converges strongly to the minimum-norm solution $x^\dagger \in \Gamma$ of CMP (2); see [7, Theorem 6.1]. Very recently, the hybrid GPA with regularization is extended to develop new extragradient methods with regularization in Ceng et al. [8, 9] for finding a common solution of the split feasibility problem (SFP) and the fixed point problem of a nonexpansive mapping in a real Hilbert space.

On the other hand, consider the following variational inequality problem (VIP): find a $\bar{x} \in C$ such that

$$\langle A\bar{x}, y - \bar{x} \rangle \geq 0, \quad \forall y \in C. \tag{7}$$

The solution set of VIP (7) is denoted by VI(C, A).

The VIP (7) was first discussed by Lions [10] and now is well known; there are a lot of different approaches towards solving VIP (7) in finite-dimensional and infinite-dimensional spaces, and the research is intensively continued. The VIP (7) has many applications in computational mathematics, mathematical physics, operations research, mathematical economics, optimization theory, and other fields; see,

for example, [11–14]. It is well known that if A is a strongly monotone and Lipschitz-continuous mapping on C, then VIP (7) has a unique solution. Not only the existence and uniqueness of solutions are important topics in the study of VIP (7), but also how to actually find a solution of VIP (7) is important.

Motivated by the idea of Korpelevič's extragradient method [15], Nadezhkina and Takahashi [16] introduced an extragradient iterative scheme:

$$\begin{aligned} x_0 &= x \in C \text{ chosen arbitrary,} \\ y_n &= P_C (x_n - \lambda_n A x_n), \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) S P_C (x_n - \lambda_n A y_n), \quad \forall n \geq 0, \end{aligned} \tag{8}$$

where $A : C \to H$ is a monotone, L-Lipschitz continuous mapping, $S : C \to C$ is a nonexpansive mapping, and $\{\lambda_n\} \subset [a, b]$ for some $a, b \in (0, 1/L)$ and $\{\alpha_n\} \subset [c, d]$ for some $c, d \in (0, 1)$. They proved the weak convergence of $\{x_n\}$ to an element of Fix(S) $\cap$ VI(C, A). Recently, inspired by Nadezhkina and Takahashi's iterative scheme [16], Zeng and Yao [17] introduced another iterative scheme for finding an element of Fix(S) $\cap$ VI(C, A) and derived the weak convergence result. Furthermore, by combining the CQ method and extragradient method, Nadezhkina and Takahashi [18] introduced an iterative process:

$$\begin{aligned} x_0 &= x \in C \text{ chosen arbitrary,} \\ y_n &= P_C (x_n - \lambda_n A x_n), \\ z_n &= \alpha_n x_n + (1 - \alpha_n) P_C (x_n - \lambda_n A y_n), \\ C_n &= \{z \in C : \|z_n - z\| \leq \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x, \quad \forall n \geq 0. \end{aligned} \tag{8}'$$

They proved the strong convergence of $\{x_n\}$ to an element of Fix(S) $\cap$ VI(C, A) under appropriate conditions. Later on, Ceng and Yao [19] introduced an extragradient-like approximation method which is based on the above extragradient method and viscosity approximation method and derived a strong convergence result as well. Next, recall some concepts. A mapping $A : C \to H$ is called L-Lipschitz continuous if there exists a constant $L \geq 0$ such that

$$\|Ax - Ay\| \leq L \|x - y\|, \quad \forall x, y \in C. \tag{9}$$

In particular, if $L = 1$, then A is called a nonexpansive mapping; if $L \in [0, 1)$, then A is called a contraction.

Recall that a mapping $A : C \to H$ is called

(i) monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in C; \tag{10}$$

(ii) η-strongly monotone if there exists a constant $\eta > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \eta \|x - y\|^2, \quad \forall x, y \in C; \tag{11}$$

(iii) α-inverse-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \quad (12)$$

It is obvious that if A is α-inverse-strongly monotone, then A is monotone and $(1/\alpha)$-Lipschitz continuous.

Definition 1. Let C be a nonempty subset of a normed space X and let $S : C \to C$ be a self-mapping on C.

(i) S is asymptotically nonexpansive (cf. [20]) if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n\to\infty} k_n = 1$ and

$$\|S^n x - S^n y\| \leq k_n \|x - y\|, \quad \forall n \geq 1, \ \forall x, y \in C; \quad (13)$$

(ii) S is asymptotically nonexpansive in the intermediate sense [21] provided S is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|S^n x - S^n y\| - \|x - y\|) \leq 0; \quad (14)$$

(iii) S is uniformly Lipschitzian if there exists a constant $\mathscr{L} > 0$ such that

$$\|S^n x - S^n y\| \leq \mathscr{L} \|x - y\|, \quad \forall n \geq 1, \ \forall x, y \in C. \quad (15)$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian.

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [20] as an important generalization of the class of nonexpansive mappings. The existence of fixed points of asymptotically nonexpansive mappings was proved by Goebel and Kirk [20] as follows.

Theorem GK (see [20, Theorem 1]). *If C is a nonempty closed convex bounded subset of a uniformly convex Banach space, then every asymptotically nonexpansive mapping $S : C \to C$ has a fixed point in C.*

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck et al. [21]. Recently, Kim and Xu [22] introduced the concept of asymptotically κ-strict pseudocontractive mappings in a Hilbert space as follows.

Definition 2. Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ if there exists a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\|S^n x - S^n y\|^2 \leq (1 + \gamma_n) \|x - y\|^2$$
$$+ \kappa \|x - S^n x - (y - S^n y)\|^2, \quad (16)$$
$$\forall n \geq 1, \quad \forall x, y \in C.$$

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$ is a uniformly $\mathscr{L}$-Lipschitzian mapping with $\mathscr{L} = \sup\{((\kappa + \sqrt{1 + (1-\kappa)\gamma_n})/(1 + \kappa)) : n \geq 1\}$.

Recently, Sahu et al. [23] considered the concept of asymptotically κ-strict pseudocontractive mappings in the intermediate sense, which are not necessarily Lipschitzian.

Definition 3. Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to C$ is said to be an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ if there exist a constant $\kappa \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, \infty)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that

$$\limsup_{n\to\infty} \sup_{x,y\in C} \left(\|S^n x - S^n y\|^2 - (1 + \gamma_n) \|x - y\|^2 \right.$$
$$\left. - \kappa \|x - S^n x - (y - S^n y)\|^2 \right) \leq 0. \quad (17)$$

Put $c_n := \max\{0, \sup_{x,y\in C}(\|S^n x - S^n y\|^2 - (1+\gamma_n)\|x - y\|^2 - \kappa\|x - S^n x - (y - S^n y)\|^2)\}$. Then $c_n \geq 0$ ($\forall n \geq 1$), $c_n \to 0 (n \to \infty)$ and (17) reduces to the relation

$$\|S^n x - S^n y\|^2 \leq (1 + \gamma_n) \|x - y\|^2$$
$$+ \kappa \|x - S^n x - (y - S^n y)\|^2 + c_n, \quad (18)$$
$$\forall n \geq 1, \ \forall x, y \in C.$$

Whenever $c_n = 0$ for all $n \geq 1$ in (18), then S is an asymptotically κ-strict pseudocontractive mapping with sequence $\{\gamma_n\}$. In 2009, Sahu et al. [23] derived the weak and strong convergence of the modified Mann iteration process for the class of asymptotically κ-strictly pseudocontractive mappings in the intermediate sense with sequence $\{\gamma_n\}$. More precisely, they established the following theorems.

Theorem SXY 1. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $\text{Fix}(S) \neq \emptyset$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa - \delta$ and $\sum_{n=1}^{\infty} \alpha_n c_n < \infty$. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in C generated by the modified Mann iteration process:*

$$x_1 = x \in C \ chosen \ arbitrary,$$
$$y_n = (1 - \alpha_n) x_n + \alpha_n S^n x_n,$$
$$C_n = \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\}, \quad (19)$$
$$Q_n = \left\{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \right\},$$
$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n S^n x_n, \quad \forall n \geq 1.$$

Then $\{x_n\}$ converges weakly to an element of $\text{Fix}(S)$.

Theorem SXY 2. *Let C be a nonempty closed convex subset of a real Hilbert space H and let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $\mathrm{Fix}(S)$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ such that $0 < \delta \leq \alpha_n \leq 1 - \kappa$ for all $n \geq 1$. Let $\{x_n\}$ be the sequence in C generated by the following (CQ) algorithm:*

$$x_1 = x \in C \text{ chosen arbitrary,}$$

$$y_n = (1 - \alpha_n) x_n + \alpha_n S^n x_n,$$

$$C_n = \left\{ z \in C : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\}, \qquad (20)$$

$$Q_n = \left\{ z \in C : \langle x_n - z, x - x_n \rangle \geq 0 \right\},$$

$$x_{n+1} = P_{C_n \cap Q_n} x, \quad \forall n \geq 1,$$

where $\theta_n = c_n + \gamma_n \Delta_n$ and $\Delta_n = \sup\{\|x_n - z\|^2 : z \in \mathrm{Fix}(S)\} < \infty$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{Fix}(S)} x$.

Subsequently, the iterative algorithms in Theorems SXY1 and SXY2 are extended to develop new iterative algorithms for finding a common solution of the VIP and the fixed point problem of an asymptotically strict pseudocontractive mapping in the intermediate sense in a real Hilbert space; see, for example, [24, 25].

On the other hand, Yao et al. [26] introduced two iterative algorithms for finding a common element of the set of fixed points of an asymptotically κ-strict pseudocontraction and the set of solutions of a mixed equilibrium problem in a real Hilbert space. Then they obtained some weak and strong convergence theorems for the proposed iterative algorithms. Very recently, motivated by Yao et al. [26], Cai and Bu [3] introduced two iterative algorithms for finding a common element of the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inequalities for inverse strong monotone mappings, and the set of fixed points of an asymptotically κ-strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. Then they proved some strong and weak convergence theorems for the proposed iterative algorithms under appropriate conditions.

In this paper, inspired by the above facts, we introduce two iterative algorithms by hybrid extragradient method with regularization for finding a common element of the set of solutions of the CMP (2) for a convex functional $f : C \to \mathbf{R}$ with L-Lipschitz continuous gradient ∇f, the set of solutions of finite GMEPs, the set of solutions of finite VIPs for inverse strong monotone mappings, and the set of fixed points of an asymptotically κ-strict pseudocontractive mapping in the intermediate sense in a real Hilbert space. Then we prove some strong and weak convergence theorems for the proposed iterative algorithms under mild conditions. For recent related results, see, for example, [7, 24, 27–31] and ther references therein.

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x and $x_n \to x$ to indicate that the sequence $\{x_n\}$ converges strongly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$; that is,

$$\omega_w(x_n) := \left\{ x \in H : x_{n_i} \rightharpoonup x \right.$$
$$\left. \text{for some subsequence } \left\{ x_{n_i} \right\} \text{ of } \{x_n\} \right\}. \qquad (21)$$

The metric (or nearest point) projection from H onto C is the mapping $P_C : H \to C$ which assigns to each point $x \in H$ the unique point $P_C x \in C$ satisfying the property

$$\|x - P_C x\| = \inf_{y \in C} \|x - y\| =: d(x, C). \qquad (22)$$

Some important properties of projections are gathered in the following proposition.

Proposition 4. *For given $x \in H$ and $z \in C$:*

(i) $z = P_C x \Leftrightarrow \langle x - z, y - z \rangle \leq 0,\ \text{for all}\ y \in C$;

(ii) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2,\ \text{for all}\ y \in C$;

(iii) $\langle P_C x - P_C y, x - y \rangle \geq \|P_C x - P_C y\|^2,\ \text{for all}\ y \in H$.

Consequently, P_C is nonexpansive and monotone.

If A is an α-inverse-strongly monotone mapping of C into H, then it is obvious that A is $(1/\alpha)$-Lipschitz continuous. We also have that, for all $u, v \in C$ and $\lambda > 0$,

$$\|(I - \lambda A) u - (I - \lambda A) v\|^2$$
$$= \|(u - v) - \lambda (Au - Av)\|^2$$
$$= \|u - v\|^2 - 2\lambda \langle Au - Av, u - v \rangle \qquad (23)$$
$$+ \lambda^2 \|Au - Av\|^2$$
$$\leq \|u - v\|^2 + \lambda (\lambda - 2\alpha) \|Au - Av\|^2.$$

So if $\lambda \leq 2\alpha$, then $I - \lambda A$ is a nonexpansive mapping from C to H.

Definition 5. A mapping $T : H \to H$ is said to be

(a) nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in H; \qquad (24)$$

(b) firmly nonexpansive if $2T - I$ is nonexpansive, or equivalently, if T is 1-inverse strongly monotone (1-ism),

$$\langle x - y, Tx - Ty \rangle \geq \|Tx - Ty\|^2, \quad \forall x, y \in H; \qquad (25)$$

alternatively, T is firmly nonexpansive if and only if T can be expressed as

$$T = \frac{1}{2}(I + S), \tag{26}$$

where $S : H \to H$ is nonexpansive; projections are firmly nonexpansive.

It can be easily seen that if T is nonexpansive, then $I - T$ is monotone. It is also easy to see that a projection P_C is 1-ism. Inverse strongly monotone (also referred to as co-coercive) operators have been applied widely in solving practical problems in various fields.

Definition 6. A mapping $T : H \to H$ is said to be an averaged mapping if it can be written as the average of the identity I and a nonexpansive mapping; that is,

$$T \equiv (1 - \alpha)I + \alpha S, \tag{27}$$

where $\alpha \in (0, 1)$ and $S : H \to H$ is nonexpansive. More precisely, when the last equality holds, we say that T is α-averaged. Thus, firmly nonexpansive mappings (in particular, projections) are $(1/2)$-averaged maps.

Proposition 7 (see [32]). *Let $T : H \to H$ be a given mapping.*

(i) *T is nonexpansive if and only if the complement $I - T$ is $(1/2)$-ism.*

(ii) *If T is ν-ism, then for $\gamma > 0$, γT is (ν/γ)-ism.*

(iii) *T is averaged if and only if the complement $I - T$ is ν-ism for some $\nu > 1/2$. Indeed, for $\alpha \in (0, 1)$, T is α-averaged if and only if $I - T$ is $(1/2\alpha)$-ism.*

Proposition 8 (see [32, 33]). *Let $S, T, V : H \to H$ be given operators.*

(i) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is averaged and V is nonexpansive, then T is averaged.*

(ii) *T is firmly nonexpansive if and only if the complement $I - T$ is firmly nonexpansive.*

(iii) *If $T = (1 - \alpha)S + \alpha V$ for some $\alpha \in (0, 1)$ and if S is firmly nonexpansive and V is nonexpansive, then T is averaged.*

(iv) *The composite of finitely many averaged mappings is averaged. That is, if each of the mappings $\{T_i\}_{i=1}^{N}$ is averaged, then so is the composite $T_1 \cdots T_N$. In particular, if T_1 is α_1-averaged and T_2 is α_2-averaged, where $\alpha_1, \alpha_2 \in (0, 1)$, then the composite $T_1 T_2$ is α-averaged, where $\alpha = \alpha_1 + \alpha_2 - \alpha_1\alpha_2$.*

(v) *If the mappings $\{T_i\}_{i=1}^{N}$ are averaged and have a common fixed point, then*

$$\bigcap_{i=1}^{N} \text{Fix}(T_i) = \text{Fix}(T_1 \cdots T_N). \tag{28}$$

The notation $\text{Fix}(T)$ denotes the set of all fixed points of the mapping T; that is, $\text{Fix}(T) = \{x \in H : Tx = x\}$.

We need some facts and tools in a real Hilbert space H which are listed as lemmas below.

Lemma 9. *Let X be a real inner product space. Then there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall x, y \in X. \tag{29}$$

Lemma 10. *Let $\{x_n\}$ be a bounded sequence on a reflexive Banach space X. If $\omega_w(\{x_n\}) = \{x\}$, then $x_n \to x$.*

Lemma 11. *Let $A : C \to H$ be a monotone mapping. In the context of the variational inequality problem the characterization of the projection (see Proposition 4 (i)) implies*

$$u \in \text{VI}(C, A) \Longleftrightarrow u = P_C(u - \lambda Au), \quad \text{for some } \lambda > 0. \tag{30}$$

Lemma 12. *Let H be a real Hilbert space. Then the following hold:*

(i) *$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y\rangle$ for all $x, y \in H$;*

(ii) *$\|(1 - t)x + ty\|^2 = (1-t)\|x\|^2 + t\|y\|^2 - t(1-t)\|x - y\|^2$ for all $t \in [0, 1]$ and for all $x, y \in H$;*

(iii) *If $\{x_n\}$ is a sequence in H such that $x_n \to x$, it follows that*

$$\limsup_{n \to \infty} \|x_n - y\|^2 = \limsup_{n \to \infty} \|x_n - x\|^2 + \|x - y\|^2, \quad \forall y \in H. \tag{31}$$

Lemma 13 ([23, Lemma 2.5]). *Let H be a real Hilbert space. Given a nonempty closed convex subset of H and points $x, y, z \in H$ and given also a real number $a \in \mathbf{R}$, the set*

$$\left\{ v \in C : \|y - v\|^2 \leq \|x - v\|^2 + \langle z, v\rangle + a \right\} \tag{32}$$

is convex (and closed).

Lemma 14 ([23, Lemma 2.6]). *Let C be a nonempty subset of a Hilbert space H and let $S : C \to C$ be an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then*

$$\|S^n x - S^n y\|$$

$$\leq \frac{1}{1 - \kappa}$$

$$\times \left(\kappa \|x - y\| \sqrt{(1 + (1 - \kappa)\gamma_n)\|x - y\|^2 + (1 - \kappa)c_n}\right) \tag{33}$$

for all $x, y \in C$ and $n \geq 1$.

Lemma 15 ([23, Lemma 2.7]). *Let C be a nonempty subset of a Hilbert space H and let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - S^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Sx_n\| \to 0$ as $n \to \infty$.*

Lemma 16 (demiclosedness principle [23, Proposition 3.1]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $S : C \to C$ be a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Then $I - S$ is demiclosed at zero in the sense that if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{m\to\infty}\limsup_{n\to\infty}\|x_n - S^m x_n\| = 0$, then $(I - S)x = 0$.*

Lemma 17 ([23, Proposition 3.2]). *Let C be a nonempty closed convex subset of a Hilbert space H and let $S : C \to C$ be a continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$ such that $\mathrm{Fix}(S) \neq \emptyset$. Then $\mathrm{Fix}(S)$ is closed and convex.*

Remark 18. Lemmas 16 and 17 give some basic properties of an asymptotically κ-strict pseudocontractive mapping in the intermediate sense with sequence $\{\gamma_n\}$. Moreover, Lemma 16 extends the demiclosedness principles studied for certain classes of nonlinear mappings in Kim and Xu [22], Górnicki [34], Marino and Xu [35], and Xu [36].

