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Research Article

Homoclinic Solutions of a Class of Nonperiodic Discrete Nonlinear Systems in Infinite Higher Dimensional Lattices

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By using critical point theory, we obtain a new sufficient condition on the existence of homoclinic solutions of a class of nonperiodic discrete nonlinear systems in infinite lattices. The classical Ambrosetti-Rabinowitz superlinear condition is improved by a general superlinear one. Some results in the literature are improved.

1. Introduction

Assume that m is a positive integer. Consider the following difference equation in infinite higher dimensional lattices:

$$Lu_n - \omega u_n = \sigma \gamma_n f_n(u_n), \quad n \in Z^m, \tag{1}$$

where $f_n(u)$ is continuous in u, $\sigma = \pm 1$, $n = (n_1, n_2, \ldots, n_m) \in Z^m$, $\gamma = \{\gamma_n\}$ is a positive real valued sequence, $\omega \in R$, and L is a Jacobi operator [1] given by

$$\begin{aligned} Lu_n = {} & a_{1(n_1,n_2,\ldots,n_m)} u_{(n_1+1,n_2,\ldots,n_m)} \\ & + a_{1(n_1-1,n_2,\ldots,n_m)} u_{(n_1-1,n_2,\ldots,n_m)} \\ & + a_{2(n_1,n_2,\ldots,n_m)} u_{(n_1,n_2+1,\ldots,n_m)} \\ & + a_{2(n_1,n_2-1,\ldots,n_m)} u_{(n_1,n_2-1,\ldots,n_m)} \\ & + \cdots + a_{m(n_1,n_2,\ldots,n_m)} u_{(n_1,n_2,\ldots,n_m+1)} \\ & + a_{m(n_1,n_2,\ldots,n_m-1)} u_{(n_1,n_2,\ldots,n_m-1)} \\ & + b_{(n_1,n_2,\ldots,n_m)} u_{(n_1,n_2,\ldots,n_m)}; \end{aligned} \tag{2}$$

here, $\{a_{in}\}$ $(i = 1, 2, \ldots, m)$ and $\{b_n\}$ are real valued bounded sequences.

Assume that $f_n(0) = 0$ for $n \in Z^m$; then $\{u_n\} = \{0\}$ is a solution of (1), which is called the trivial solution. As usual, we say that $u = \{u_n\}$, a solution of (1), is homoclinic (to 0) if

$$\lim_{|n| \to \infty} u_n = 0, \tag{3}$$

where $|n| = |n_1| + |n_2| + \cdots + |n_m|$ is the length of multi-index n. In addition, if $\{u_n\} \neq \{0\}$, then u is called a nontrivial homoclinic solution. We are interested in the existence of the nontrivial homoclinic solutions for (1). This problem appears when we seek the discrete solitons of nonperiodic discrete nonlinear Schrödinger (DNLS) equation

$$i\dot{\psi}_n = -\Delta\psi_n - \sigma\gamma_n f_n(\psi_n), \quad n \in Z^m, \tag{4}$$

where $\sigma = \pm 1$ and

$$\begin{aligned} \Delta\psi_n = {} & \psi_{(n_1+1,n_2,\ldots,n_m)} \\ & + \psi_{(n_1-1,n_2,\ldots,n_m)} + \psi_{(n_1,n_2+1,\ldots,n_m)} \\ & + \psi_{(n_1,n_2-1,\ldots,n_m)} + \cdots + \psi_{(n_1,n_2,\ldots,n_m+1)} \\ & + \psi_{(n_1,n_2,\ldots,n_m-1)} - 2m\psi_{(n_1,n_2,\ldots,n_m)} \end{aligned} \tag{5}$$

is the discrete Laplacian in m spatial dimension. Typical representatives of power nonlinearities are

$$f_n(u) = l_n |u|^p u, \quad l_n, p > 0. \tag{6}$$

Primarily, we are interested in spatially localized, or solitary, standing waves. Such waves are often called breathers or gap solitons. The origin of the last name is that typically such solutions do exist for frequencies in gaps of linear spectrum. Considering (4), we suppose that the nonlinearity is gauge invariant; that is,

$$f_n\left(e^{i\theta}u\right) = e^{i\theta} f_n(u), \quad \theta \in R, \tag{7}$$

and, in addition, $f_n(u) \geq 0$ for $u \geq 0$ for $n \in Z^m$.