To prove a weak convergence theorem by a modified extragradient method with regularization for the CMP (2) and the fixed point problem of an asymptotically κ-strict pseudocontractive mapping in the intermediate sense, we need the following lemma due to Osilike et al. [37].

Lemma 19 (see [37, page. 80]). *Let $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ and $\{\delta_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1 + \delta_n) a_n + b_n, \quad \forall n \ge 1. \tag{34}$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists. If, in addition, $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges to zero, then $\lim_{n\to\infty} a_n = 0$.

Corollary 20 (see [38, page. 303]). *Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le a_n + b_n, \quad \forall n \ge 0. \tag{35}$$

If $\sum_{n=0}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.

Recall that a Banach space X is said to satisfy the Opial condition [39] if for any given sequence $\{x_n\} \subset X$ which converges weakly to an element $x \in X$, there holds the inequality

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, \ y \neq x. \tag{36}$$

It is well known in [39] that every Hilbert space H satisfies the Opial condition.

Lemma 21 (see [24, Proposition 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H and let $\{x_n\}$ be a sequence in H. Suppose that*

$$\|x_{n+1} - p\|^2 \le (1 + \lambda_n) \|x_n - p\|^2 + \delta_n, \quad \forall p \in C, \ n \ge 1, \tag{37}$$

where $\{\lambda_n\}$ and $\{\delta_n\}$ are sequences of nonnegative real numbers such that $\sum_{n=1}^{\infty} \lambda_n < \infty$ and $\sum_{n=1}^{\infty} \delta_n < \infty$. Then $\{P_C x_n\}$ converges strongly in C.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x - y, f - g \rangle \ge 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H, \langle x - y, f - g \rangle \ge 0$ for all $(y, g) \in G(T)$ implies $f \in Tx$. Let $A : C \to H$ be a monotone, L-Lipschitz continuous mapping, and let $N_C v$ be the normal cone to C at $v \in C$; that is, $N_C v = \{w \in H : \langle v - u, w \rangle \ge 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C. \end{cases} \tag{38}$$

It is known that in this case T is maximal monotone, and $0 \in Tv$ if and only if $v \in \Omega$; see [40].

For solving the equilibrium problem, let us assume that the bifunction F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \le 0$ for any $x, y \in C$;

(A3) F is upper-hemicontinuous, that is, for each $x, y, z \in C$,

$$\limsup_{t \to 0^+} F(tz + (1 - t)x, y) \le F(x, y); \tag{39}$$

(A4) $F(x, \cdot)$ is convex and lower semicontinuous for each $x \in C$;

(B1) for each $x \in H$ and $r > 0$, there exists a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \backslash D_x$,

$$F(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{r} \langle y_x - z, z - x \rangle < 0; \tag{40}$$

(B2) C is a bounded set.

Lemma 22 (see [41]). *Assume that $F : C \times C \to \mathbf{R}$ satisfies (A1)–(A4) and let $\varphi : C \to \mathbf{R}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r^{(F,\varphi)} : H \to C$ as follows:*

$$T_r^{(F,\varphi)}(x) = \Big\{ z \in C : F(z, y) + \varphi(y) - \varphi(z) + \frac{1}{r} \langle y - z, z - x \rangle \ge 0, \forall y \in C \Big\} \tag{41}$$

for all $x \in H$. Then the following hold:

(1) *for each $x \in H, T_r^{(F,\varphi)}(x) \neq \emptyset$;*

(2) *$T_r^{(F,\varphi)}$ is single-valued;*

(3) $T_r^{(F,\varphi)}$ is firmly nonexpansive; that is, for any $x, y \in H$,

$$\left\| T_r^{(F,\varphi)} x - T_r^{(F,\varphi)} y \right\|^2 \leq \left\langle T_r^{(F,\varphi)} x - T_r^{(F,\varphi)} y, x - y \right\rangle; \quad (42)$$

(4) $\text{Fix}(T_r^{(F,\varphi)}) = \text{MEP}(F, \varphi)$;

(5) $\text{MEP}(F, \varphi)$ is closed and convex.

Lemma 23 (see [42]). *Let C be a closed convex subset of a real Hilbert space H. Let $\{x_n\}$ be a sequence in H and $u \in H$. Let $q = P_C u$. If $\{x_n\}$ is such that $\omega_w(x_n) \subset C$ and satisfies the condition*

$$\|x_n - u\| \leq \|u - q\|, \quad \forall n, \quad (43)$$

then $x_n \to q$ as $n \to \infty$.

3. Strong Convergence Theorem

In this section, we prove a strong convergence theorem for a hybrid extragradient iterative algorithm with regularization for finding a common element of the set of solutions of the CMP (2) for a convex functional $f : C \to \mathbf{R}$ with L-Lipschitz continuous gradient ∇f, the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inequalities for inverse strong monotone mappings, and the set of fixed points of an asymptotically κ-strict pseudocontractive mapping $S : C \to C$ in the intermediate sense in a real Hilbert space. This iterative algorithm with regularization is based on the extragradient method, shrinking projection method, Mann-type iterative method, and hybrid gradient projection algorithm (GPA) with regularization.

Theorem 24. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let M, N be two integers. Let F_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \dots, M\}$. Let B_k and A_i be μ_k-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, where $k \in \{1, 2, \dots, M\}$, $i \in \{1, 2, \dots, N\}$. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\lim_{n \to \infty} \gamma_n = 0$ and $\{c_n\} \subset [0, \infty)$ such that $\lim_{n \to \infty} c_n = 0$. Assume that $F := \cap_{k=1}^M \text{GMEP}(F_k, \varphi_k, A_k) \cap \cap_{i=1}^N \text{VI}(C, A_i) \cap \text{Fix}(S) \cap \Gamma$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0, \infty)$ and let $\{\beta_n\}, \{\delta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \beta_n \leq 1$ and $\kappa \leq \delta_n \leq b < 1$. Pick any $x_0 \in H$ and set $C_1 = C$, $x_1 = P_{C_1} x_0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$u_n = T_{r_{M,n}}^{(F_M, \varphi_M)} \left(I - r_{M,n} B_M \right) T_{r_{M,n}}^{(F_{M-1}, \varphi_{M-1})}$$
$$\times \left(I - r_{M-1,n} B_{M-1} \right) \cdots T_{r_{1,n}}^{(F_1, \varphi_1)} \left(I - r_{1,n} B_1 \right) x_n,$$

$$\tilde{x}_n = P_C \left(I - \lambda_{N,n} A_N \right) P_C \left(I - \lambda_{N-1,n} A_{N-1} \right)$$
$$\cdots P_C \left(I - \lambda_{2,n} A_2 \right) P_C \left(I - \lambda_{1,n} A_1 \right) u_n,$$

$$t_n = P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} (\tilde{x}_n) \right),$$

$$z_n = P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} (t_n) \right),$$

$$k_n = \delta_n z_n + (1 - \delta_n) S^n z_n,$$

$$y_n = (1 - \beta_n) x_n + \beta_n k_n,$$

$$C_{n+1} = \left\{ z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\},$$

$$x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0,$$

$$(44)$$

where $\theta_n = (\gamma_n + \lambda_n \alpha_n) \Delta_n^2 + 3 \lambda_n \alpha_n (1 + \gamma_n)^2 \varrho^2 + c_n$, $\Delta_n = \sup\{\|x_n - p\| : p \in F\} < \infty$ and $= \sup\{\|p\| : p \in F\} < \infty$. Assume that the following conditions hold:

(i) $\lim_{n \to \infty} \alpha_n = 0$;

(ii) $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, (1/L))$;

(iii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, for all $i \in \{1, 2, \dots, N\}$;

(iv) $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, for all $k \in \{1, 2, \dots, M\}$.

Then $\{x_n\}$ converge strongly to $P_F x_0$ provided either (B1) or (B2) holds.

Proof. First of all, one can show that $P_C(I - \lambda \nabla f_\alpha)$ is ξ-averaged for each $\lambda \in (0, (2/(\alpha + L)))$, where

$$\xi = \frac{2 + \lambda (\alpha + L)}{4} \in (0, 1), \quad (45)$$

which shows that $P_C(I - \lambda \nabla f_\alpha)$ is nonexpansive. Furthermore, for $\{\lambda_n\} \subset [c, d]$ with $c, d \in (0, (1/L))$, we have

$$c \leq \inf_{n \geq 1} \lambda_n \leq \sup_{n \geq 1} \lambda_n \leq d < \frac{1}{L} = \lim_{n \to \infty} \frac{1}{\alpha_n + L}. \quad (46)$$

Without loss of generality, we may assume that

$$c \leq \inf_{n \geq 1} \lambda_n \leq \sup_{n \geq 1} \lambda_n \leq d < \frac{1}{\alpha_n + L}, \quad \forall n \geq 1. \quad (47)$$

Consequently, it follows that for each integer $n \geq 1$, $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is ξ_n-averaged with

$$\xi_n = \frac{1}{2} + \frac{\lambda_n (\alpha_n + L)}{2} - \frac{1}{2} \cdot \frac{\lambda_n (\alpha_n + L)}{2}$$
$$= \frac{2 + \lambda_n (\alpha_n + L)}{4} \in (0, 1). \quad (48)$$

This immediately implies that $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is nonexpansive for all $n \geq 1$.

We divide the proof into several steps.

Step 1. We show that $\{x_n\}$ is well defined. It is obvious that C_n is closed and convex. As the defining inequality in C_n is equivalent to the inequality

$$\langle 2 (x_n - z_n), z \rangle \leq \|x_n\|^2 - \|z_n\|^2 + \theta_n, \quad (49)$$

by Lemma 13 we know that C_n is convex for every $n \geq 1$.

Next we show that $F \subset C_n$ for all $n \geq 1$. Put

$$\Theta_n^k = T_{r_{k,n}}^{(F_k, \varphi_k)} \left(I - r_{k,n} B_M \right) T_{r_{k-1,n}}^{(F_{k-1}, \varphi_{k-1})}$$
$$\times \left(I - r_{k-1,n} B_{k-1} \right) \cdots T_{r_{1,n}}^{(F_1, \varphi_1)} \left(I - r_{1,n} B_1 \right) x_n \quad (50)$$

for all $k \in \{1, 2, \ldots, M\}$ and $n \geq 1$,

$$\Omega_n^i = P_C \left(I - \lambda_{i,n} A_i \right) P_C \left(I - \lambda_{i-1,n} A_{i-1} \right)$$
$$\cdots P_C \left(I - \lambda_{2,n} A_2 \right) P_C \left(I - \lambda_{1,n} A_1 \right) \quad (51)$$

for all $i \in \{1, 2, \ldots, N\}$ and $n \geq 1$, and $\Theta_n^0 = \Omega_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Theta_n^M x_n$ and $\widetilde{x}_n = \Omega_n^N u_n$. Suppose that $F \subset C_n$ for some $n \geq 1$. Take $p \in F$ arbitrarily. Then from (23) and Lemma 22 we have

$$\|u_n - p\|$$
$$= \left\| T_{r_{M,n}}^{(F_M, \varphi_M)} \left(I - r_{M,n} B_M \right) \Theta_n^{M-1} x_n \right.$$
$$\left. - T_{r_{M,n}}^{(F_M, \varphi_M)} \left(I - r_{M,n} B_M \right) \Theta_n^{M-1} p \right\|$$
$$\leq \left\| \left(I - r_{M,n} B_M \right) \Theta_n^{M-1} x_n - \left(I - r_{M,n} B_M \right) \Theta_n^{M-1} p \right\| \quad (52)$$
$$\leq \left\| \Theta_n^{M-1} x_n - \Theta_n^{M-1} p \right\|$$
$$\vdots$$
$$\leq \left\| \Theta_n^0 x_n - \Theta_n^0 p \right\| = \|x_n - p\|.$$

Similarly, we have

$$\|\widetilde{x}_n - p\|$$
$$= \left\| P_C \left(I - \lambda_{N,n} A_N \right) \Omega_n^{N-1} u_n - P_C \left(I - \lambda_{N,n} A_N \right) \Omega_n^{N-1} p \right\|$$
$$\leq \left\| \left(I - \lambda_{N,n} A_N \right) \Omega_n^{N-1} u_n - \left(I - \lambda_{N,n} A_N \right) \Omega_n^{N-1} p \right\|$$
$$\leq \left\| \Omega_n^{N-1} u_n - \Omega_n^{N-1} p \right\|$$
$$\vdots$$
$$\leq \left\| \Omega_n^0 x_n - \Omega_n^0 p \right\| = \|u_n - p\|. \quad (53)$$

Combining (52) and (53), we have

$$\|\widetilde{x}_n - p\| \leq \|x_n - p\|. \quad (54)$$

Also, it follows from (44) that

$$\|t_n - p\|$$
$$= \left\| P_C \left(I - \lambda_n \nabla f_{\alpha_n} \right) \widetilde{x}_n - P_C \left(I - \lambda_n \nabla f \right) p \right\|$$
$$\leq \left\| P_C \left(I - \lambda_n \nabla f_{\alpha_n} \right) \widetilde{x}_n - P_C \left(I - \lambda_n \nabla f_{\alpha_n} \right) p \right\|$$
$$+ \left\| P_C \left(I - \lambda_n \nabla f_{\alpha_n} \right) p - P_C \left(I - \lambda_n \nabla f \right) p \right\| \quad (55)$$
$$\leq \|\widetilde{x}_n - p\| + \left\| \left(I - \lambda_n \nabla f_{\alpha_n} \right) p - \left(I - \lambda_n \nabla f \right) p \right\|$$
$$= \|\widetilde{x}_n - p\| + \lambda_n \alpha_n \|p\|.$$

Note that $z_n = P_C(\widetilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n))$ for every $n = 1, 2, \ldots$. Then, by Proposition 4(ii), we have

$$\|z_n - p\|^2$$
$$\leq \left\| \widetilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n) - p \right\|^2 - \left\| \widetilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n) - z_n \right\|^2$$
$$= \|\widetilde{x}_n - p\|^2 - \|\widetilde{x}_n - z_n\|^2 + 2\lambda_n \left\langle \nabla f_{\alpha_n}(t_n), p - z_n \right\rangle$$
$$= \|\widetilde{x}_n - p\|^2 - \|\widetilde{x}_n - z_n\|^2$$
$$\quad + 2\lambda_n \left(\left\langle \nabla f_{\alpha_n}(t_n) - \nabla f_{\alpha_n}(p), p - t_n \right\rangle \right.$$
$$\left. + \left\langle \nabla f_{\alpha_n}(p), p - t_n \right\rangle + \left\langle \nabla f_{\alpha_n}(t_n), t_n - z_n \right\rangle \right)$$
$$\leq \|\widetilde{x}_n - p\|^2 - \|\widetilde{x}_n - z_n\|^2$$
$$\quad + 2\lambda_n \left(\left\langle \nabla f_{\alpha_n}(p), p - t_n \right\rangle + \left\langle \nabla f_{\alpha_n}(t_n), t_n - z_n \right\rangle \right)$$
$$= \|\widetilde{x}_n - p\|^2 - \|\widetilde{x}_n - z_n\|^2$$
$$\quad + 2\lambda_n \left[\left\langle (\alpha_n I + \nabla f) p, p - t_n \right\rangle + \left\langle \nabla f_{\alpha_n}(t_n), t_n - z_n \right\rangle \right]$$
$$\leq \|\widetilde{x}_n - p\|^2 - \|\widetilde{x}_n - z_n\|^2$$
$$\quad + 2\lambda_n \left[\alpha_n \left\langle p, p - t_n \right\rangle + \left\langle \nabla f_{\alpha_n}(t_n), t_n - z_n \right\rangle \right]$$
$$= \|\widetilde{x}_n - p\|^2 - \|\widetilde{x}_n - t_n\|^2$$
$$\quad - 2 \left\langle \widetilde{x}_n - t_n, t_n - z_n \right\rangle - \|t_n - z_n\|^2$$
$$\quad + 2\lambda_n \left[\alpha_n \left\langle p, p - t_n \right\rangle + \left\langle \nabla f_{\alpha_n}(t_n), t_n - z_n \right\rangle \right]$$
$$= \|\widetilde{x}_n - p\|^2 - \|\widetilde{x}_n - t_n\|^2 - \|t_n - z_n\|^2$$
$$\quad + 2 \left\langle \widetilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n) - t_n, z_n - t_n \right\rangle$$
$$\quad + 2\lambda_n \alpha_n \left\langle p, p - t_n \right\rangle. \quad (56)$$

Further, by Proposition 4(i), we have

$$\langle \tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n) - t_n, z_n - t_n \rangle$$

$$= \langle \tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(\tilde{x}_n) - t_n, z_n - t_n \rangle$$

$$\quad + \langle \lambda_n \nabla f_{\alpha_n}(\tilde{x}_n) - \lambda_n \nabla f_{\alpha_n}(t_n), z_n - t_n \rangle$$

$$\leq \langle \lambda_n \nabla f_{\alpha_n}(\tilde{x}_n) - \lambda_n \nabla f_{\alpha_n}(t_n), z_n - t_n \rangle$$

$$\leq \lambda_n \left\| \nabla f_{\alpha_n}(\tilde{x}_n) - \nabla f_{\alpha_n}(t_n) \right\| \left\| z_n - t_n \right\|$$

$$\leq \lambda_n (\alpha_n + L) \left\| \tilde{x}_n - t_n \right\| \left\| z_n - t_n \right\|. \tag{57}$$

So from (54) and (55), we obtain

$$\|z_n - p\|^2 \leq \|\tilde{x}_n - p\|^2 - \|\tilde{x}_n - t_n\|^2 - \|t_n - z_n\|^2$$

$$\quad + 2\lambda_n (\alpha_n + L) \|\tilde{x}_n - t_n\| \|z_n - t_n\|$$

$$\quad + 2\lambda_n \alpha_n \langle p, p - t_n \rangle$$

$$\leq \|\tilde{x}_n - p\|^2 - \|\tilde{x}_n - t_n\|^2 - \|t_n - z_n\|^2$$

$$\quad + \lambda_n^2 (\alpha_n + L)^2 \|\tilde{x}_n - t_n\|^2 + \|t_n - z_n\|^2$$

$$\quad + 2\lambda_n \alpha_n \|p\| \|p - t_n\|$$

$$= \|\tilde{x}_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\|$$

$$\quad + \left(\lambda_n^2 (\alpha_n + L)^2 - 1 \right) \|\tilde{x}_n - t_n\|^2$$

$$\leq \|\tilde{x}_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\|$$

$$\leq \|\tilde{x}_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \left[\|\tilde{x}_n - p\| + \lambda_n \alpha_n \|p\| \right]$$

$$\leq \|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \left[\|x_n - p\| + \lambda_n \alpha_n \|p\| \right]$$

$$= \|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|x_n - p\| + 2\lambda_n^2 \alpha_n^2 \|p\|^2. \tag{58}$$