Making use of the standing wave ansatz,

$$\begin{gathered} \psi_n = u_n e^{-i\omega t}, \\ \lim_{|n|\to\infty} \psi_n = 0, \end{gathered} \tag{8}$$

where $\{u_n\}$ is a real valued sequence and $\omega \in R$ is the temporal frequency. Then (4) becomes

$$-\Delta u_n - \omega u_n = \sigma \gamma_n f_n(u_n), \quad n \in Z^m, \tag{9}$$

and (3) holds. This is an equation of the form (1) with $a_{in} = -1$ $(i = 1, 2, \ldots, m)$ and $b_n = 2m$.

When $f_n(u)$ has the form of (6), the homoclinic solutions of (9) were obtained by Karachalios in [2] by assuming that $\gamma \in l^\rho$, $\rho = (q-1)/(q-2)$, for some $q > 2$. We note that $\gamma \in l^\rho$ implies that $\lim_{|n|\to\infty}\gamma_n = 0$. Moreover, (6) satisfies the classical Ambrosetti-Rabinowitz superlinear condition [3], and $f_n(u)/|u|$ is nondecreasing with respect to $|u|$, both of which played important roles in the existence of homoclinic solutions of (1.7) in [2].

The aim of this paper is to improve both the monotone condition of $f_n(u)/|u|$ and the classical Ambrosetti-Rabinowitz superlinear condition by general ones; see Remarks 8 and 9 for details. Moreover, in this paper, we only need $\lim_{|n|\to\infty}\gamma_n = 0$. Particularly, our results improved the results in [2]; see Remarks 3 and 7 for details.

In the past years, there has been large growth in the study of DNLS equation, which is a nonlinear lattice system that appears in many areas of physics. Discrete solitons which exist in DNLS systems, that is, solitary waves and localized structures in spatially discrete media, are also of particular interest in their own right. Among these, one can mention photorefractive media [4], biomolecular chains [5], and Bose-Einstein condensates [6]. The experimental observations of discrete solitons in nonlinear lattice systems have been reported [7–11]. To mention that, many authors have studied the existence of discrete solitons of the DNLS equations [12–17]. The fruitful methods include centre manifold reduction [16], variational methods [12, 14], the principle of anticontinuity [13, 17], and the Nehari manifold approach [18]. However, most of the existing literature is devoted to the DNLS equations with constant coefficients or periodic coefficients. Results on such DNLS equations have been summarized in [19–23]. And we also want to mention that, in recent years, the existence of homoclinic solutions for difference equations has been studied by many authors, and we refer to [24–36].

Since the operator L is bounded and self-adjoint in the space l^2 (defined in Section 2), we consider (1) as a nonlinear equation in l^2 with (3) being satisfied automatically. The spectrum $\sigma(L)$ of L is closed. Thus, the complement $R \setminus \sigma(L)$ consists of a finite number of open intervals called spectral gaps and two of them are semi-infinite which are denoted by $(-\infty, \beta)$ and (α, ∞), respectively. In this paper, we consider the homoclinic solutions of (1) in l^2 for the case where $\omega \in (-\infty, \beta)$ and $\sigma = 1$. The case where $\omega \in (\alpha, \infty)$ and $\sigma = -1$ is omitted, since, in this case, we can replace L by $-L$.

The main idea in this paper is as follows. First, we assume that $\{\gamma_n\}$ converges to zero at infinity; that is, $\lim_{|n|\to\infty}\gamma_n = 0$. After that, we prove a compact inclusion between ordinary sequence spaces l^2 and weighted sequence spaces l^2_γ (defined in Section 2), in order to come over lack of compactness for the so-called $(C)_c$ condition (defined in Section 2). Finally, by making use of the Mountain Pass Lemma [37], we prove the existence of homoclinic solutions of (1) in l^2.