By Lemma 12 and (58), we have

$$\|k_n - p\|^2$$

$$= \|\delta_n (z_n - p) + (1 - \delta_n)(S^n z_n - p)\|^2$$

$$= \delta_n \|z_n - p\|^2 + (1 - \delta_n) \|S^n z_n - p\|^2$$

$$\quad - \delta_n (1 - \delta_n) \|z_n - S^n z_n\|^2$$

$$\leq \delta_n \|z_n - p\|^2 + (1 - \delta_n)$$

$$\quad \times \left[(1 + \gamma_n) \|z_n - p\|^2 + \kappa \|z_n - S^n z_n\|^2 + c_n \right]$$

$$\quad - \delta_n (1 - \delta_n) \|z_n - S^n z_n\|^2$$

$$= \left[1 + \gamma_n (1 - \delta_n) \right] \|z_n - p\|^2$$

$$\quad + (1 - \delta_n)(\kappa - \delta_n) \|z_n - S^n z_n\|^2 + (1 - \delta_n) c_n$$

$$\leq \left[1 + \gamma_n (1 - \delta_n) \right]$$

$$\quad \times \left[\|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|x_n - p\| + 2\lambda_n^2 \alpha_n^2 \|p\|^2 \right]$$

$$\quad + (1 - \delta_n)(\kappa - \delta_n) \|z_n - S^n z_n\|^2 + c_n$$

$$\leq \left[1 + \gamma_n (1 - \delta_n) \right] \left[\|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \right.$$

$$\quad \left. \times \|x_n - p\| + 2\lambda_n^2 \alpha_n^2 \|p\|^2 \right] + c_n$$

$$\leq (1 + \gamma_n) \left[\|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|x_n - p\| \right.$$

$$\quad \left. + 2\lambda_n^2 \alpha_n^2 \|p\|^2 \right] + c_n. \tag{59}$$

It follows from (59) and $\lambda_n(\alpha_n + L) < 1$ that

$$\|y_n - p\|^2$$

$$= \left\| (1 - \beta_n)(x_n - p) + \beta_n (k_n - p) \right\|^2$$

$$\leq (1 - \beta_n) \|x_n - p\|^2 + \beta_n \|k_n - p\|^2$$

$$\leq (1 - \beta_n) \|x_n - p\|^2$$

$$\quad + \beta_n \left\{ (1 + \gamma_n) \left[\|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|x_n - p\| \right. \right.$$

$$\quad \left. \left. + 2\lambda_n^2 \alpha_n^2 \|p\|^2 \right] + c_n \right\}$$

$$\leq (1 + \gamma_n) \left[\|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|x_n - p\| \right.$$

$$\quad \left. + 2\lambda_n^2 \alpha_n^2 \|p\|^2 \right] + c_n$$

$$= (1 + \gamma_n) \|x_n - p\|^2 + 2\lambda_n \alpha_n (1 + \gamma_n) \|p\| \|x_n - p\|$$

$$\quad + 2(1 + \gamma_n) \lambda_n^2 \alpha_n^2 \|p\|^2 + c_n$$

$$\leq (1 + \gamma_n) \|x_n - p\|^2$$

$$\quad + \lambda_n \alpha_n \left[(1 + \gamma_n)^2 \|p\|^2 + \|x_n - p\|^2 \right]$$

$$\quad + 2(1 + \gamma_n) \lambda_n^2 \alpha_n^2 \|p\|^2 + c_n$$

$$\leq (1 + \gamma_n) \|x_n - p\|^2$$

$$\quad + \lambda_n \alpha_n \left[(1 + \gamma_n)^2 \|p\|^2 + \|x_n - p\|^2 \right]$$

$$\quad + 2(1 + \gamma_n)^2 \lambda_n \alpha_n \|p\|^2 + c_n$$

$$= (1 + \gamma_n) \|x_n - p\|^2$$

$$+ \lambda_n \alpha_n \left[3(1 + \gamma_n)^2 \|p\|^2 + \|x_n - p\|^2 \right] + c_n$$

$$\leq \|x_n - p\|^2 + (\gamma_n + \lambda_n \alpha_n) \Delta_n^2$$

$$+ 3\lambda_n \alpha_n (1 + \gamma_n)^2 \varrho^2 + c_n = \|x_n - p\|^2 + \theta_n. \tag{60}$$

Hence $p \in C_{n+1}$. This implies that $F \subset C_n$ for all $n \geq 1$. Therefore, $\{x_n\}$ is well defined.

Step 2. We prove that $\|x_n - k_n\| \to 0$ as $n \to \infty$.

Indeed, let $v = P_F x_0$. From $x_n = P_{C_n} x_0$ and $v \in F \subset C_n$, we obtain

$$\|x_n - x_0\| \leq \|v - x_0\|. \tag{61}$$

This implies that $\{x_n\}$ is bounded and hence $\{u_n\}$, $\{\tilde{x}_n\}, \{t_n\}, \{z_n\}, \{k_n\}$, and $\{y_n\}$ are also bounded. Since $x_{n+1} \in C_{n+1} \subset C_n$ and $x_n = P_{C_n} x_0$, we have

$$\|x_n - x_0\| \leq \|x_{n+1} - x_0\|, \quad \forall n \geq 1. \tag{62}$$

Therefore, $\lim_{n \to \infty} \|x_n - x_0\|$ exists. From $x_n = P_{C_n} x_0, x_{n+1} \in C_{n+1} \subset C_n$, we obtain

$$\|x_{n+1} - x_n\|^2 \leq \|x_0 - x_{n+1}\|^2 - \|x_0 - x_n\|^2, \tag{63}$$

which implies

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{64}$$

It follows from $x_{n+1} \in C_{n+1}$ that $\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \theta_n$ and hence

$$\|x_n - y_n\|^2 \leq 2 \left(\|x_n - x_{n+1}\|^2 + \|x_{n+1} - y_n\|^2 \right)$$

$$\leq 2 \left(\|x_n - x_{n+1}\|^2 + \|x_n - x_{n+1}\|^2 + \theta_n \right) \tag{65}$$

$$= 2 \left(2\|x_n - x_{n+1}\|^2 + \theta_n \right).$$

From (64) and $\lim_{n \to \infty} \theta_n = 0$, we have

$$\lim_{n \to \infty} \|x_n - y_n\| = 0. \tag{66}$$

Note that

$$y_n - x_n = \beta_n (k_n - x_n). \tag{67}$$

Since $0 < a \leq \beta_n$ and (66), we obtain

$$\lim_{n \to \infty} \|x_n - k_n\| = 0. \tag{68}$$

Step 3. We prove that $\|x_n - u_n\| \to 0, \|u_n - z_n\| \to 0$, and $\|z_n - S^n z_n\| \to 0$ as $n \to \infty$.

Indeed, from (58) and (59) it follows that

$$\|k_n - p\|^2 \leq [1 + \gamma_n (1 - \delta_n)] \|z_n - p\|^2$$

$$+ (1 - \delta_n) (\kappa - \delta_n) \|z_n - S^n z_n\|^2$$

$$+ (1 - \delta_n) c_n$$

$$\leq \|z_n - p\|^2 + \gamma_n \|z_n - p\|^2 + c_n$$

$$\leq \|z_n - p\|^2$$

$$+ \gamma_n \left[\|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|x_n - p\| \right.$$

$$\left. + 2\lambda_n^2 \alpha_n^2 \|p\|^2 \right] + c_n \tag{69}$$

$$\leq \|z_n - p\|^2$$

$$+ \gamma_n \left[\|x_n - p\|^2 + 2\sqrt{2}\lambda_n \alpha_n \|p\| \|x_n - p\| \right.$$

$$\left. + 2\lambda_n^2 \alpha_n^2 \|p\|^2 \right] + c_n$$

$$\leq \|z_n - p\|^2$$

$$+ \gamma_n \left(\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\| \right)^2 + c_n.$$

Next we prove that

$$\lim_{n \to \infty} \left\| \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\| = 0, \quad k = 1, 2, \ldots, M. \tag{70}$$

For $p \in F$, it follows from (23) that

$$\left\| \Theta_n^k x_n - p \right\|^2 = \left\| T_{r_{k,n}}^{(F_k, \varphi_k)} (I - r_{k,n} B_k) \Theta_n^{k-1} x_n \right.$$

$$\left. - T_{r_{k,n}}^{(F_k, \varphi_k)} (I - r_{k,n} B_k) p \right\|^2$$

$$\leq \left\| (I - r_{k,n} B_k) \Theta_n^{k-1} x_n - (I - r_{k,n} B_k) p \right\|^2$$

$$\leq \left\| \Theta_n^{k-1} x_n - p \right\|^2 + r_{k,n} (r_{k,n} - 2\mu_k) \tag{71}$$

$$\times \left\| B_k \Theta_n^{k-1} x_n - B_k p \right\|^2$$

$$\leq \|x_n - p\|^2 + r_{k,n} (r_{k,n} - 2\mu_k)$$

$$\times \left\| B_k \Theta_n^{k-1} x_n - B_k p \right\|^2.$$

By (52), (53), (58), (69), and (71), we obtain

$$
\begin{aligned}
\|k_n - p\|^2 \\
&\leq \|z_n - p\|^2 + \gamma_n\big(\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\|\big)^2 + c_n \\
&\leq \|\tilde{x}_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\big[\|\tilde{x}_n - p\| + \lambda_n\alpha_n\|p\|\big] \\
&\quad + \gamma_n\big(\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\|\big)^2 + c_n \\
&\leq \|u_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\big[\|u_n - p\| + \lambda_n\alpha_n\|p\|\big] \\
&\quad + \gamma_n\big(\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\|\big)^2 + c_n \\
&\leq \|u_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\big[\|x_n - p\| + \lambda_n\alpha_n\|p\|\big] \\
&\quad + \gamma_n\big(\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\|\big)^2 + c_n \\
&\leq \|\Theta_n^k x_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\big[\|x_n - p\| + \lambda_n\alpha_n\|p\|\big] \\
&\quad + \gamma_n\big(\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\|\big)^2 + c_n \\
&\leq \|x_n - p\|^2 + r_{k,n}\big(r_{k,n} - 2\mu_k\big)\big\|B_k\Theta_n^{k-1} x_n - B_k p\big\|^2 \\
&\quad + 2\lambda_n\alpha_n\|p\|\big[\|x_n - p\| + \lambda_n\alpha_n\|p\|\big] \\
&\quad + \gamma_n\big(\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\|\big)^2 + c_n,
\end{aligned}
\tag{72}
$$

which implies that

$$
\begin{aligned}
r_{k,n}\big(2\mu_k - r_{k,n}\big)\big\|B_k\Theta_n^{k-1} x_n - B_k p\big\|^2 \\
&\leq \|x_n - p\|^2 - \|k_n - p\|^2 \\
&\quad + 2\lambda_n\alpha_n\|p\|\big[\|x_n - p\| + \lambda_n\alpha_n\|p\|\big] \\
&\quad + \gamma_n\big(\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\|\big)^2 + c_n \\
&\leq \|x_n - k_n\|\big(\|x_n - p\| + \|k_n - p\|\big) \\
&\quad + 2\lambda_n\alpha_n\|p\|\big[\|x_n - p\| + \lambda_n\alpha_n\|p\|\big] \\
&\quad + \gamma_n\big(\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\|\big)^2 + c_n.
\end{aligned}
\tag{73}
$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, 1/L)$, $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $k \in \{1, 2, \ldots, M\}$, $\alpha_n \to 0$, $\gamma_n \to 0$, and $c_n \to 0$, we conclude from (68) and the boundedness of $\{x_n\}, \{k_n\}$ that

$$
\lim_{n\to\infty}\big\|B_k\Theta_n^{k-1} x_n - B_k p\big\| = 0, \quad k = 1, 2, \ldots, M.
\tag{74}
$$

By Lemmas 12 and 22, we have

$$
\begin{aligned}
\big\|\Theta_n^k x_n - p\big\|^2 \\
&= \big\|T_{r_{k,n}}^{(F_k, \varphi_k)}(I - r_{k,n}B_k)\Theta_n^{k-1} x_n - T_{r_{k,n}}^{(F_k, \varphi_k)}(I - r_{k,n}B_k)p\big\|^2 \\
&\leq \big\langle (I - r_{k,n}B_k)\Theta_n^{k-1} x_n - (I - r_{k,n}B_k)p, \Theta_n^k x_n - p\big\rangle \\
&= \frac{1}{2}\Big(\big\|(I - r_{k,n}B_k)\Theta_n^{k-1} x_n - (I - r_{k,n}B_k)p\big\|^2 \\
&\quad + \big\|\Theta_n^k x_n - p\big\|^2 - \big\|(I - r_{k,n}B_k)\Theta_n^{k-1} x_n \\
&\qquad\qquad - (I - r_{k,n}B_k)p - (\Theta_n^k x_n - p)\big\|^2\Big) \\
&\leq \frac{1}{2}\Big(\big\|\Theta_n^{k-1} x_n - p\big\|^2 + \big\|\Theta_n^k x_n - p\big\|^2 \\
&\quad -\big\|\Theta_n^{k-1} x_n - \Theta_n^k x_n - r_{k,n}(B_k\Theta_n^{k-1} x_n - B_k p)\big\|^2\Big),
\end{aligned}
\tag{75}
$$

which implies that

$$
\begin{aligned}
\big\|\Theta_n^k x_n - p\big\|^2 &\leq \big\|\Theta_n^{k-1} x_n - p\big\|^2 \\
&\quad - \big\|\Theta_n^{k-1} x_n - \Theta_n^k x_n \\
&\qquad\quad - r_{k,n}\big(B_k\Theta_n^{k-1} x_n - B_k p\big)\big\|^2 \\
&= \big\|\Theta_n^{k-1} x_n - p\big\|^2 - \big\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\big\|^2 \\
&\quad - r_{k,n}^2\big\|B_k\Theta_n^{k-1} x_n - B_k p\big\|^2 \\
&\quad + 2r_{k,n}\big\langle \Theta_n^{k-1} x_n - \Theta_n^k x_n, B_k\Theta_n^{k-1} x_n - B_k p\big\rangle \\
&\leq \big\|\Theta_n^{k-1} x_n - p\big\|^2 - \big\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\big\|^2 \\
&\quad + 2r_{k,n}\big\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\big\|\big\|B_k\Theta_n^{k-1} x_n - B_k p\big\| \\
&\leq \|x_n - p\|^2 - \big\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\big\|^2 \\
&\quad + 2r_{k,n}\big\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\big\|\big\|B_k\Theta_n^{k-1} x_n - B_k p\big\|.
\end{aligned}
\tag{76}
$$

Utilizing (72) and (76), we have

$$
\begin{aligned}
\|k_n - p\|^2 \\
&\leq \big\|\Theta_n^k x_n - p\big\|^2 + 2\lambda_n\alpha_n\|p\|\big[\|x_n - p\| + \lambda_n\alpha_n\|p\|\big] \\
&\quad + \gamma_n\big(\|x_n - p\| + \sqrt{2}\lambda_n\alpha_n\|p\|\big)^2 + c_n
\end{aligned}
$$

$$\leq \|x_n - p\|^2 - \left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\|^2$$
$$+ 2r_{k,n} \left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\| \left\|B_k \Theta_n^{k-1} x_n - B_k p\right\|$$
$$+ 2\lambda_n \alpha_n \|p\| \left[\|x_n - p\| + \lambda_n \alpha_n \|p\|\right]$$
$$+ \gamma_n \left(\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|\right)^2 + c_n,$$

(77)

which implies

$$\left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\|^2$$
$$\leq \|x_n - p\|^2 - \|k_n - p\|^2$$
$$+ 2r_{k,n} \left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\| \left\|B_k \Theta_n^{k-1} x_n - B_k p\right\|$$
$$+ 2\lambda_n \alpha_n \|p\| \left[\|x_n - p\| + \lambda_n \alpha_n \|p\|\right]$$
$$+ \gamma_n \left(\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|\right)^2 + c_n$$

(78)

$$\leq \|x_n - k_n\| \left(\|x_n - p\| + \|k_n - p\|\right)$$
$$+ 2r_{k,n} \left\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\right\| \left\|B_k \Theta_n^{k-1} x_n - B_k p\right\|$$
$$+ 2\lambda_n \alpha_n \|p\| \left[\|x_n - p\| + \lambda_n \alpha_n \|p\|\right]$$
$$+ \gamma_n \left(\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|\right)^2 + c_n.$$

Since $\{\lambda_n\} \subset [c,d] \subset (0,1/L)$, $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $k \in \{1,2,\ldots,M\}$, $\alpha_n \to 0$, $\gamma_n \to 0$, and $c_n \to 0$, we conclude from (68) and (74) and the boundedness of $\{x_n\}$, $\{k_n\}$ that (70) holds. Hence we obtain

$$\|x_n - u_n\| = \left\|\Theta_n^0 x_n - \Theta_n^M x_n\right\|$$
$$\leq \left\|\Theta_n^0 x_n - \Theta_n^1 x_n\right\| + \left\|\Theta_n^1 x_n - \Theta_n^2 x_n\right\|$$
$$+ \cdots + \left\|\Theta_n^{M-1} x_n - \Theta_n^M x_n\right\|$$

(79)

$$\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Next we show that $\lim_{n\to\infty} \|A_i \Omega_n^i u_n - A_i p\| = 0$, $i = 1,2,\ldots,N$. It follows from (23) that

$$\left\|\Omega_n^i u_n - p\right\|^2$$
$$= \left\|P_C \left(I - \lambda_{i,n} A_i\right) \Omega_n^{i-1} u_n - P_C \left(I - \lambda_{i,n} A_i\right) p\right\|^2$$
$$\leq \left\|\left(I - \lambda_{i,n} A_i\right) \Omega_n^{i-1} u_n - \left(I - \lambda_{i,n} A_i\right) p\right\|^2$$
$$\leq \left\|\Omega_n^{i-1} u_n - p\right\|^2 + \lambda_{i,n} \left(\lambda_{i,n} - 2\eta_i\right) \left\|A_i \Omega_n^{i-1} u_n - A_i p\right\|^2$$
$$\leq \|u_n - p\|^2 + \lambda_{i,n} \left(\lambda_{i,n} - 2\eta_i\right) \left\|A_i \Omega_n^{i-1} u_n - A_i p\right\|^2$$
$$\leq \|x_n - p\|^2 + \lambda_{i,n} \left(\lambda_{i,n} - 2\eta_i\right) \left\|A_i \Omega_n^{i-1} u_n - A_i p\right\|^2.$$

(80)

Utilizing (54), (72), and (80), we get

$$\|k_n - p\|^2$$
$$\leq \|\tilde{x}_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \left[\|\tilde{x}_n - p\| + \lambda_n \alpha_n \|p\|\right]$$
$$+ \gamma_n \left(\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|\right)^2 + c_n$$
$$\leq \|\tilde{x}_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \left[\|x_n - p\| + \lambda_n \alpha_n \|p\|\right]$$
$$+ \gamma_n \left(\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|\right)^2 + c_n$$
$$\leq \left\|\Omega_n^i u_n - p\right\|^2 + 2\lambda_n \alpha_n \|p\| \left[\|x_n - p\| + \lambda_n \alpha_n \|p\|\right]$$
$$+ \gamma_n \left(\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|\right)^2 + c_n$$
$$\leq \|x_n - p\|^2 + \lambda_{i,n} \left(\lambda_{i,n} - 2\eta_i\right) \left\|A_i \Omega_n^{i-1} u_n - A_i p\right\|^2$$
$$+ 2\lambda_n \alpha_n \|p\| \left[\|x_n - p\| + \lambda_n \alpha_n \|p\|\right]$$
$$+ \gamma_n \left(\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|\right)^2 + c_n,$$

(81)

which implies

$$\lambda_{i,n} \left(2\eta_i - \lambda_{i,n}\right) \left\|A_i \Omega_n^{i-1} u_n - A_i p\right\|^2$$
$$\leq \|x_n - p\|^2 - \|k_n - p\|^2$$
$$+ 2\lambda_n \alpha_n \|p\| \left[\|x_n - p\| + \lambda_n \alpha_n \|p\|\right]$$
$$+ \gamma_n \left(\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|\right)^2 + c_n$$

(82)

$$\leq \|x_n - k_n\| \left(\|x_n - p\| + \|k_n - p\|\right)$$
$$+ 2\lambda_n \alpha_n \|p\| \left[\|x_n - p\| + \lambda_n \alpha_n \|p\|\right]$$
$$+ \gamma_n \left(\|x_n - p\| + \sqrt{2}\lambda_n \alpha_n \|p\|\right)^2 + c_n.$$

Since $\{\lambda_n\} \subset [c,d] \subset (0,1/L)$, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i \in \{1,2,\ldots,N\}$, $\alpha_n \to 0$, $\gamma_n \to 0$, and $c_n \to 0$, we conclude from (68) and the boundedness of $\{x_n\}$, $\{k_n\}$ that

$$\lim_{n\to\infty} \left\|A_i \Omega_n^{i-1} u_n - A_i p\right\| = 0, \quad i = 1,2,\ldots,N.$$

(83)

By Proposition 4 and Lemma 12, we obtain

$$\left\|\Omega_n^i u_n - p\right\|^2$$
$$= \left\|P_C(I - \lambda_{i,n} A_i) \Omega_n^{i-1} u_n - P_C(I - \lambda_{i,n} A_i) p\right\|^2$$
$$\leq \left\langle \left(I - \lambda_{i,n} A_i\right) \Omega_n^{i-1} u_n - \left(I - \lambda_{i,n} A_i\right) p, \Omega_n^i u_n - p\right\rangle$$

$$= \frac{1}{2} \left(\left\| \left(I - \lambda_{i,n} A_i \right) \Omega_n^{i-1} u_n - \left(I - \lambda_{i,n} A_i \right) p \right\|^2 \right.$$

$$+ \left\| \Omega_n^i u_n - p \right\|^2$$

$$- \left\| \left(I - \lambda_{i,n} A_i \right) \Omega_n^{i-1} u_n \right.$$

$$\left. \left. - \left(I - \lambda_{i,n} A_i \right) p - \left(\Omega_n^i u_n - p \right) \right\|^2 \right)$$

$$\leq \frac{1}{2} \left(\left\| \Omega_n^{i-1} u_n - p \right\|^2 + \left\| \Omega_n^i u_n - p \right\|^2 \right.$$

$$\left. - \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n - \lambda_{i,n} \left(A_i \Omega_n^{i-1} u_n - A_i p \right) \right\|^2 \right)$$

$$\leq \frac{1}{2} \left(\left\| u_n - p \right\|^2 + \left\| \Omega_n^i u_n - p \right\|^2 \right.$$

$$\left. - \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n - \lambda_{i,n} (A_i \Omega_n^{i-1} u_n - A_i p) \right\|^2 \right)$$

$$\leq \frac{1}{2} \left(\left\| x_n - p \right\|^2 + \left\| \Omega_n^i u_n - p \right\|^2 \right.$$

$$\left. - \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n - \lambda_{i,n} \left(A_i \Omega_n^{i-1} u_n - A_i p \right) \right\|^2 \right),$$

$$(84)$$

which implies

$$\left\| \Omega_n^i u_n - p \right\|^2 \leq \left\| x_n - p \right\|^2$$

$$- \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n - \lambda_{i,n} \left(A_i \Omega_n^{i-1} u_n - A_i p \right) \right\|^2$$

$$= \left\| x_n - p \right\|^2 - \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n \right\|^2$$

$$- \lambda_{i,n}^2 \left\| A_i \Omega_n^{i-1} u_n - A_i p \right\|^2$$

$$+ 2\lambda_{i,n} \left\langle \Omega_n^{i-1} u_n - \Omega_n^i u_n, A_i \Omega_n^{i-1} u_n - A_i p \right\rangle$$

$$\leq \left\| x_n - p \right\|^2 - \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n \right\|^2$$

$$+ 2\lambda_{i,n} \left\langle \Omega_n^{i-1} u_n - \Omega_n^i u_n, A_i \Omega_n^{i-1} u_n - A_i p \right\rangle.$$

$$(85)$$

Utilizing (81) and (85), we get

$$\left\| k_n - p \right\|^2 \leq \left\| \Omega_n^i u_n - p \right\|^2 + 2\lambda_n \alpha_n \left\| p \right\|$$

$$\times \left[\left\| x_n - p \right\| + \lambda_n \alpha_n \left\| p \right\| \right]$$

$$+ \gamma_n \left(\left\| x_n - p \right\| + \sqrt{2} \lambda_n \alpha_n \left\| p \right\| \right)^2 + c_n$$

$$\leq \left\| x_n - p \right\|^2 - \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n \right\|^2$$

$$+ 2\lambda_{i,n} \left\langle \Omega_n^{i-1} u_n - \Omega_n^i u_n, A_i \Omega_n^{i-1} u_n - A_i p \right\rangle$$

$$+ 2\lambda_n \alpha_n \left\| p \right\| \left[\left\| x_n - p \right\| + \lambda_n \alpha_n \left\| p \right\| \right]$$

$$+ \gamma_n \left(\left\| x_n - p \right\| + \sqrt{2} \lambda_n \alpha_n \left\| p \right\| \right)^2 + c_n,$$

$$(86)$$

which implies

$$\left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n \right\|^2$$

$$\leq \left\| x_n - p \right\|^2 - \left\| k_n - p \right\|^2$$

$$+ 2\lambda_{i,n} \left\langle \Omega_n^{i-1} u_n - \Omega_n^i u_n, A_i \Omega_n^{i-1} u_n - A_i p \right\rangle$$

$$+ 2\lambda_n \alpha_n \left\| p \right\| \left[\left\| x_n - p \right\| + \lambda_n \alpha_n \left\| p \right\| \right]$$

$$+ \gamma_n \left(\left\| x_n - p \right\| + \sqrt{2} \lambda_n \alpha_n \left\| p \right\| \right)^2 + c_n \quad (87)$$

$$\leq \left\| x_n - k_n \right\| \left(\left\| x_n - p \right\| + \left\| k_n - p \right\| \right)$$

$$+ 2\lambda_{i,n} \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n \right\| \left\| A_i \Omega_n^{i-1} u_n - A_i p \right\|$$

$$+ 2\lambda_n \alpha_n \left\| p \right\| \left[\left\| x_n - p \right\| + \lambda_n \alpha_n \left\| p \right\| \right]$$

$$+ \gamma_n \left(\left\| x_n - p \right\| + \sqrt{2} \lambda_n \alpha_n \left\| p \right\| \right)^2 + c_n.$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, 1/L)$, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, $\alpha_n \to 0$, $\gamma_n \to 0$, and $c_n \to 0$, we conclude from (68) and (83) and the boundedness of $\{x_n\}$, $\{k_n\}$, and $\{u_n\}$ that

$$\lim_{n \to \infty} \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n \right\| = 0, \quad i = 1, 2, \ldots, N. \quad (88)$$

From (88) we get

$$\left\| u_n - \tilde{x}_n \right\| = \left\| \Omega_n^0 u_n - \Omega_n^N u_n \right\|$$

$$\leq \left\| \Omega_n^0 u_n - \Omega_n^1 u_n \right\| + \left\| \Omega_n^1 u_n - \Omega_n^2 u_n \right\|$$

$$+ \cdots + \left\| \Omega_n^{N-1} u_n - \Omega_n^N u_n \right\| \quad (89)$$

$$\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

By (79) and (89), we have

$$\left\| x_n - \tilde{x}_n \right\| \leq \left\| x_n - u_n \right\| + \left\| u_n - \tilde{x}_n \right\| \quad (90)$$

$$\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

From (64) and (90), we have

$$\left\| \tilde{x}_{n+1} - \tilde{x}_n \right\| \leq \left\| \tilde{x}_{n+1} - x_{n+1} \right\|$$

$$+ \left\| x_{n+1} - x_n \right\| + \left\| x_n - \tilde{x}_n \right\| \quad (91)$$

$$\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

By (68), (79), and (89), we get

$$\|k_n - \tilde{x}_n\| \le \|k_n - x_n\| + \|x_n - u_n\| + \|u_n - \tilde{x}_n\| \tag{92}$$
$$\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

On the other hand, utilizing (58) and (59) we have

$$\begin{aligned}
\|k_n - p\|^2 &\le [1 + \gamma_n(1 - \delta_n)]\|z_n - p\|^2 \\
&\quad + (1 - \delta_n)(\kappa - \delta_n)\|z_n - S^n z_n\|^2 \\
&\quad + (1 - \delta_n)c_n \\
&\le \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n \\
&\le \|\tilde{x}_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|p - t_n\| \\
&\quad + \left(\lambda_n^2(\alpha_n + L)^2 - 1\right)\|\tilde{x}_n - t_n\|^2 \\
&\quad + \gamma_n\|z_n - p\|^2 + c_n,
\end{aligned} \tag{93}$$

which yields

$$\begin{aligned}
\left(1 - d^2(\alpha_n + L)^2\right)&\|\tilde{x}_n - t_n\|^2 \\
&\le \left(1 - \lambda_n^2(\alpha_n + L)^2\right)\|\tilde{x}_n - t_n\|^2 \\
&\le \|\tilde{x}_n - p\|^2 - \|k_n - p\|^2 \\
&\quad + 2\lambda_n\alpha_n\|p\|\|p - t_n\| + \gamma_n\|z_n - p\|^2 + c_n \\
&\le \|\tilde{x}_n - k_n\|\left(\|\tilde{x}_n - p\| + \|k_n - p\|\right) \\
&\quad + 2\lambda_n\alpha_n\|p\|\|p - t_n\| + \gamma_n\|z_n - p\|^2 + c_n.
\end{aligned} \tag{94}$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, 1/L)$, $\alpha_n \to 0$, $\gamma_n \to 0$, and $c_n \to 0$, we conclude from (92) and the boundedness of $\{\tilde{x}_n\}$, $\{k_n\}$, $\{t_n\}$, and $\{z_n\}$ that

$$\lim_{n \to \infty}\|\tilde{x}_n - t_n\| = 0. \tag{95}$$

Also, utilizing the similar arguments to those of (58), we obtain

$$\begin{aligned}
\|z_n - p\|^2 \\
&\le \|\tilde{x}_n - p\|^2 - \|\tilde{x}_n - t_n\|^2 \\
&\quad - \|t_n - z_n\|^2 + 2\lambda_n(\alpha_n + L)\|\tilde{x}_n - t_n\|\|z_n - t_n\| \\
&\quad + 2\lambda_n\alpha_n\langle p, p - t_n\rangle \\
&\le \|\tilde{x}_n - p\|^2 - \|\tilde{x}_n - t_n\|^2 - \|t_n - z_n\|^2 \\
&\quad + \lambda_n^2(\alpha_n + L)^2\|t_n - z_n\|^2 + \|\tilde{x}_n - t_n\|^2 \\
&\quad + 2\lambda_n\alpha_n\|p\|\|p - t_n\| \\
&= \|\tilde{x}_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|p - t_n\| \\
&\quad + \left(\lambda_n^2(\alpha_n + L)^2 - 1\right)\|t_n - z_n\|^2,
\end{aligned} \tag{96}$$

which together with (59) leads to

$$\begin{aligned}
\|k_n - p\|^2 \\
&\le [1 + \gamma_n(1 - \delta_n)]\|z_n - p\|^2 \\
&\quad + (1 - \delta_n)(\kappa - \delta_n)\|z_n - S^n z_n\|^2 + (1 - \delta_n)c_n \\
&\le \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n \\
&\le \|\tilde{x}_n - p\|^2 + 2\lambda_n\alpha_n\|p\|\|p - t_n\| \\
&\quad + \left(\lambda_n^2(\alpha_n + L)^2 - 1\right)\|t_n - z_n\|^2 + \gamma_n\|z_n - p\|^2 + c_n.
\end{aligned} \tag{97}$$

So we have

$$\begin{aligned}
\left(1 - d^2(\alpha_n + L)^2\right)&\|t_n - z_n\|^2 \\
&\le \left(1 - \lambda_n^2(\alpha_n + L)^2\right)\|t_n - z_n\|^2 \\
&\le \|\tilde{x}_n - p\|^2 - \|k_n - p\|^2 \\
&\quad + 2\lambda_n\alpha_n\|p\|\|p - t_n\| + \gamma_n\|z_n - p\|^2 + c_n \\
&\le \|\tilde{x}_n - k_n\|\left(\|\tilde{x}_n - p\| + \|k_n - p\|\right) \\
&\quad + 2\lambda_n\alpha_n\|p\|\|p - t_n\| + \gamma_n\|z_n - p\|^2 + c_n.
\end{aligned} \tag{98}$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, 1/L)$, $\alpha_n \to 0$, $\gamma_n \to 0$, and $c_n \to 0$, we conclude from (92) and the boundedness of $\{\tilde{x}_n\}$, $\{k_n\}$, $\{t_n\}$, and $\{z_n\}$ that

$$\lim_{n \to \infty}\|t_n - z_n\| = 0. \tag{99}$$

Utilizing (92)–(99), we get

$$\|k_n - z_n\| \le \|k_n - \tilde{x}_n\| + \|\tilde{x}_n - t_n\| + \|t_n - z_n\| \tag{100}$$
$$\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Since $k_n = \delta_n z_n + (1 - \delta_n)S^n z_n$, we get

$$(1 - b)\|S^n z_n - z_n\| \le (1 - \delta_n)\|S^n z_n - z_n\| = \|k_n - z_n\|, \tag{101}$$

which together with (100) implies

$$\lim_{n \to \infty}\|S^n z_n - z_n\| = 0. \tag{102}$$

In addition, observe that

$$\|u_n - z_n\| \le \|u_n - x_n\| + \|x_n - k_n\| + \|k_n - z_n\|. \tag{103}$$

From (68), (79), and (100), it immediately follows that

$$\lim_{n \to \infty}\|u_n - z_n\| = 0. \tag{104}$$

Moreover, note that

$$\begin{aligned}
\|z_{n+1} - z_n\| &\le \|z_{n+1} - k_{n+1}\| + \|k_{n+1} - \tilde{x}_{n+1}\| \\
&\quad + \|\tilde{x}_{n+1} - \tilde{x}_n\| + \|\tilde{x}_n - k_n\| + \|k_n - z_n\|.
\end{aligned} \tag{105}$$

From (91), (92), and (100), it immediately follows that

$$\lim_{n \to \infty} \|z_{n+1} - z_n\| = 0. \tag{106}$$

Meantime, it is clear that

$$\begin{aligned}
\left\|S^n z_n - S^{n+1} z_n\right\| &\le \left\|S^n z_n - z_n\right\| + \left\|z_n - z_{n+1}\right\| \\
&\quad + \left\|z_{n+1} - S^{n+1} z_{n+1}\right\| + \left\|S^{n+1} z_{n+1} - S^{n+1} z_n\right\|.
\end{aligned} \tag{107}$$

From (102) and (106) and Lemma 14, we obtain

$$\lim_{n \to \infty} \left\|S^n z_n - S^{n+1} z_n\right\| = 0. \tag{108}$$

Furthermore, we note that

$$\begin{aligned}
\|z_n - S z_n\| &\le \|z_n - S^n z_n\| + \left\|S^n z_n - S^{n+1} z_n\right\| \\
&\quad + \left\|S^{n+1} z_n - S z_n\right\|.
\end{aligned} \tag{109}$$

From (102) and (108) and the uniform continuity of S, we have

$$\lim_{n \to \infty} \|z_n - S z_n\| = 0. \tag{110}$$

Step 4. Finally we prove that $x_n \to v = P_F x_0$ as $n \to \infty$.