2. Preliminaries

In this section, we first establish the variational setting associated with (1). Let

$$l^p = l^p(Z^m) = \left\{ u = \{u_n\}_{n\in Z^m} : \forall n \in Z^m,\ u_n \in R, \right.$$
$$\left. \|u\|_{l^p} = \left(\sum_{n\in Z^m} |u_n|^p \right)^{1/p} < \infty \right\}. \tag{10}$$

Then the following embedding between l^p spaces holds:

$$l^q \subset l^p, \quad \|u\|_{l^p} \leq \|u\|_{l^q}, \quad 1 \leq q \leq p \leq \infty. \tag{11}$$

For $p = 2$, we get the usual Hilbert space of square-summable sequences, with the real scalar product

$$(u, v)_{l^2} = \sum_{n\in Z^m} u_n v_n, \quad u, v \in l^2. \tag{12}$$

For a positive real valued bounded sequence $\gamma = \{\gamma_n : 0 < \gamma_n \leq \overline{\gamma} < \infty\}_{n\in Z^m}$, we define the weighted sequence spaces l^2_γ:

$$l^2_\gamma = \left\{ u = \{u_n\}_{n\in Z^m} : \forall n \in Z^m,\ u_n \in R, \right.$$
$$\left. \|u\|_{l^2_\gamma} = \left(\sum_{n\in Z^m} \gamma_n |u_n|^2 \right)^{1/2} < \infty \right\}. \tag{13}$$

It is not hard to see that l^2_γ is a Hilbert space, with the scalar product

$$(u, v)_{l^2_\gamma} = \sum_{n\in Z^m} \gamma_n u_n v_n, \quad u, v \in l^2_\gamma. \tag{14}$$

For a certain class of weight γ, we have the following lemmas, which will play a crucial role in our analysis.

Lemma 1. *Let $\kappa = \{\kappa_n : |\kappa_n| \le \overline{\kappa} < \infty\}_{n\in Z^m}$ be a multiplication operator from l^2_γ to l^2_γ defined by $\kappa u = \{\kappa_n u_n\}_{n\in Z^m}$. If $\lim_{|n|\to\infty}\kappa_n = 0$, then the operator κ is compact.*

Proof of Lemma 1. Let

$$\Lambda = \left\{\kappa u : \|u\|_{l^2_\gamma} \le 1\right\}. \tag{15}$$

We only need to prove that Λ is precompact in l^2_γ. By assumption, for any $\varepsilon > 0$, there exists $N > 0$ such that $|\kappa_n| \le \varepsilon$ for any $|n| > N$. Define a cutting sequence $\chi = \{\chi_n\}$ by

$$\chi_n = \begin{cases} 1, & |n| \le N, \\ 0, & |n| > N. \end{cases} \tag{16}$$

Denote by $\chi^c = 1 - \chi$ the anticutting sequence. Then for any $\kappa u \in \Lambda$

$$\begin{aligned} \left\|\chi^c \kappa u\right\|^2_{l^2_\gamma} &= \sum_{|n|>N} \gamma_n \left|\kappa_n u_n\right|^2 \le \varepsilon^2, \\ \left\|\chi \kappa u\right\|^2_{l^2_\gamma} &= \sum_{|n|\le N} \gamma_n \left|\kappa_n u_n\right|^2 \le \overline{\kappa}^2. \end{aligned} \tag{17}$$

For arbitrary $\varepsilon > 0$ and $\Lambda_\varepsilon = \{\chi\kappa u : \|u\|_{l^2_\gamma} \le 1\}$ finite-dimensional and bounded, we know that Λ is precompact. The proof is complete. □

Lemma 2. *One assumes positive sequence of real numbers γ with $\lim_{|n|\to\infty}\gamma_n = 0$. Then $l^2 \hookrightarrow l^2_\gamma$ with compact inclusion.*