Indeed, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ which converges weakly to some w. From (70), (88)–(90), and (104), we have that $\Theta_{n_i}^k x_{n_i} \rightharpoonup w$, $\Omega_{n_i}^m u_{n_i} \rightharpoonup w$, $z_{n_i} \rightharpoonup w$, $\tilde{x}_{n_i} \rightharpoonup w$, where $k \in \{1, 2, \ldots, M\}, m \in \{1, 2, \ldots, N\}$. Since S is uniformly continuous, by (110) we get $\lim_{n \to \infty} \|z_n - S^m z_n\| = 0$ for any $m \ge 1$. Hence from Lemma 16, we obtain $w \in \mathrm{Fix}(S)$. Next we prove that $w \in \cap_{m=1}^{N} \mathrm{VI}(C, A_m)$. Let

$$T_m v = \begin{cases} A_m v + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases} \tag{111}$$

where $m \in \{1, 2, \ldots, N\}$. Let $(v, u) \in G(T_m)$. Since $u - A_m v \in N_C v$ and $\Omega_n^m u_n \in C$, we have

$$\langle v - \Omega_n^m u_n, u - A_m v \rangle \ge 0. \tag{112}$$

On the other hand, from $\Omega_n^m u_n = P_C(I - \lambda_{m,n} A_m) \Omega_n^{m-1} u_n$ and $v \in C$, we have

$$\left\langle v - \Omega_n^m u_n, \Omega_n^m u_n - \left(\Omega_n^{m-1} u_n - \lambda_{m,n} A_m \Omega_n^{m-1} u_n\right) \right\rangle \ge 0, \tag{113}$$

and hence

$$\left\langle v - \Omega_n^m u_n, \frac{\Omega_n^m u_n - \Omega_n^{m-1} u_n}{\lambda_{m,n}} + A_m \Omega_n^{m-1} u_n \right\rangle \ge 0. \tag{114}$$

Therefore, we have

$$\begin{aligned}
\left\langle v - \Omega_{n_i}^m u_{n_i}, u \right\rangle &\ge \left\langle v - \Omega_{n_i}^m u_{n_i}, A_m v \right\rangle \\
&\ge \left\langle v - \Omega_{n_i}^m u_{n_i}, A_m v \right\rangle \\
&\quad - \left\langle v - \Omega_{n_i}^m u_{n_i}, \frac{\Omega_{n_i}^m u_{n_i} - \Omega_{n_i}^{m-1} u_{n_i}}{\lambda_{m,n_i}} \right. \\
&\qquad\qquad \left. + A_m \Omega_{n_i}^{m-1} u_{n_i} \right\rangle \\
&= \left\langle v - \Omega_{n_i}^m u_{n_i}, A_m v - A_m \Omega_{n_i}^m u_{n_i} \right\rangle \\
&\quad + \left\langle v - \Omega_{n_i}^m u_{n_i}, A_m \Omega_{n_i}^m u_{n_i} - A_m \Omega_{n_i}^{m-1} u_{n_i} \right\rangle \\
&\quad - \left\langle v - \Omega_{n_i}^m u_{n_i}, \frac{\Omega_{n_i}^m u_{n_i} - \Omega_{n_i}^{m-1} u_{n_i}}{\lambda_{m,n_i}} \right\rangle \\
&\ge \left\langle v - \Omega_{n_i}^m u_{n_i}, A_m \Omega_{n_i}^m u_{n_i} - A_m \Omega_{n_i}^{m-1} u_{n_i} \right\rangle \\
&\quad - \left\langle v - \Omega_{n_i}^m u_{n_i}, \frac{\Omega_{n_i}^m u_{n_i} - \Omega_{n_i}^{m-1} u_{n_i}}{\lambda_{m,n_i}} \right\rangle.
\end{aligned} \tag{115}$$

From (88) and since A_m is continuous, we obtain that $\lim_{n \to \infty} \|A_m \Omega_n^m u_n - A_m \Omega_n^{m-1} u_n\| = 0$. From $\Omega_{n_i}^m u_{n_i} \rightharpoonup w$, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, for all $i \in \{1, 2, \ldots, N\}$ and (88), we have

$$\langle v - w, u \rangle \ge 0. \tag{116}$$

Since T_m is maximal monotone, we have $w \in T_m^{-1} 0$ and hence $w \in \mathrm{VI}(C, A_m)$, $m = 1, 2, \ldots, N$, which implies $w \in \cap_{m=1}^{N} \mathrm{VI}(C, A_m)$. Next we prove that $w \in \cap_{k=1}^{M} \mathrm{GMEP}(F_k, \varphi_k, B_k)$. Since $\Theta_n^k x_n = T_{r_{k,n}}^{(F_k, \varphi_k)}(I - r_{k,n} B_k) \Theta_n^{k-1} x_n, n \ge 1, k \in \{1, 2, \ldots, M\}$, we have

$$\begin{aligned}
&F_k\left(\Theta_n^k x_n, y\right) + \varphi_k(y) - \varphi_k\left(\Theta_n^k x_n\right) + \left\langle B_k \Theta_n^{k-1} x_n, y - \Theta_n^k x_n \right\rangle \\
&\quad + \frac{1}{r_{k,n}} \left\langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\rangle \ge 0.
\end{aligned} \tag{117}$$

By (A2), we have

$$\begin{aligned}
&\varphi_k(y) - \varphi_k\left(\Theta_n^k x_n\right) + \left\langle B_k \Theta_n^{k-1} x_n, y - \Theta_n^k x_n \right\rangle \\
&\quad + \frac{1}{r_{k,n}} \left\langle y - \Theta_n^k x_n, \Theta_n^k x_n - \Theta_n^{k-1} x_n \right\rangle \ge F_k\left(y, \Theta_n^k x_n\right).
\end{aligned} \tag{118}$$

Let $z_t = ty + (1-t)w$ for all $t \in (0,1]$ and $y \in C$. This implies that $z_t \in C$. Then we have

$$
\begin{aligned}
&\left\langle z_t - \Theta_n^k x_n, B_k z_t \right\rangle \\
&\quad \geq \varphi_k\left(\Theta_n^k x_n\right) - \varphi_k\left(z_t\right) + \left\langle z_t - \Theta_n^k x_n, B_k z_t \right\rangle \\
&\qquad - \left\langle z_t - \Theta_n^k x_n, B_k \Theta_n^{k-1} x_n \right\rangle \\
&\qquad - \left\langle z_t - \Theta_n^k x_n, \frac{\Theta_n^k x_n - \Theta_n^{k-1} x_n}{r_{k,n}} \right\rangle + F_k\left(z_t, \Theta_n^k x_n\right) \\
&\quad = \varphi_k\left(\Theta_n^k x_n\right) - \varphi_k\left(z_t\right) + \left\langle z_t - \Theta_n^k x_n, B_k z_t - B_k \Theta_n^k x_n \right\rangle \\
&\qquad + \left\langle z_t - \Theta_n^k x_n, B_k \Theta_n^k x_n - B_k \Theta_n^{k-1} x_n \right\rangle \\
&\qquad - \left\langle z_t - \Theta_n^k x_n, \frac{\Theta_n^k x_n - \Theta_n^{k-1} x_n}{r_{k,n}} \right\rangle + F_k\left(z_t, \Theta_n^k x_n\right).
\end{aligned}
\tag{119}
$$

By (70), we have $\|B_k \Theta_n^k x_n - B_k \Theta_n^{k-1} x_n\| \to 0$ as $n \to \infty$. Furthermore, by the monotonicity of B_k, we obtain $\langle z_t - \Theta_n^k x_n, B_k z_t - B_k \Theta_n^k x_n \rangle \geq 0$. Then by (A4), we obtain

$$
\left\langle z_t - w, B_k z_t \right\rangle \geq \varphi_k(w) - \varphi_k(z_t) + F_k(z_t, w). \tag{120}
$$

Utilizing (A1), (A4), and (120), we obtain

$$
\begin{aligned}
0 &= F_k\left(z_t, z_t\right) + \varphi_k\left(z_t\right) - \varphi_k\left(z_t\right) \\
&\leq t F_k\left(z_t, y\right) + (1-t) F_k\left(z_t, w\right) \\
&\quad + t\varphi_k(y) + (1-t)\varphi_k(w) - \varphi_k\left(z_t\right) \\
&\leq t\left[F_k\left(z_t, y\right) + \varphi_k(y) - \varphi_k\left(z_t\right)\right] \\
&\quad + (1-t)\left\langle z_t - w, B_k z_t \right\rangle \\
&= t\left[F_k\left(z_t, y\right) + \varphi_k(y) - \varphi_k\left(z_t\right)\right] \\
&\quad + (1-t) t\left\langle y - w, B_k z_t \right\rangle,
\end{aligned}
\tag{121}
$$

and hence

$$
0 \leq F_k\left(z_t, y\right) + \varphi_k(y) - \varphi_k\left(z_t\right) + (1-t)\left\langle y - w, B_k z_t \right\rangle. \tag{122}
$$

Letting $t \to 0$, we have, for each $y \in C$,

$$
0 \leq F_k(w, y) + \varphi_k(y) - \varphi_k(w) + \left\langle y - w, B_k w \right\rangle. \tag{123}
$$

This implies that $w \in \text{GMEP}(F_k, \varphi_k, B_k)$ and hence $w \in \cap_{k=1}^M \text{GMEP}(F_k, \varphi_k, B_k)$.

Further, let us show that $w \in \Gamma$. As a matter of fact, since $\tilde{x}_n - t_n \to 0$ and $\tilde{x}_n - z_n \to 0$ (due to (95) and (99)), we have $t_{n_i} \rightharpoonup w$ and $z_{n_i} \rightharpoonup w$. Let

$$
Tv = \begin{cases} \nabla f(v) + N_C v, & \text{if } v \in C, \\ \emptyset, & \text{if } v \notin C, \end{cases} \tag{124}
$$

where $N_C v$ is the normal cone to C at $v \in C$. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tv$ if and only if $v \in \text{VI}(C, \nabla f)$; see [40] for more details. Let $G(T)$ be the graph of T and let $(v, u) \in G(T)$. Then, we have $u \in Tv = \nabla f(v) + N_C v$ and hence $u - \nabla f(v) \in N_C v$. So we have $\langle v - t, u - \nabla f(v) \rangle \geq 0$ for all $t \in C$. On the other hand, from $z_n = P_C(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n))$ and $v \in C$, we have

$$
\left\langle \tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n) - z_n, z_n - v \right\rangle \geq 0, \tag{125}
$$

and hence

$$
\left\langle v - z_n, \frac{z_n - \tilde{x}_n}{\lambda_n} + \nabla f_{\alpha_n}(t_n) \right\rangle \geq 0. \tag{126}
$$

Therefore, from $\langle v - t, u - \nabla f(v) \rangle \geq 0$ for all $t \in C$ and $z_{n_i} \in C$, we have

$$
\begin{aligned}
\left\langle v - z_{n_i}, u \right\rangle &\geq \left\langle v - z_{n_i}, \nabla f(v) \right\rangle \\
&\geq \left\langle v - z_{n_i}, \nabla f(v) \right\rangle \\
&\quad - \left\langle v - z_{n_i}, \frac{z_{n_i} - \tilde{x}_{n_i}}{\lambda_{n_i}} + \nabla f_{\alpha_{n_i}}(t_{n_i}) \right\rangle \\
&= \left\langle v - z_{n_i}, \nabla f(v) \right\rangle \\
&\quad - \left\langle v - z_{n_i}, \frac{z_{n_i} - \tilde{x}_{n_i}}{\lambda_{n_i}} + \nabla f(t_{n_i}) \right\rangle \\
&\quad - \alpha_{n_i} \left\langle v - z_{n_i}, t_{n_i} \right\rangle \\
&= \left\langle v - z_{n_i}, \nabla f(v) - \nabla f(z_{n_i}) \right\rangle \\
&\quad + \left\langle v - z_{n_i}, \nabla f(z_{n_i}) - \nabla f(t_{n_i}) \right\rangle \\
&\quad - \left\langle v - z_{n_i}, \frac{z_{n_i} - \tilde{x}_{n_i}}{\lambda_{n_i}} \right\rangle - \alpha_{n_i} \left\langle v - z_{n_i}, t_{n_i} \right\rangle \\
&\geq \left\langle v - z_{n_i}, \nabla f(z_{n_i}) - \nabla f(t_{n_i}) \right\rangle \\
&\quad - \left\langle v - z_{n_i}, \frac{z_{n_i} - \tilde{x}_{n_i}}{\lambda_{n_i}} \right\rangle - \alpha_{n_i} \left\langle v - z_{n_i}, t_{n_i} \right\rangle.
\end{aligned}
\tag{127}
$$

Note that $\alpha_n \to 0$, $\|z_n - \tilde{x}_n\| \to 0$, and $\|\nabla f(z_n) - \nabla f(t_n)\| \to 0$ (due to the L-Lipschitz continuity of ∇f). Thus, we obtain $\langle v - w, u \rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in \text{VI}(C, \nabla f)$. Clearly, $w \in \Gamma$. Consequently, $w \in F$. This shows that $\omega_w(\{x_n\}) \subset F$. From (61) and Lemma 23 we infer that $x_n \to v = P_F x_0$ as $n \to \infty$. This completes the proof. $\square$

Corollary 25 (i.e., [3, Theorem 3.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let M, N be two integers. Let F_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex*

function, where $k \in \{1, 2, \ldots, M\}$. Let B_k and let A_i be μ_k-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, where $k \in \{1, 2, \ldots, M\}, i \in \{1, 2, \ldots, N\}$. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\lim_{n \to \infty} \gamma_n = 0$ and $\{c_n\} \subset [0, \infty)$ such that $\lim_{n \to \infty} c_n = 0$. Assume that $F := \cap_{k=1}^{M} \text{GMEP}(F_k, \varphi_k, A_k) \cap \cap_{i=1}^{N} \text{VI}(C, A_i) \cap \text{Fix}(S) \cap \Gamma$ is nonempty and bounded. Let $\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \beta_n \leq 1$ and $\kappa \leq \delta_n \leq b < 1$. Pick any $x_0 \in H$ and set $C_1 = C$, $x_1 = P_{C_1} x_0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$
\begin{aligned}
u_n &= T_{r_{M,n}}^{(F_M, \varphi_M)} \left(I - r_{M,n} B_M \right) T_{r_{M,n}}^{(F_{M-1}, \varphi_{M-1})} \\
&\quad \times \left(I - r_{M-1,n} B_{M-1} \right) \cdots T_{r_{1,n}}^{(F_1, \varphi_1)} \left(I - r_{1,n} B_1 \right) x_n, \\
z_n &= P_C \left(I - \lambda_{N,n} A_N \right) P_C \left(I - \lambda_{N-1,n} A_{N-1} \right) \\
&\quad \cdots P_C \left(I - \lambda_{2,n} A_2 \right) P_C \left(I - \lambda_{1,n} A_1 \right) u_n, \\
k_n &= \delta_n z_n + \left(1 - \delta_n \right) S^n z_n, \\
y_n &= \left(1 - \beta_n \right) x_n + \beta_n k_n, \\
C_{n+1} &= \left\{ z \in C_n : \left\| y_n - z \right\|^2 \leq \left\| x_n - z \right\|^2 + \theta_n \right\}, \\
x_{n+1} &= P_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{aligned}
\tag{128}
$$

where $\theta_n = \gamma_n \Delta_n^2 + c_n$, $\Delta_n = \sup\{\|x_n - p\| : p \in F\} < \infty$, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $i \in \{1, 2, \ldots, N\}$, $k \in \{1, 2, \ldots, M\}$. Assume that either (B1) or (B2) holds. Then $\{x_n\}$ converge strongly to $P_F x_0$.

Proof. In Theorem 24, put $\nabla f = 0$ and $\alpha_n = 0$ for all $n \geq 1$. Then $\Gamma = C$ and $F = \cap_{k=1}^{M} \text{GMEP}(F_k, \varphi_k, A_k) \cap \cap_{i=1}^{N} \text{VI}(C, A_i) \cap \text{Fix}(S) \cap \Gamma = \cap_{k=1}^{M} \text{GMEP}(F_k, \varphi_k, A_k) \cap \cap_{i=1}^{N} \text{VI}(C, A_i) \cap \text{Fix}(S)$. In this case, we obtain from (44) that

$$
\begin{aligned}
t_n &= P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} (\tilde{x}_n) \right) = \tilde{x}_n, \\
z_n &= P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} (t_n) \right) = \tilde{x}_n.
\end{aligned}
\tag{129}
$$

Thus, the iterative scheme (44) reduces to (128). Since $\alpha_n = 0$ for all $n \geq 1$ and F is bounded, we know that $\varrho = \sup\{\|p\| : p \in F\} < \infty$, $\Delta_n = \sup\{\|x_n - p\| : p \in F\} < \infty$, and

$$
\theta_n = \left(\gamma_n + \lambda_n \alpha_n \right) \Delta_n^2 + 3 \lambda_n \alpha_n \left(1 + \gamma_n \right)^2 \varrho^2 + c_n = \gamma_n \Delta_n^2 + c_n.
\tag{130}
$$

It is easy to see that all the conditions of Theorem 24 are satisfied. Therefore, in terms of Theorem 24, we derive the desired result. □

Corollary 26. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let F be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let B and A_i be μ-inverse strongly monotone and*

η_i-inverse-strongly monotone, respectively, where $i = 1, 2$. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\lim_{n \to \infty} \gamma_n = 0$ and $\{c_n\} \subset [0, \infty)$ such that $\lim_{n \to \infty} c_n = 0$. Assume that $F := \text{GMEP}(F, \varphi, A) \cap \text{VI}(C, A_1) \cap \text{VI}(C, A_2) \cap \text{Fix}(S) \cap \Gamma$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0, \infty)$, and let $\{\beta_n\}, \{\delta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \beta_n \leq 1$ and $\kappa \leq \delta_n \leq b < 1$. Pick any $x_0 \in H$ and set $C_1 = C$, $x_1 = P_{C_1} x_0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$
\begin{aligned}
F \left(u_n, y \right) &+ \varphi \left(y \right) - \varphi \left(u_n \right) + \left\langle B x_n, y - u_n \right\rangle \\
&+ \frac{1}{r_n} \left\langle y - u_n, u_n - x_n \right\rangle \geq 0, \quad \forall y \in C, \\
\tilde{x}_n &= P_C \left(I - \lambda_{2,n} A_2 \right) P_C \left(I - \lambda_{1,n} A_1 \right) u_n, \\
t_n &= P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} (\tilde{x}_n) \right), \\
z_n &= P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} (t_n) \right), \\
k_n &= \delta_n z_n + \left(1 - \delta_n \right) S^n z_n, \\
y_n &= \left(1 - \beta_n \right) x_n + \beta_n k_n, \\
C_{n+1} &= \left\{ z \in C_n : \left\| y_n - z \right\|^2 \leq \left\| x_n - z \right\|^2 + \theta_n \right\}, \\
x_{n+1} &= P_{C_{n+1}} x_0, \quad \forall n \geq 0,
\end{aligned}
\tag{131}
$$

where $\theta_n = \left(\gamma_n + \lambda_n \alpha_n \right) \Delta_n^2 + 3 \lambda_n \alpha_n (1 + \gamma_n)^2 \varrho^2 + c_n$, $\Delta_n = \sup\{\|x_n - p\| : p \in F\} < \infty$ and $\varrho = \sup\{\|p\| : p \in F\} < \infty$. Assume that the following conditions hold:

(i) $\lim_{n \to \infty} \alpha_n = 0$;

(ii) $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, 1/L)$;

(iii) $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i = 1, 2$;

(iv) $\{r_n\} \subset [e, f] \subset (0, 2\mu)$.

Then $\{x_n\}$ converge strongly to $P_F x_0$ provided either (B1) or (B2) holds.

Corollary 27. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let F be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let B and A be μ-inverse strongly monotone and η-inverse-strongly monotone, respectively. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\lim_{n \to \infty} \gamma_n = 0$ and $\{c_n\} \subset [0, \infty)$ such that $\lim_{n \to \infty} c_n = 0$. Assume that $F := \text{GMEP}(F, \varphi, A) \cap \text{VI}(C, A) \cap \text{Fix}(S) \cap \Gamma$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0, \infty)$, and let $\{\beta_n\}, \{\delta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \beta_n \leq 1$ and $\kappa \leq \delta_n \leq b < 1$. Pick any $x_0 \in H$*

and set $C_1 = C$, $x_1 = P_{C_1} x_0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle$$

$$+ \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$\tilde{x}_n = P_C(u_n - \rho_n A u_n),$$

$$t_n = P_C\left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(\tilde{x}_n)\right),$$

$$z_n = P_C\left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n)\right), \quad (132)$$

$$k_n = \delta_n z_n + (1 - \delta_n) S^n z_n,$$

$$y_n = (1 - \beta_n) x_n + \beta_n k_n,$$

$$C_{n+1} = \left\{ z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\},$$

$$x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0,$$

where $\theta_n = (\gamma_n + \lambda_n \alpha_n)\Delta_n^2 + 3\lambda_n \alpha_n (1 + \gamma_n)^2 \varrho^2 + c_n$, $\Delta_n = \sup\{\|x_n - p\| : p \in F\} < \infty$, and $\varrho = \sup\{\|p\| : p \in F\} < \infty$. Assume that the following conditions hold:

(i) $\lim_{n \to \infty} \alpha_n = 0$;

(ii) $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, 1/L)$;

(iii) $\{\rho_n\} \subset [a, b] \subset (0, 2\eta)$;

(iv) $\{r_n\} \subset [e, f] \subset (0, 2\mu)$.

Then $\{x_n\}$ converge strongly to $P_F x_0$ provided either (B1) or (B2) holds.

Corollary 28. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let F be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let B and A be μ-inverse strongly monotone and η-inverse-strongly monotone, respectively. Let $S : C \to C$ be a uniformly continuous asymptotically κ-strict pseudocontractive mapping for some $0 \leq \kappa < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$ such that $\lim_{n \to \infty} \gamma_n = 0$. Assume that $F := \mathrm{GMEP}(F, \varphi, A) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \Gamma$ is nonempty and bounded. Let $\{\alpha_n\}$ be a sequence in $[0, \infty)$, and let $\{\beta_n\}, \{\delta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \beta_n \leq 1$ and $\kappa \leq \delta_n \leq b < 1$. Pick any $x_0 \in H$ and set $C_1 = C$, $x_1 = P_{C_1} x_0$. Let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle$$

$$+ \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C,$$

$$\tilde{x}_n = P_C(u_n - \rho_n A u_n),$$

$$t_n = P_C\left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(\tilde{x}_n)\right),$$

$$z_n = P_C\left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n)\right),$$

$$k_n = \delta_n z_n + (1 - \delta_n) S^n z_n,$$

$$y_n = (1 - \beta_n) x_n + \beta_n k_n,$$

$$C_{n+1} = \left\{ z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n \right\},$$

$$x_{n+1} = P_{C_{n+1}} x_0, \quad \forall n \geq 0,$$

$$(133)$$

where $\theta_n = (\gamma_n + \lambda_n \alpha_n)\Delta_n^2 + 3\lambda_n \alpha_n (1 + \gamma_n)^2 \varrho^2$, $\Delta_n = \sup\{\|x_n - p\| : p \in F\} < \infty$, and $\varrho = \sup\{\|p\| : p \in F\} < \infty$. Assume that the following conditions hold:

(i) $\lim_{n \to \infty} \alpha_n = 0$;

(ii) $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, 1/L)$;

(iii) $\{\rho_n\} \subset [a, b] \subset (0, 2\eta)$;

(iv) $\{r_n\} \subset [e, f] \subset (0, 2\mu)$.

Then $\{x_n\}$ converge strongly to $P_F x_0$ provided either (B1) or (B2) holds.

Remark 29. Theorem 24 extends, improves, supplements, and develops Yao et al. [26, Theorem 3.1] in the following aspects.

(i) Theorem 24 generalizes and extends [26, Theorem 3.1] from the asymptotically k-strict pseudocontractive mapping to the asymptotically k-strict pseudocontractive mapping in the intermediate sense and from one mixed equilibrium problem to finite GMEPs.

(ii) We add finite VIPs and the CMP (2) in our algorithm such that it can be applied to find a common element of the set of solutions of finite GMEPs, the set of solutions of finite VIPs for inverse strongly monotone mappings, the set of fixed points of an asymptotically k-strict pseudocontractive mapping in the intermediate sense, and the CMP (2) for a convex functional f with L-Lipschitz continuous gradient ∇f.

(iii) Theorem 24 also removes the condition (ii) in [26, Theorem 3.1].

4. Weak Convergence Theorem

In this section, we prove a new weak convergence theorem by a modified extragradient method with regularization for finding a common element of the set of solutions of the CMP (2) for a convex functional $f : C \to \mathbf{R}$ with L-Lipschitz continuous gradient ∇f, the set of solutions of finite generalized mixed equilibrium problems, the set of solutions of finite variational inequalities for inverse strong monotone mappings, and the set of fixed points of an asymptotically κ-strict pseudocontractive mapping $S : C \to C$ in the intermediate sense in a real Hilbert space. This iterative algorithm with regularization is based on the extragradient method, Mann-type iterative method, and hybrid gradient projection algorithm (GPA) with regularization.

Theorem 30. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let M, N be two integers. Let F_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \ldots, M\}$. Let B_k and A_i be μ_k-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, where $k \in \{1, 2, \ldots, M\}$, $i \in \{1, 2, \ldots, N\}$. Let $S : C \to C$ be the uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \le \kappa < 1$ with sequences $\{\gamma_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, \infty)$. Assume that $F := \cap_{k=1}^M \mathrm{GMEP}(F_k, \varphi_k, A_k) \cap \cap_{i=1}^N \mathrm{VI}(C, A_i) \cap \mathrm{Fix}(S) \cap \Gamma$ is nonempty. Let $\{\alpha_n\}$ be a sequence in $(0, \infty)$, and let $\{\beta_n\}, \{\delta_n\}$ be sequences in $[0, 1]$ such that $0 < a \le \beta_n \le \beta < 1$ and $\kappa \le \delta_n \le b < 1$. Pick any $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{aligned}
u_n &= T_{r_{M,n}}^{(F_M, \varphi_M)} \left(I - r_{M,n} B_M \right) T_{r_{M,n}}^{(F_{M-1}, \varphi_{M-1})} \\
&\quad \times \left(I - r_{M-1,n} B_{M-1} \right) \cdots T_{r_{1,n}}^{(F_1, \varphi_1)} \left(I - r_{1,n} B_1 \right) x_n, \\
\tilde{x}_n &= P_C \left(I - \lambda_{N,n} A_N \right) P_C \left(I - \lambda_{N-1,n} A_{N-1} \right) \\
&\quad \cdots P_C \left(I - \lambda_{2,n} A_2 \right) P_C \left(I - \lambda_{1,n} A_1 \right) u_n, \\
t_n &= P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} \left(\tilde{x}_n \right) \right), \\
z_n &= P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} \left(t_n \right) \right), \\
k_n &= \delta_n z_n + \left(1 - \delta_n \right) S^n z_n, \\
x_{n+1} &= \left(1 - \beta_n \right) x_n + \beta_n k_n,
\end{aligned}
\tag{134}
$$

where $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $i \in \{1, 2, \ldots, N\}$, and $k \in \{1, 2, \ldots, M\}$. Assume that either (B1) or (B2) holds and that the following conditions are satisfied:

(i) $\sum_{n=1}^\infty \alpha_n < \infty$, $\sum_{n=1}^\infty \gamma_n < \infty$ and $\sum_{n=1}^\infty c_n < \infty$;

(ii) $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, 1/L)$.

Then

(a) $\{x_n\}$ converges weakly to an element $w \in F$;

(b) $\{x_n\}$ converges weakly to $w = \lim_{n \to \infty} P_F x_n$ provided F is bounded.

Proof. First of all, again one can show that $P_C(I - \lambda \nabla f_\alpha)$ is ξ-averaged for each $\lambda \in (0, 2/(\alpha + L))$, where

$$
\xi = \frac{2 + \lambda(\alpha + L)}{4} \in (0, 1).
\tag{135}
$$

This shows that $P_C(I - \lambda \nabla f_\alpha)$ is nonexpansive. Furthermore, for $\{\lambda_n\} \subset [a, b]$ with $a, b \in (0, 1/L)$, without loss of generality, we may assume that

$$
a \le \inf_{n \ge 1} \lambda_n \le \sup_{n \ge 1} \lambda_n \le b < \frac{1}{\alpha_n + L}, \quad \forall n \ge 1.
\tag{136}
$$

Consequently, it follows that for each integer $n \ge 1$, $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is ξ_n-averaged with

$$
\begin{aligned}
\xi_n &= \frac{1}{2} + \frac{\lambda_n (\alpha_n + L)}{2} - \frac{1}{2} \cdot \frac{\lambda_n (\alpha_n + L)}{2} \\
&= \frac{2 + \lambda_n (\alpha_n + L)}{4} \in (0, 1).
\end{aligned}
\tag{137}
$$

This immediately implies that $P_C(I - \lambda_n \nabla f_{\alpha_n})$ is nonexpansive for all $n \ge 1$.

Next let us show that $\lim_{n \to \infty} \|x_n - p\|$ exists for any $p \in F$. Put

$$
\begin{aligned}
\Theta_n^k &= T_{r_{k,n}}^{(F_k, \varphi_k)} \left(I - r_{k,n} B_k \right) T_{r_{k-1,n}}^{(F_{k-1}, \varphi_{k-1})} \\
&\quad \times \left(I - r_{k-1,n} B_{M-1} \right) \cdots T_{r_{1,n}}^{(F_1, \varphi_1)} \left(I - r_{1,n} B_1 \right) x_n,
\end{aligned}
\tag{138}
$$

for all $k \in \{1, 2, \ldots, M\}$, $n \ge 1$, and $\Theta_n^0 = \Omega_n^0 = I$, where I is the identity mapping on H. Then we have that $u_n = \Theta_n^M x_n$ and $\tilde{x}_n = \Omega_n^N u_n$. Take $p \in F$ arbitrarily. Similarly to the proof of Theorem 24, we obtain that

$$
\|u_n - p\| \le \|x_n - p\|,
\tag{139}
$$

$$
\|\tilde{x}_n - p\| \le \|u_n - p\|,
\tag{140}
$$

$$
\|t_n - p\| \le \|x_n - p\| + \lambda_n \alpha_n \|p\|,
\tag{141}
$$

$$
\begin{aligned}
\|z_n - p\|^2 &\le \|\tilde{x}_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\| \\
&\quad + \left(\lambda_n^2 (\alpha_n + L)^2 - 1 \right) \|\tilde{x}_n - t_n\|^2 \\
&\le \|\tilde{x}_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\| \\
&\le \|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|x_n - p\| \\
&\quad + 2\lambda_n^2 \alpha_n^2 \|p\|^2,
\end{aligned}
\tag{142}
$$

$$
\begin{aligned}
\|z_n - p\|^2 &\le \|\tilde{x}_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\| \\
&\quad + \left(\lambda_n^2 (\alpha_n + L)^2 - 1 \right) \|t_n - z_n\|^2,
\end{aligned}
\tag{4.5$'$}
$$

$$
\begin{aligned}
\left\| \Theta_n^k x_n - p \right\|^2 &\le \|x_n - p\|^2 + r_{k,n} \left(r_{k,n} - 2\mu_k \right) \\
&\quad \times \left\| B_k \Theta_n^{k-1} x_n - B_k p \right\|^2, \quad k \in \{1, 2, \ldots, M\},
\end{aligned}
\tag{143}
$$

$$\left\| \Theta_n^k x_n - p \right\|^2 \leq \left\| x_n - p \right\|^2 - \left\| \Theta_n^{k-1} x_n - \Theta_n^k x_n \right\|^2$$
$$+ 2r_{k,n} \left\| \Theta_n^{k-1} x_n - \Theta_n^k x_n \right\| \left\| B_k \Theta_n^{k-1} x_n - B_k p \right\|,$$
$$k = 1, 2, \ldots, M, \tag{144}$$

$$\left\| \Omega_n^i u_n - p \right\|^2 \leq \left\| x_n - p \right\|^2 + \lambda_{i,n} \left(\lambda_{i,n} - 2\eta_i \right)$$
$$\times \left\| A_i \Omega_n^{i-1} u_n - A_i p \right\|^2, \quad i \in \{1, 2, \ldots, N\}, \tag{145}$$

$$\left\| \Omega_n^i u_n - p \right\|^2 \leq \left\| x_n - p \right\|^2 - \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n \right\|^2$$
$$+ 2\lambda_{i,n} \left\| \Omega_n^{i-1} u_n - \Omega_n^i u_n \right\| \left\| A_i \Omega_n^{i-1} u_n - A_i p \right\|,$$
$$i \in \{1, 2, \ldots, N\}. \tag{146}$$

We observe that

$$\left\| k_n - p \right\|^2 = \left\| \delta_n \left(z_n - p \right) + \left(1 - \delta_n \right) \left(S^n z_n - p \right) \right\|^2$$
$$= \delta_n \left\| z_n - p \right\|^2 + \left(1 - \delta_n \right) \left\| S^n z_n - p \right\|^2$$
$$- \delta_n \left(1 - \delta_n \right) \left\| z_n - S^n z_n \right\|^2$$
$$\leq \delta_n \left\| z_n - p \right\|^2 + \left(1 - \delta_n \right)$$
$$\times \left[\left(1 + \gamma_n \right) \left\| z_n - p \right\|^2 + \kappa \left\| z_n - S^n z_n \right\|^2 + c_n \right]$$
$$- \delta_n \left(1 - \delta_n \right) \left\| z_n - S^n z_n \right\|^2$$
$$= \left[1 + \gamma_n \left(1 - \delta_n \right) \right] \left\| z_n - p \right\|^2$$
$$+ \left(1 - \delta_n \right) \left(\kappa - \delta_n \right) \left\| z_n - S^n z_n \right\|^2 + \left(1 - \delta_n \right) c_n$$
$$\leq \left(1 + \gamma_n \right) \left\| z_n - p \right\|^2 + c_n. \tag{147}$$

It follows from (142) and (147) and $\lambda_n(\alpha_n + L) < 1$ that

$$\left\| x_{n+1} - p \right\|^2$$
$$= \left\| \left(1 - \beta_n \right) \left(x_n - p \right) + \beta_n \left(k_n - p \right) \right\|^2$$
$$\leq \left(1 - \beta_n \right) \left\| x_n - p \right\|^2 + \beta_n \left\| k_n - p \right\|^2$$
$$\leq \left(1 - \beta_n \right) \left\| x_n - p \right\|^2 + \beta_n \left[\left(1 + \gamma_n \right) \left\| z_n - p \right\|^2 + c_n \right]$$
$$\leq \left(1 - \beta_n \right) \left\| x_n - p \right\|^2$$
$$+ \beta_n \left\{ \left(1 + \gamma_n \right) \left[\left\| x_n - p \right\|^2 + 2\lambda_n \alpha_n \left\| p \right\| \left\| x_n - p \right\| \right. \right.$$
$$\left. \left. + 2\lambda_n^2 \alpha_n^2 \left\| p \right\|^2 \right] + c_n \right\}$$

$$\leq \left(1 + \gamma_n \right) \left[\left\| x_n - p \right\|^2 + 2\lambda_n \alpha_n \left\| p \right\| \left\| x_n - p \right\| \right.$$
$$\left. + 2\lambda_n^2 \alpha_n^2 \left\| p \right\|^2 \right] + c_n$$
$$= \left(1 + \gamma_n \right) \left\| x_n - p \right\|^2 + 2 \left(1 + \gamma_n \right) \lambda_n \alpha_n \left\| p \right\| \left\| x_n - p \right\|$$
$$+ 2 \left(1 + \gamma_n \right) \lambda_n^2 \alpha_n^2 \left\| p \right\|^2 + c_n$$
$$\leq \left(1 + \gamma_n \right) \left\| x_n - p \right\|^2 + \left(1 + \gamma_n \right) \lambda_n \alpha_n \left(\left\| p \right\|^2 + \left\| x_n - p \right\|^2 \right)$$
$$+ 2 \left(1 + \gamma_n \right) \lambda_n \alpha_n \left\| p \right\|^2 + c_n$$
$$= \left[1 + \gamma_n + \left(1 + \gamma_n \right) \lambda_n \alpha_n \right] \left\| x_n - p \right\|^2$$
$$+ 3 \left(1 + \gamma_n \right) \lambda_n \alpha_n \left\| p \right\|^2 + c_n. \tag{148}$$

From $\{\lambda_n\} \subset [c, d] \subset (0, 1/L)$ and condition (i) we have

$$\sum_{n=1}^{\infty} \left[\gamma_n + \left(1 + \gamma_n \right) \lambda_n \alpha_n \right] < \infty,$$
$$\sum_{n=1}^{\infty} \left[3 \left(1 + \gamma_n \right) \lambda_n \alpha_n \left\| p \right\|^2 + c_n \right] < \infty. \tag{149}$$