Proof of Lemma 2. Note that $\|u\|_{l^2_\gamma} \le \sqrt{\overline{\gamma}}\|u\|_{l^2}$ for any $u \in l^2$ and $\gamma^{1/2}$ is compact. Thus, $l^2 \hookrightarrow l^2_\gamma$ with compact inclusion by Lemma 1. The proof is complete. □

Remark 3. Karachalios [2] proved $l^2 \hookrightarrow l^2_\gamma$ with compact inclusion assuming that $\gamma \in l^\rho$, $\rho = (q-1)/(q-2)$, for some $q > 2$. Note that $\gamma \in l^\rho$ implies that $\lim_{|n|\to\infty}\gamma_n = 0$. Thus, we find that Lemma 2 improves Lemma 2.1 in [2].

On the Hilbert space l^2, we consider the functional

$$J(u) = \frac{1}{2}((L-\omega)u, u)_{l^2} - \sigma \sum_{n\in Z^m} \gamma_n F_n(u_n), \tag{18}$$

where

$$F_n(u) = \int_0^u f_n(s)\, ds \tag{19}$$

is the primitive function of $f_n(u)$. Then $J \in C^1(l^2, R)$ and

$$\begin{aligned} \left\langle J'(u), v\right\rangle &= ((L-\omega)u, v)_{l^2} \\ &\quad - \sigma \sum_{n\in Z^m} \gamma_n f_n(u_n) v_n, \quad u, v \in l^2. \end{aligned} \tag{20}$$

Equation (20) implies that (1) is the corresponding Euler-Lagrange equation for J. Therefore, we have reduced the problem of finding a nontrivial homoclinic solution of (1) to that of seeking a nonzero critical point of the functional J on l^2.

Let δ be the distance from ω to the spectrum $\sigma(L)$; that is,

$$\delta = \beta - \omega. \tag{21}$$

Then, we have

$$((L-\omega)u, u) \ge \delta\|u\|^2_{l^2}, \quad u \in l^2. \tag{22}$$

We also consider a norm in l^2 defined by

$$\|u\|_{l^2_\omega} = [((L-\omega)u, u)]^{1/2}, \quad u \in l^2. \tag{23}$$

Since

$$\delta\|u\|^2_{l^2} \le \|u\|^2_{l^2_\omega} \le \|L-\omega\|\,\|u\|^2_{l^2}, \quad u \in l^2, \tag{24}$$

norm (23) is an equivalent norm with the usual one of l^2.

In order to obtain the existence of critical points of J on l^2, we cite some basic notations and some known results from critical point theory.

Let H be a Hilbert space and $C^1(H, R)$ denote the set of functionals that are Fréchet differentiable and their Fréchet derivatives are continuous on H.

Let $J \in C^1(H, R)$. A sequence $\{u_j\} \subset H$ is called a $(C)_c$ sequence for J if $J(u_j) \to c$ for some $c \in R$ and $(1 + \|u_j\|)\|J'(u_j)\|$ as $j \to \infty$. We say J satisfies the $(C)_c$ condition if any $(C)_c$ sequence for J possesses a convergent subsequence.

Let B_r be the open ball in H with radius r and center 0, and let ∂B_r denote its boundary. The following lemma is taken from [37].

Lemma 4 (Mountain Pass Lemma). *If $J \in C^1(H, R)$ and satisfies the following conditions: there exist $e \in H \setminus \{0\}$ and $r \in (0, \|e\|)$ such that $\max\{J(0), J(e)\} < \inf_{\|u\|=r} J(u)$, then there exists a $(C)_c$ sequence $\{u_n\}$ for the mountain pass level c which is defined by*

$$c = \inf_{h\in\Gamma} \max_{s\in[0,1]} J(h(s)), \tag{25}$$

where

$$\Gamma = \{h \in C([0,1], H) : h(0) = 0,\ h(1) = e\}. \tag{26}$$

3. Main Results

In this section, we will establish some sufficient conditions on the existence of nontrivial solutions of (1) in l^2.

Theorem 5. *Assume that $\sigma = 1$, $\omega \in (-\infty, \beta)$, and the following conditions hold.*

- $(H1)$ *$f_n(u)$ is continuous in u, $f_n(u) = o(u)$ as $u \to 0$ uniformly for $n \in Z^m$.*
- $(H2)$ *There exist $b > 0$, $p > 2$ such that*

$$\left|f_n(u)\right| \le b\left(1 + |u|^{p-1}\right) \tag{27}$$

uniformly for $n \in Z^m$ and $u \in R$.

($H3$) *There exists some* $\theta \geq 1$ *such that* $\theta G_n(u) \geq G_n(tu)$, *for* $n \in Z^m$, $u \in R$, *and* $t \in [0,1]$, *where* $G_n(u) = (1/2) f_n(u)u - F_n(u)$.

($H4$) $F_n(u) \geq 0$ *for* $u \in R$, *and* $\lim_{|u|\to\infty}(F_n(u)/u^2) = \infty$, *uniformly for* $n \in Z^m$.

($H5$) *Positive real valued sequence* $\gamma = \{\gamma_n\}_{n\in Z^m}$ *with* $\lim_{|n|\to\infty}\gamma_n = 0$.

Then (1) *has at least a nontrivial solution* u *in* l^2 *and the solution decays exponentially at infinity. That is, there exist two positive constants* C *and* τ *such that*

$$|u_n| \leq Ce^{-\tau|n|}, \quad n \in Z^m. \tag{28}$$

Theorem 5 gives some sufficient conditions on the existence of nontrivial solutions of (1) in l^2. However, (1) may have no nontrivial solutions in l^2. In fact, we have the following proposition.

Proposition 6. *Assume that* $\sigma = -1$, $\omega \leq \beta$, *and* $\gamma_n f_n(u)u > 0$ *when* $u \neq 0$ *for all* $n \in Z^m$. *Then* (1) *has no nontrivial solutions in* l^2.

Proof of Proposition 6. By way of contradiction, we assume that (1) has a nontrivial solution $u = \{u_n\} \in l^2$. Then u is a nonzero critical point of J, and

$$\begin{aligned}\left\langle J'(u), u\right\rangle &= ((L-\omega)u, u)_{l^2} - \sigma \sum_{n\in Z^m} \gamma_n f_n(u_n) u_n \\ &\geq \sum_{n\in Z^m} \gamma_n f_n(u_n) u_n > 0.\end{aligned} \tag{29}$$

This is a contradiction as $\langle J'(u), u\rangle = 0$, so the conclusion holds. □

Remark 7. It is easy to see that the function f_n defined by

$$f_n(u) = |u|^{2\lambda} u, \tag{30}$$

where $\lambda > 0$ and $\gamma \in l^\rho$, $\rho = (q-1)/(q-2)$, for some $q > 2$, satisfies all conditions in Theorem 5. This case was studied by [2], and we find that Theorem 5 improves Theorem 2.3 in [2].

Remark 8. We will introduce another condition ($\overline{H}3$): $f_n(u)/|u|$ is nondecreasing with respect to $|u|$. We want to point out that condition ($H3$) is equivalent to ($\overline{H}3$) when $\theta = 1$ and ($H3$) gives "better monotony" when $\theta > 1$, since ($\overline{H}3$) implies ($H3$) (see [38]). Moreover, we can find that $f_n(u) = 5u\ln(1+u^2) + 9\sin u$ satisfies ($H3$) but not ($\overline{H}3$) for some $\theta \geq 100$.

Remark 9. As we know, the condition

($\overline{H}4$) $f(u)u > qF(u) > 0$ for some $q > 2$ and $u \neq 0$

is often called Ambrosetti-Rabinowitz superlinear condition [3]. Clearly, ($\overline{H}4$) implies ($H4$). Actually, ($H4$) is more general than ($\overline{H}4$). Let $\{a_n\}$ be a positive sequence, and $f_n(u) = a_n u \ln(1+|u|)$. Then f_n satisfies ($H4$). However, f_n does not satisfy ($\overline{H}4$).

The proof of Theorem 5 is based on a direct application of the following lemmas. The key points read as follows.