So, applying Lemma 19 to (148), we deduce that $\lim_{n \to \infty} \| x_n - p \|$ exists. This implies that $\{x_n\}$ is bounded and hence $\{u_n\}, \{\tilde{x}_n\}, \{t_n\}, \{z_n\}$, and $\{k_n\}$ are also bounded. In addition, by Lemma 12 and (59) we obtain from $\lambda_n(\alpha_n + L) < 1$ that

$$\left\| x_{n+1} - p \right\|^2 = \left(1 - \beta_n \right) \left\| x_n - p \right\|^2 + \beta_n \left\| k_n - p \right\|^2$$
$$- \beta_n \left(1 - \beta_n \right) \left\| x_n - k_n \right\|^2$$
$$\leq \left(1 - \beta_n \right) \left\| x_n - p \right\|^2$$
$$+ \beta_n \left\{ \left(1 + \gamma_n \right) \left[\left\| x_n - p \right\|^2 \right. \right.$$
$$+ 2\lambda_n \alpha_n \left\| p \right\| \left\| x_n - p \right\|$$
$$\left. + 2\lambda_n^2 \alpha_n^2 \left\| p \right\|^2 \right] + c_n \bigg\}$$
$$- \beta_n \left(1 - \beta_n \right) \left\| x_n - k_n \right\|^2$$
$$\leq \left(1 + \gamma_n \right) \left[\left\| x_n - p \right\|^2 + 2\lambda_n \alpha_n \left\| p \right\| \right.$$
$$\left. \times \left\| x_n - p \right\| + 2\lambda_n^2 \alpha_n^2 \left\| p \right\|^2 \right]$$
$$+ c_n - \beta_n \left(1 - \beta_n \right) \left\| x_n - k_n \right\|^2$$
$$\leq \left(1 + \gamma_n \right) \left[\left\| x_n - p \right\|^2 + 2\lambda_n \alpha_n \left\| p \right\| \left\| x_n - p \right\| \right.$$
$$\left. + 2\lambda_n \alpha_n \left\| p \right\|^2 \right] + c_n$$
$$- \beta_n \left(1 - \beta_n \right) \left\| x_n - k_n \right\|^2$$

$$= (1 + \gamma_n) \|x_n - p\|^2 + 2 (1 + \gamma_n) \lambda_n \alpha_n$$
$$\times \|p\| (\|x_n - p\| + \|p\|)$$
$$+ c_n - \beta_n (1 - \beta_n) \|x_n - k_n\|^2. \tag{150}$$

Thus, it is easy to see from $0 < a \leq \beta_n \leq \beta < 1$ that

$$a (1 - \beta) \|x_n - k_n\|^2$$
$$\leq \beta_n (1 - \beta_n) \|x_n - k_n\|^2$$
$$\leq \|x_n - p\|^2 - \|x_{n+1} - p\|^2 + \gamma_n \|x_n - p\|^2$$
$$+ 2 (1 + \gamma_n) \lambda_n \alpha_n \|p\| (\|x_n - p\| + \|p\|) + c_n. \tag{151}$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, 1/L), \lim_{n \to \infty} \|x_n - p\|$ exists, $\alpha_n \to 0, \gamma_n \to 0, c_n \to 0$, and the sequence $\{x_n\}$ is bounded, we obtain

$$\lim_{n \to \infty} \|x_n - k_n\| = 0. \tag{152}$$

Taking into consideration $x_{n+1} = (1 - \beta_n)x_n + \beta_n k_n$, we get $x_{n+1} - x_n = \beta_n(k_n - x_n)$, which together with (152) leads to

$$\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0. \tag{153}$$

Combining (140), (142), (143), and (147), we have

$$\|k_n - p\|^2$$
$$\leq \|z_n - p\|^2 + \gamma_n \|z_n - p\|^2 + c_n$$
$$\leq \|\tilde{x}_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n$$
$$\leq \|u_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n$$
$$\leq \|\Theta_n^k x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n$$
$$\leq \|x_n - p\|^2 + r_{k,n} (r_{k,n} - 2\mu_k) \|B_k \Theta_n^{k-1} x_n - B_k p\|^2$$
$$+ 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n, \tag{154}$$

which yields

$$r_{k,n} (2\mu_k - r_{k,n}) \|B_k \Theta_n^{k-1} x_n - B_k p\|^2$$
$$\leq \|x_n - p\|^2 - \|k_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\|$$
$$+ \gamma_n \|z_n - p\|^2 + c_n \tag{155}$$
$$\leq \|x_n - k_n\| (\|x_n - p\| + \|k_n - p\|)$$
$$+ 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n.$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, 1/L)$, $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $k \in \{1, 2, \ldots, M\}$, $\alpha_n \to 0, \gamma_n \to 0$, and $c_n \to 0$, we conclude

from (152) and the boundedness of $\{x_n\}, \{k_n\}, \{t_n\}$, and $\{z_n\}$ that

$$\lim_{n \to \infty} \|B_k \Theta_n^{k-1} x_n - B_k p\| = 0, \quad k = 1, 2, \ldots, M. \tag{156}$$

Combining (140), (142), (144), and (147), we have

$$\|k_n - p\|^2$$
$$\leq \|\Theta_n^k x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\|$$
$$+ \gamma_n \|z_n - p\|^2 + c_n$$
$$\leq \|x_n - p\|^2 - \|\Theta_n^{k-1} x_n - \Theta_n^k x_n\|^2$$
$$+ 2r_{k,n} \|\Theta_n^{k-1} x_n - \Theta_n^k x_n\| \|B_k \Theta_n^{k-1} x_n - B_k p\|$$
$$+ 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n, \tag{157}$$

which implies

$$\|\Theta_n^{k-1} x_n - \Theta_n^k x_n\|^2$$
$$\leq \|x_n - p\|^2 - \|k_n - p\|^2$$
$$+ 2r_{k,n} \|\Theta_n^{k-1} x_n - \Theta_n^k x_n\| \|B_k \Theta_n^{k-1} x_n - B_k p\|$$
$$+ 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n \tag{158}$$
$$\leq \|x_n - k_n\| (\|x_n - p\| + \|k_n - p\|)$$
$$+ 2r_{k,n} \|\Theta_n^{k-1} x_n - \Theta_n^k x_n\| \|B_k \Theta_n^{k-1} x_n - B_k p\|$$
$$+ 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n.$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, (1/L))$, $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $k \in \{1, 2, \ldots, M\}$, $\alpha_n \to 0, \gamma_n \to 0$, and $c_n \to 0$, we conclude from (152) and (156) and the boundedness of $\{x_n\}, \{k_n\}, \{t_n\}$, and $\{z_n\}$ that

$$\lim_{n \to \infty} \|\Theta_n^{k-1} x_n - \Theta_n^k x_n\| = 0, \quad k = 1, 2, \ldots, M. \tag{159}$$

From (159), we have

$$\|x_n - u_n\| = \|\Theta_n^0 x_n - \Theta_n^M x_n\|$$
$$\leq \|\Theta_n^0 x_n - \Theta_n^1 x_n\| + \|\Theta_n^1 x_n - \Theta_n^2 x_n\|$$
$$+ \cdots + \|\Theta_n^{M-1} x_n - \Theta_n^M x_n\| \tag{160}$$
$$\longrightarrow \text{ as } n \longrightarrow \infty.$$

Combining (140), (142), (145), and (147), we have

$$
\begin{aligned}
\|k_n - &p\|^2 \\
&\leq \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n \\
&\leq \|\tilde{x}_n - p\|^2 + 2\lambda_n\alpha_n \|p\| \|p - t_n\| \\
&\quad + \gamma_n\|z_n - p\|^2 + c_n \\
&\leq \|\Omega_n^i u_n - p\|^2 + 2\lambda_n\alpha_n \|p\| \|p - t_n\| \\
&\quad + \gamma_n\|z_n - p\|^2 + c_n \\
&\leq \|x_n - p\|^2 + \lambda_{i,n}\left(\lambda_{i,n} - 2\eta_i\right) \|A_i\Omega_n^{i-1}u_n - A_ip\|^2 \\
&\quad + 2\lambda_n\alpha_n \|p\| \|p - t_n\| + \gamma_n\|z_n - p\|^2 + c_n,
\end{aligned}
\tag{161}
$$

where $i \in \{1, 2, \ldots, N\}$, which implies

$$
\begin{aligned}
\lambda_{i,n}&\left(2\eta_i - \lambda_{i,n}\right) \|A_i\Omega_n^{i-1}u_n - A_ip\|^2 \\
&\leq \|x_n - p\|^2 - \|k_n - p\|^2 \\
&\quad + 2\lambda_n\alpha_n \|p\| \|p - t_n\| + \gamma_n\|z_n - p\|^2 + c_n \\
&\leq \|x_n - k_n\| \left(\|x_n - p\| + \|k_n - p\|\right) \\
&\quad + 2\lambda_n\alpha_n \|p\| \|p - t_n\| + \gamma_n\|z_n - p\|^2 + c_n.
\end{aligned}
\tag{162}
$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, 1/L)$, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, $\alpha_n \to 0$, $\gamma_n \to 0$, and $c_n \to 0$, we conclude from (152) and the boundedness of $\{x_n\}, \{k_n\}, \{t_n\}$, and $\{z_n\}$ that

$$
\lim_{n \to \infty} \|A_i\Omega_n^{i-1}u_n - A_ip\| = 0, \quad i \in \{1, 2, \ldots, N\}.
\tag{163}
$$

Combining (140), (142), (146), and (147), we have

$$
\begin{aligned}
\|k_n - &p\|^2 \\
&\leq \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n \\
&\leq \|\tilde{x}_n - p\|^2 + 2\lambda_n\alpha_n \|p\| \|p - t_n\| \\
&\quad + \gamma_n\|z_n - p\|^2 + c_n \\
&\leq \|\Omega_n^i u_n - p\|^2 + 2\lambda_n\alpha_n \|p\| \|p - t_n\| \\
&\quad + \gamma_n\|z_n - p\|^2 + c_n \\
&\leq \|x_n - p\|^2 - \|\Omega_n^{i-1}u_n - \Omega_n^i u_n\|^2 \\
&\quad + 2\lambda_{i,n} \|\Omega_n^{i-1}u_n - \Omega_n^i u_n\| \|A_i\Omega_n^{i-1}u_n - A_ip\| \\
&\quad + 2\lambda_n\alpha_n \|p\| \|p - t_n\| + \gamma_n\|z_n - p\|^2 + c_n,
\end{aligned}
\tag{164}
$$

which implies

$$
\begin{aligned}
\|\Omega_n^{i-1}&u_n - \Omega_n^i u_n\|^2 \\
&\leq \|x_n - p\|^2 - \|k_n - p\|^2 \\
&\quad + 2\lambda_{i,n} \|\Omega_n^{i-1}u_n - \Omega_n^i u_n\| \|A_i\Omega_n^{i-1}u_n - A_ip\| \\
&\quad + 2\lambda_n\alpha_n \|p\| \|p - t_n\| + \gamma_n\|z_n - p\|^2 + c_n \\
&\leq \|x_n - k_n\| \left(\|x_n - p\| + \|k_n - p\|\right) \\
&\quad + 2\lambda_{i,n} \|\Omega_n^{i-1}u_n - \Omega_n^i u_n\| \|A_i\Omega_n^{i-1}u_n - A_ip\| \\
&\quad + 2\lambda_n\alpha_n \|p\| \|p - t_n\| + \gamma_n\|z_n - p\|^2 + c_n.
\end{aligned}
\tag{165}
$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, 1/L)$, $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $i \in \{1, 2, \ldots, N\}$, $\alpha_n \to 0$, $\gamma_n \to 0$, and $c_n \to 0$, we conclude from (152) and (163) and the boundedness of $\{x_n\}, \{k_n\}, \{u_n\}, \{t_n\}$, and $\{z_n\}$ that

$$
\lim_{n \to \infty} \|\Omega_n^{i-1}u_n - \Omega_n^i u_n\| = 0, \quad i \in \{1, 2, \ldots, N\}.
\tag{166}
$$

By (166), we have

$$
\begin{aligned}
\|u_n - \tilde{x}_n\| &= \|\Omega_n^0 u_n - \Omega_n^N u_n\| \\
&\leq \|\Omega_n^0 u_n - \Omega_n^1 u_n\| + \|\Omega_n^1 u_n - \Omega_n^2 u_n\| \\
&\quad + \cdots + \|\Omega_n^{N-1}u_n - \Omega_n^N u_n\| \\
&\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\end{aligned}
\tag{167}
$$

From (160) and (167), we have

$$
\|x_n - \tilde{x}_n\| \leq \|x_n - u_n\| + \|u_n - \tilde{x}_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\tag{168}
$$

By (152) and (168), we obtain

$$
\|k_n - \tilde{x}_n\| \leq \|k_n - x_n\| + \|x_n - \tilde{x}_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.
\tag{169}
$$

Furthermore, combining (139), (140), (142), and (147), we have

$$
\begin{aligned}
\|k_n - p\|^2 &\leq \|z_n - p\|^2 + \gamma_n\|z_n - p\|^2 + c_n \\
&\leq \|\tilde{x}_n - p\|^2 + 2\lambda_n\alpha_n \|p\| \|p - t_n\| \\
&\quad + \left(\lambda_n^2(\alpha_n + L)^2 - 1\right) \|\tilde{x}_n - t_n\|^2 \\
&\quad + \gamma_n\|z_n - p\|^2 + c_n \\
&\leq \|x_n - p\|^2 + 2\lambda_n\alpha_n \|p\| \|p - t_n\| \\
&\quad + \left(\lambda_n^2(\alpha_n + L)^2 - 1\right) \|\tilde{x}_n - t_n\|^2 \\
&\quad + \gamma_n\|z_n - p\|^2 + c_n,
\end{aligned}
\tag{170}
$$

which yields

$$\left(1 - \lambda_n^2 (\alpha_n + L)^2\right) \|\tilde{x}_n - t_n\|^2$$

$$\leq \|x_n - p\|^2 - \|k_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\|$$

$$+ \gamma_n \|z_n - p\|^2 + c_n \tag{171}$$

$$\leq \|x_n - k_n\| \left(\|x_n - p\| + \|k_n - p\|\right)$$

$$+ 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n.$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, 1/L), \alpha_n \to 0, \gamma_n \to 0$, and $c_n \to 0$, we obtain from (152) and the boundedness of $\{x_n\}, \{k_n\}\{t_n\}$, and $\{z_n\}$ that

$$\lim_{n \to \infty} \|\tilde{x}_n - t_n\| = 0. \tag{172}$$

Also, combining (139), (140), (4.5)′, and (147), we have

$$\|k_n - p\|^2 \leq \|z_n - p\|^2 + \gamma_n \|z_n - p\|^2 + c_n$$

$$\leq \|\tilde{x}_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\|$$

$$+ \left(\lambda_n^2 (\alpha_n + L)^2 - 1\right) \|\tilde{x}_n - t_n\|^2$$

$$+ \gamma_n \|z_n - p\|^2 + c_n \tag{173}$$

$$\leq \|x_n - p\|^2 + 2\lambda_n \alpha_n \|p\| \|p - t_n\|$$

$$+ \left(\lambda_n^2 (\alpha_n + L)^2 - 1\right) \|t_n - z_n\|^2$$

$$+ \gamma_n \|z_n - p\|^2 + c_n,$$

which leads to

$$\left(1 - \lambda_n^2 (\alpha_n + L)^2\right) \|t_n - z_n\|^2$$

$$\leq \|x_n - p\|^2 - \|k_n - p\|^2$$

$$+ 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n \tag{174}$$

$$\leq \|x_n - k_n\| \left(\|x_n - p\| + \|k_n - p\|\right)$$

$$+ 2\lambda_n \alpha_n \|p\| \|p - t_n\| + \gamma_n \|z_n - p\|^2 + c_n.$$

Since $\{\lambda_n\} \subset [c, d] \subset (0, 1/L), \alpha_n \to 0, \gamma_n \to 0$, and $c_n \to 0$, we obtain from (152) and the boundedness of $\{x_n\}, \{k_n\}, \{t_n\}$, and $\{z_n\}$ that

$$\lim_{n \to \infty} \|t_n - z_n\| = 0. \tag{175}$$

Hence, combining (172) and (175), we get

$$\|\tilde{x}_n - z_n\| \leq \|\tilde{x}_n - t_n\| + \|t_n - z_n\| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{176}$$

We note that

$$k_n - \tilde{x}_n = (1 - \delta_n) \left(S^n z_n - z_n\right). \tag{177}$$

From $\delta_n \leq b < 1$ and (169), we have

$$\lim_{n \to \infty} \|S^n z_n - z_n\| = 0. \tag{178}$$

On the other hand, we observe that

$$\|z_{n+1} - z_n\| \leq \|z_{n+1} - \tilde{x}_{n+1}\| + \|\tilde{x}_{n+1} - x_{n+1}\|$$

$$+ \|x_{n+1} - x_n\| + \|x_n - \tilde{x}_n\| + \|\tilde{x}_n - z_n\|. \tag{179}$$

By (153), (168), and (176), we have

$$\lim_{n \to \infty} \|z_{n+1} - z_n\| = 0. \tag{180}$$

We note that

$$\|z_n - Sz_n\| \leq \|z_n - z_{n+1}\| + \|z_{n+1} - S^{n+1} z_{n+1}\|$$

$$+ \|S^{n+1} z_{n+1} - S^{n+1} z_n\| + \|S^{n+1} z_n - Sz_n\|. \tag{181}$$

From (178), (180), Lemma 14, and the uniform continuity of S, we obtain

$$\lim_{n \to \infty} \|z_n - Sz_n\| = 0. \tag{182}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ which converges weakly to w. From (168) and (176), we have that $z_{n_i} \rightharpoonup w$. From (182) and the uniform continuity of S, we have $\lim_{n \to \infty} \|z_n - S^m z_n\| = 0$ for any $m \geq 1$. So, from Lemma 16, we have $w \in \text{Fix}(S)$. Similarly to the arguments in the proof of Theorem 24, we can derive $w \in \cap_{k=1}^M \text{GMEP}(F_k, \varphi_k, B_k) \cap \cap_{i=1}^N \text{VI}(C, A_i) \cap \Gamma$. Consequently, $w \in F$. This shows that $\omega_w(x_n) \subset F$.

Next let us show that $\omega_w(x_n)$ is a single-point set. As a matter of fact, let $\{x_{n_j}\}$ be another subsequence of $\{x_n\}$ such that $x_{n_j} \rightharpoonup w'$. Then we get $w' \in F$. If $w \neq w'$, from the Opial condition, we have

$$\lim_{n \to \infty} \|x_n - w\| = \lim_{i \to \infty} \|x_{n_i} - w\| < \lim_{i \to \infty} \|x_{n_i} - w'\|$$

$$= \lim_{n \to \infty} \|x_n - w'\| = \lim_{j \to \infty} \|x_{n_j} - w'\| \tag{183}$$

$$< \lim_{j \to \infty} \|x_{n_j} - w\| = \lim_{n \to \infty} \|x_n - w\|.$$

This attains a contraction. So we have $w = w'$. This shows that $\omega_w(x_n) = \{w\}$. Therefore, by Lemma 10, we know that $x_n \rightharpoonup w$.

Finally, we claim that $w = \lim_{n \to \infty} P_F x_n$ provided F is bounded. Put $v_n = P_F(x_n)$. Since $w \in F$, we have $\langle x_n - v_n, v_n - w \rangle \geq 0$. By (148) and Lemma 21, we have $\{v_n\}$ which converges strongly to some $w_0 \in F$. Since $\{x_n\}$ converges weakly to w, we have

$$\langle w - w_0, w_0 - w \rangle \geq 0. \tag{184}$$

Therefore, we obtain $w = w_0 = \lim_{n \to \infty} P_F x_n$. This completes the proof. □

Corollary 31 (i.e., [3, Theorem 4.1]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let M, N be two integers. Let F_k be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi_k : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function, where $k \in \{1, 2, \ldots, M\}$. Let B_k and A_i be μ_k-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, where $k \in \{1, 2, \ldots, M\}$, $i \in \{1, 2, \ldots, N\}$. Let $S : C \to C$ be the uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa < 1$ with sequences $\{\gamma_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, \infty)$. Assume that $F := \cap_{k=1}^{M} \mathrm{GMEP}(F_k, \varphi_k, A_k) \cap \cap_{i=1}^{N} \mathrm{VI}(C, A_i) \cap \mathrm{Fix}(S)$ is nonempty. Let $\{\beta_n\}$ and $\{\delta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \beta_n \leq \beta < 1$ and $\kappa \leq \delta_n \leq b < 1$. Pick any $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{aligned}
u_n &= T_{r_{M,n}}^{(F_M, \varphi_M)} \left(I - r_{M,n} B_M \right) T_{r_{M,n}}^{(F_{M-1}, \varphi_{M-1})} \\
&\quad \times \left(I - r_{M-1,n} B_{M-1} \right) \cdots T_{r_{1,n}}^{(F_1, \varphi_1)} \left(I - r_{1,n} B_1 \right) x_n, \\
z_n &= P_C \left(I - \lambda_{N,n} A_N \right) P_C \left(I - \lambda_{N-1,n} A_{N-1} \right) \\
&\quad \cdots P_C \left(I - \lambda_{2,n} A_2 \right) P_C \left(I - \lambda_{1,n} A_1 \right) u_n, \\
k_n &= \delta_n z_n + \left(1 - \delta_n \right) S^n z_n, \\
x_{n+1} &= \left(1 - \beta_n \right) x_n + \beta_n k_n,
\end{aligned}
\tag{185}
$$

where $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\{r_{k,n}\} \subset [e_k, f_k] \subset (0, 2\mu_k)$, $i \in \{1, 2, \ldots, N\}$, and $k \in \{1, 2, \ldots, M\}$. Assume that $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$, and that either (B1) or (B2) holds. Then $\{x_n\}$ converges weakly to $w = \lim_{n \to \infty} P_F x_n$.

Proof. In Theorem 30, put $\nabla f = 0$ and $\alpha_n = 0$ for all $n \geq 1$. Then $\Gamma = C$ and $F = \cap_{k=1}^{M} \mathrm{GMEP}(F_k, \varphi_k, A_k) \cap \cap_{i=1}^{N} \mathrm{VI}(C, A_i) \cap \mathrm{Fix}(S) \cap \Gamma = \cap_{k=1}^{M} \mathrm{GMEP}(F_k, \varphi_k, A_k) \cap \cap_{i=1}^{N} \mathrm{VI}(C, A_i) \cap \mathrm{Fix}(S)$. In this case, we obtain from (134) that

$$
\begin{aligned}
t_n &= P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} \left(\tilde{x}_n \right) \right) = \tilde{x}_n, \\
z_n &= P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} \left(t_n \right) \right) = \tilde{x}_n.
\end{aligned}
\tag{186}
$$

Thus, the iterative scheme (134) reduces to (95). It is easy to see that all the conditions of Theorem 24 are satisfied. In terms of Theorem 24, we have that $\{x_n\}$ converges weakly to an element $w \in F$. Now, put $v_n = P_F(x_n)$. Since $w \in F$, we have $\langle x_n - v_n, v_n - w \rangle \geq 0$. Taking into account that $\alpha_n = 0$ for all $n \geq 1$, we conclude from (148) that

$$
\begin{aligned}
\left\| x_{n+1} - p \right\|^2 &\leq \left[1 + \gamma_n + \left(1 + \gamma_n \right) \lambda_n \alpha_n \right] \left\| x_n - p \right\|^2 \\
&\quad + 3 \left(1 + \gamma_n \right) \lambda_n \alpha_n \| p \|^2 + c_n \\
&\leq \left(1 + \gamma_n \right) \left\| x_n - p \right\|^2 + c_n.
\end{aligned}
\tag{187}
$$

By Lemma 21, we have that $\{v_n\}$ converges strongly to some $w_0 \in F$. Since $\{x_n\}$ converges weakly to w, we have

$$
\langle w - w_0, w_0 - w \rangle \geq 0.
\tag{188}
$$

Therefore, we obtain $w = w_0 = \lim_{n \to \infty} P_F x_n$. This completes the proof. $\square$

Corollary 32. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let F be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let B and A_i be μ-inverse strongly monotone and η_i-inverse-strongly monotone, respectively, where $i \in \{1, 2\}$. Let $S : C \to C$ be the uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa < 1$ with sequences $\{\gamma_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, \infty)$. Assume that $F := \mathrm{GMEP}(F, \varphi, B) \cap \mathrm{VI}(C, A_1) \cap \mathrm{VI}(C, A_2) \cap \mathrm{Fix}(S) \cap \Gamma$ is nonempty. Let $\{\alpha_n\}$ is a sequence in $(0, \infty)$, and let $\{\beta_n\}, \{\delta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \beta_n \leq \beta < 1$ and $\kappa \leq \delta_n \leq b < 1$. Pick any $x_1 \in H$ and let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$
\begin{aligned}
& F \left(u_n, y \right) + \varphi \left(y \right) - \varphi \left(u_n \right) + \langle B x_n, y - u_n \rangle \\
& \quad + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
& \tilde{x}_n = P_C \left(I - \lambda_{2,n} A_2 \right) P_C \left(I - \lambda_{1,n} A_1 \right) u_n, \\
& t_n = P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} \left(\tilde{x}_n \right) \right), \\
& z_n = P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} \left(t_n \right) \right), \\
& k_n = \delta_n z_n + \left(1 - \delta_n \right) S^n z_n, \\
& x_{n+1} = \left(1 - \beta_n \right) x_n + \beta_n k_n,
\end{aligned}
\tag{189}
$$

where $\{\lambda_{i,n}\} \subset [a_i, b_i] \subset (0, 2\eta_i)$, $\{r_n\} \subset [e, f] \subset (0, 2\mu)$, and $i \in \{1, 2\}$. Assume that either (B1) or (B2) holds and that the following conditions are satisfied:

(i) *$\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} c_n < \infty$;*

(ii) *$\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, 1/L)$.*

Then

(a) *$\{x_n\}$ converges weakly to an element $w \in F$;*

(b) *$\{x_n\}$ converges weakly to $w = \lim_{n \to \infty} P_F x_n$ provided F is bounded.*

Corollary 33. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f. Let F be a bifunction from $C \times C$ to $\mathbf{R}$ satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let B and A be μ-inverse strongly monotone and η-inverse-strongly monotone, respectively. Let $S : C \to C$ be the uniformly continuous asymptotically κ-strict pseudocontractive mapping in the intermediate sense for some $0 \leq \kappa < 1$ with sequences $\{\gamma_n\} \subset [0, \infty)$ and $\{c_n\} \subset [0, \infty)$. Assume that $F := \mathrm{GMEP}(F, \varphi, B) \cap \mathrm{VI}(C, A) \cap \mathrm{Fix}(S) \cap \Gamma$ is nonempty. Let $\{\alpha_n\}$ be a sequence in $(0, \infty)$, and let $\{\beta_n\}, \{\delta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \beta_n \leq \beta < 1$ and*

$\kappa \leq \delta_n \leq b < 1$. *Pick any* $x_1 \in H$ *and let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$
\begin{aligned}
&F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle \\
&\quad + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
&\qquad \tilde{x}_n = P_C(u_n - \rho_n A u_n), \\
&\qquad t_n = P_C\left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(\tilde{x}_n)\right), \\
&\qquad z_n = P_C\left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n)\right), \\
&\qquad k_n = \delta_n z_n + (1 - \delta_n) S^n z_n, \\
&\qquad x_{n+1} = (1 - \beta_n) x_n + \beta_n k_n,
\end{aligned}
\tag{190}
$$

where $\{\rho_n\} \subset [a, b] \subset (0, 2\eta)$, $\{r_n\} \subset [e, f] \subset (0, 2\mu)$. *Assume that either (B1) or (B2) holds and that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$, $\sum_{n=1}^{\infty} \gamma_n < \infty$ *and* $\sum_{n=1}^{\infty} c_n < \infty$;

(ii) $\{\lambda_n\} \subset [c, d]$ *for some* $c, d \in (0, 1/L)$.

Then

(a) $\{x_n\}$ *converges weakly to an element* $w \in F$;

(b) $\{x_n\}$ *converges weakly to* $w = \lim_{n \to \infty} P_F x_n$ *provided* F *is bounded.*

Corollary 34. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* $f : C \to \mathbf{R}$ *be a convex functional with* L-*Lipschitz continuous gradient* ∇f. *Let* F *be a bifunction from* $C \times C$ *to* $\mathbf{R}$ *satisfying (A1)–(A4) and let* $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* B *and* A *be* μ-*inverse strongly monotone and* η-*inverse-strongly monotone, respectively. Let* $S : C \to C$ *be the uniformly continuous asymptotically* κ-*strict pseudocontractive mapping for some* $0 \leq \kappa < 1$ *with sequence* $\{\gamma_n\} \subset [0, \infty)$. *Assume that* $F := \text{GMEP}(F, \varphi, B) \cap \text{VI}(C, A) \cap \text{Fix}(S) \cap \Gamma$ *is nonempty. Let* $\{\alpha_n\}$ *be a sequence in* $(0, \infty)$, *and let* $\{\beta_n\}, \{\delta_n\}$ *be sequences in* $[0, 1]$ *such that* $0 < a \leq \beta_n \leq \beta < 1$ *and* $\kappa \leq \delta_n \leq b < 1$. *Pick any* $x_1 \in H$ *and let* $\{x_n\}$ *be a sequence generated by the following algorithm:*

$$
\begin{aligned}
&F(u_n, y) + \varphi(y) - \varphi(u_n) + \langle Bx_n, y - u_n \rangle \\
&\quad + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
&\qquad \tilde{x}_n = P_C(u_n - \rho_n A u_n), \\
&\qquad t_n = P_C\left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(\tilde{x}_n)\right), \\
&\qquad z_n = P_C\left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(t_n)\right), \\
&\qquad k_n = \delta_n z_n + (1 - \delta_n) S^n z_n, \\
&\qquad x_{n+1} = (1 - \beta_n) x_n + \beta_n k_n,
\end{aligned}
\tag{191}
$$

where $\{\rho_n\} \subset [a, b] \subset (0, 2\eta)$, $\{r_n\} \subset [e, f] \subset (0, 2\mu)$. *Assume that either (B1) or (B2) holds and that the following conditions are satisfied:*

(i) $\sum_{n=1}^{\infty} \alpha_n < \infty$ *and* $\sum_{n=1}^{\infty} \gamma_n < \infty$;

(ii) $\{\lambda_n\} \subset [c, d]$ *for some* $c, d \in (0, 1/L)$.

Then

(a) $\{x_n\}$ *converges weakly to an element* $w \in F$;

(b) $\{x_n\}$ *converges weakly to* $w = \lim_{n \to \infty} P_F x_n$ *provided* F *is bounded.*

Finally, we provide an example to illustrate Corollary 34.

Example 35. Let $H = \mathbf{R}^2$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ which are defined by

$$
\langle x, y \rangle = a_1 b_1 + a_2 b_2, \qquad \|x\| = \sqrt{a_1^2 + a_2^2}, \tag{192}
$$

for all $x, y \in \mathbf{R}^2$ with $x = (a_1, a_2)$ and $y = (b_1, b_2)$. Let $C = \{(a, a) : a \in \mathbf{R}\}$. Clearly, C is a nonempty closed convex subset of a real Hilbert space $H = \mathbf{R}^2$. Let $f : C \to \mathbf{R}$ be a convex functional with L-Lipschitz continuous gradient ∇f, for instance, putting $f(x) = (1/2)\langle Vx, x \rangle$, where $V = \left\{ \begin{smallmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{smallmatrix} \right\}$. Then $\nabla f = V$ is L-Lipschitz continuous with $L = 1$ (due to $\|V\| = 1$). Put $F(x, y) = 0$ and $\varphi(x) = 0$ for all $x, y \in C$. Then it is clear that F is a bifunction from $C \times C$ to R satisfying (A1)–(A4) and let $\varphi : C \to \mathbf{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let B and A be μ-inverse strongly monotone and η-inverse-strongly monotone, respectively, for instance, putting $A = \left\{ \begin{smallmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{smallmatrix} \right\}$ and $B = \left\{ \begin{smallmatrix} 3/5 & 2/5 \\ 2/5 & 3/5 \end{smallmatrix} \right\}$. Then we can take $\mu = \eta = 1/2$. Let $S : C \to C$ be the uniformly continuous asymptotically κ-strict pseudocontractive mapping for some $0 \leq \kappa < 1$ with sequence $\{\gamma_n\} \subset [0, \infty)$, for instance, putting $S = \left\{ \begin{smallmatrix} 4/7 & 3/7 \\ 3/7 & 4/7 \end{smallmatrix} \right\}$. Then $\kappa = 0$ and $\gamma_n = 0$ for all $n \geq 1$ (due to the nonexpansivity of S). Thus, we know that $F := \text{GMEP}(F, \varphi, B) \cap \text{VI}(C, A) \cap \text{Fix}(S) \cap \Gamma = \text{GMEP}(0, 0, B) \cap \text{VI}(C, A) \cap \text{Fix}(S) \cap \Gamma = \{0\}$. Take $\{\rho_n\} \subset [a, b] \subset (0, 2\eta) = (0, 1)$, $\{r_n\} \subset [e, f] \subset (0, 2\mu) = (0, 1)$, and $\{\lambda_n\} \subset [c, d] \subset (0, 1/L) = (0, 1)$. Let $\{\alpha_n\}$ be a sequence in $(0, \infty)$ such that $\lim_{n \to \infty} \alpha_n = 0$, and let $\{\beta_n\}, \{\delta_n\}$ be sequences in $[0, 1]$ such that $0 < a \leq \beta_n \leq \beta < 1$ and $0 = \kappa \leq \delta_n \leq b < 1$. Pick any $x_1 \in C$. In this case, the algorithm (191) reduces to the following algorithm:

$$
\begin{aligned}
&u_n = P_C(x_n - r_n B x_n) = (1 - r_n) x_n, \\
&\tilde{x}_n = P_C(u_n - \rho_n A u_n) = (1 - \rho_n)(1 - r_n) x_n, \\
&t_n = P_C\left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n}(\tilde{x}_n)\right) \\
&\quad = (1 - \lambda_n(1 + \alpha_n))(1 - \rho_n)(1 - r_n) x_n,
\end{aligned}
$$

$$z_n = P_C \left(\tilde{x}_n - \lambda_n \nabla f_{\alpha_n} \left(t_n \right) \right) = \left(1 - \rho_n \right) \left(1 - r_n \right) x_n$$

$$- \lambda_n \left(1 + \alpha_n \right) \left(1 - \lambda_n \left(1 + \alpha_n \right) \right) \left(1 - \rho_n \right) \left(1 - r_n \right) x_n$$

$$= \left[1 - \lambda_n \left(1 + \alpha_n \right) \left(1 - \lambda_n \left(1 + \alpha_n \right) \right) \right] \left(1 - \rho_n \right) \left(1 - r_n \right) x_n,$$

$$k_n = \delta_n z_n + \left(1 - \delta_n \right) S^n z_n = z_n,$$

$$x_{n+1} = \left(1 - \beta_n \right) x_n + \beta_n k_n$$

$$= \left\{ \left(1 - \beta_n \right) + \beta_n \left[1 - \lambda_n \left(1 + \alpha_n \right) \left(1 - \lambda_n \left(1 + \alpha_n \right) \right) \right] \right.$$

$$\left. \times \left(1 - \rho_n \right) \left(1 - r_n \right) \right\} x_n. \tag{193}$$

So it follows that

$$\| x_{n+1} \|$$

$$= \left\{ \left(1 - \beta_n \right) + \beta_n \left[1 - \lambda_n \left(1 + \alpha_n \right) \left(1 - \lambda_n \left(1 + \alpha_n \right) \right) \right] \right.$$

$$\left. \times \left(1 - \rho_n \right) \left(1 - r_n \right) \right\} \| x_n \|$$

$$\leq \left\{ \left(1 - \beta_n \right) + \beta_n \left[1 - \lambda_n \left(1 + \alpha_n \right) \left(1 - \lambda_n \left(1 + \alpha_n \right) \right) \right] \right\} \| x_n \|$$

$$= \left\{ 1 - \beta_n \lambda_n \left(1 + \alpha_n \right) \left(1 - \lambda_n \left(1 + \alpha_n \right) \right) \right\} \| x_n \|$$

$$\leq \left[1 - ac \left(1 + \alpha_n \right) \left(1 - d \left(1 + \alpha_n \right) \right) \right] \| x_n \| . \tag{194}$$

Since $\lim_{n \to \infty} ac(1 + \alpha_n)(1 - d(1 + \alpha_n)) = ac(1 - d) > (1/2)ac(1 - d)$, we deduce that there exists an integer $n_0 \geq 1$ such that

$$ac \left(1 + \alpha_n \right) \left(1 - d \left(1 + \alpha_n \right) \right) > \frac{1}{2} ac \left(1 - d \right), \quad \forall n \geq n_0. \tag{195}$$

Therefore, from (194) we obtain that for all $n \geq n_0$

$$\| x_{n+1} \| \leq \left[1 - ac \left(1 + \alpha_n \right) \left(1 - d \left(1 + \alpha_n \right) \right) \right] \| x_n \|$$

$$\leq \left[1 - \frac{1}{2} ac \left(1 - d \right) \right] \| x_n \|$$

$$\vdots$$

$$\leq \left[1 - \frac{1}{2} ac(1 - d) \right]^{n - n_0 + 1} \| x_{n_0} \| . \tag{196}$$

This shows that

    (a) $\{x_n\}$ converges to the unique point 0 in F;

    (b) $\{x_n\}$ converges to $0 = \lim_{n \to \infty} P_F x_n$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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