Lemma 10. *Assume that the conditions of Theorem 5 hold; then one has the following.*

(J_1) *There exist two constants* $a > 0$ *and* $\rho > 0$ *such that* $J|_{\partial B_\rho} \geq a$.

(J_2) *There exists an* $e \in l^2$ *such that* $J(te) \to -\infty$ *as* $|t| \to \infty$.

Proof of Lemma 10. Let $\overline{\gamma} = \max_{n\in Z^m}\{\gamma_n\}$ and $\epsilon = \delta/2\overline{\gamma}$. By ($H1$) and ($H2$), there exists $c_1 > 0$, such that

$$|f_n(u)| \leq \epsilon|u| + c_1|u|^{p-1} \tag{31}$$

for all $n \in Z^m$ and $u \in R$, and (31) implies that

$$|F_n(u)| \leq \frac{\epsilon}{2}|u|^2 + \frac{c_1}{p}|u|^p. \tag{32}$$

By (32) and the Hölder inequality, we have

$$\begin{aligned}J(u) &= \frac{1}{2}((L-\omega)u, u)_{l^2} - \sum_{n\in Z^m}\gamma_n F_n(u_n) \\ &\geq \frac{\delta}{2}\|u\|_{l^2}^2 - \overline{\gamma}\sum_{n\in Z^m}\left(\frac{\epsilon}{2}|u|^2\frac{c_1}{p}|u|^p\right) \\ &\geq \frac{\delta}{4}\|u\|_{l^2}^2 - \frac{c_1\overline{\gamma}}{p}\|u\|_{l^2}^p.\end{aligned} \tag{33}$$

Since $p > 2$, we have

$$J(u) \geq \frac{\delta\rho^2}{8} = a > 0 \quad \text{for } \|u\|_{l^2} = \rho, \tag{34}$$

where $\rho = (\delta p/8c_1\overline{\gamma})^{1/(p-2)}$.

Let $e = \{e_n\} \in l^2$ be the eigenvector of L corresponding to the eigenvalue β; that is to say, $Le = \beta e$. There exists $N > 0$, such that

$$\sum_{|n|\leq N} e_n^2 \geq \frac{1}{2}\|e\|_{l^2}^2. \tag{35}$$

Let

$$A^* = \{n \in Z^m : |n| \leq N,\ e_n \neq 0\}. \tag{36}$$

By ($H4$), for any $M > 0$, there exists $\eta = \eta(M) > 0$ such that

$$F_n(u) \geq M|u|^2 \quad \text{for } n \in Z^m,\ |u| \geq \eta. \tag{37}$$

Taking t large enough, such that $te_n > \eta$ for all $n \in A^*$, then, combining (35), (36), and (37), we have

$$\begin{aligned}J(te) &= \frac{1}{2}((L-\omega)te, te)_{l^2} - \sum_{n\in Z^m}\gamma_n F_n(te_n) \\ &\leq \frac{\delta}{2}t^2\|e\|_{l^2}^2 - \sum_{n\in A^*}\gamma_n F_n(te_n) \\ &\leq \frac{\delta}{2}t^2\|e\|_{l^2}^2 - \underline{\gamma}Mt^2\sum_{n\in A^*}e_n^2 \\ &\leq \frac{1}{2}\left(\delta - \underline{\gamma}M\right)t^2\|e\|_{l^2}^2,\end{aligned} \tag{38}$$

where $\underline{\gamma} = \min_{n\in A^*}\{\gamma_n\} > 0$. Letting M be large enough, such that $\delta \le \underline{\gamma}M$, we obtain that $J(te) \to -\infty$ as $|t| \to \infty$. The proof is complete. □

Lemma 11. *Assume that the conditions of Theorem 5 hold; then the functional J satisfies the $(C)_c$ condition for any given $c \in R$.*

Proof of Lemma 11. Let $\{u^{(k)}\} \subset l^2$ be a $(C)_c$ sequence of J; that is,

$$J\left(u^{(k)}\right) \longrightarrow c,$$
$$\left(1+\left\|u^{(k)}\right\|_{l^2}\right)\left\|J'\left(u^{(k)}\right)\right\|_{l^2} \longrightarrow 0, \quad \text{as } k \longrightarrow \infty. \tag{39}$$

First, we prove that $\{u^{(k)}\}$ is bounded in l^2. By way of contradiction, assume that $\|u^{(k)}\|_{l^2} \to \infty$ as $k \to \infty$. Set $\xi^{(k)} = u^{(k)}/\|u^{(k)}\|_{l^2}$. Up to a sequence, we have

$$\xi^{(k)} \rightharpoonup \xi, \quad \text{in } l^2, \tag{40}$$

$$\xi^{(k)} \longrightarrow \xi, \quad \text{in } l^2_\gamma. \tag{41}$$

Case 1 ($\xi \ne 0$). By $J(u^{(k)}) = c + o(1)$, where $o(1) \to 0$ as $k \to 0$, we have

$$\begin{aligned}\sum_{n\in Z^m}\gamma_n\frac{F_n\left(u_n^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^2}^2} &= \frac{1}{2}\frac{\left((L-\omega)u^{(k)},u^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^2}^2} - \frac{c+o(1)}{\left\|u^{(k)}\right\|_{l^2}^2}\\ &\le \frac{\|L-\omega\|}{2} - \frac{c+o(1)}{\left\|u^{(k)}\right\|_{l^2}^2} < \infty.\end{aligned} \tag{42}$$

Let $B^* = \{n \in Z^m : \xi_n \ne 0\}$. Obviously, B^* is nonempty. Then, for some $n_0 \in B^*$, it follows from (41) that

$$u_{n_0}^{(k)} = \xi_{n_0}^{(k)}\left\|u^{(k)}\right\|_{l^2} \longrightarrow \infty, \quad \text{as } k \longrightarrow \infty. \tag{43}$$

Combining $(H4)$ and $\gamma_{n_0} > 0$, we have

$$\gamma_{n_0}\frac{F_{n_0}\left(u_{n_0}^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^2}^2} = \gamma_{n_0}\frac{F_{n_0}\left(u_{n_0}^{(k)}\right)}{\left|u_{n_0}^{(k)}\right|^2}\left|\xi_{n_0}^{(k)}\right|^2 \longrightarrow \infty, \quad \text{as } k \longrightarrow \infty. \tag{44}$$

However,

$$\begin{aligned}\sum_{n\in Z^m}\gamma_n\frac{F_n\left(u_n^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^2}^2} &= \sum_{n\ne n_0}\gamma_n\frac{F_n\left(u_n^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^2}^2} + \gamma_{n_0}\frac{F_{n_0}\left(u_{n_0}^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^2}^2}\\ &\ge \gamma_{n_0}\frac{F_{n_0}\left(u_{n_0}^{(k)}\right)}{\left\|u^{(k)}\right\|_{l^2}^2} \longrightarrow \infty,\end{aligned} \tag{45}$$

as $k \to \infty$. This contradicts (42).

Case 2 ($\xi = 0$). Let

$$J\left(t_k u^{(k)}\right) = \max_{t\in[0,1]} J\left(tu^{(k)}\right). \tag{46}$$

For any given $M > \max\{4, \theta c/2\delta\}$, let k be large enough such that $\|u^{(k)}\|_{l^2} > M$ and $\overline{\xi}^{(k)} = 2M^{1/2}\xi^{(k)}$. Combining (32), (41), and $\xi = 0$, it is easy to see that

$$\begin{aligned}&\sum_{n\in Z^m}\gamma_n F_n\left(\overline{\xi}_n^{(k)}\right)\\ &\quad\le \frac{\epsilon}{2}\left\|\overline{\xi}^{(k)}\right\|_{l^2_\gamma}^2 + \frac{c_1\overline{\gamma}^{1/2}}{p}\left\|\overline{\xi}^{(k)}\right\|_{l^{2(p-1)}}^{p-1}\left\|\overline{\xi}^{(k)}\right\|_{l^2_\gamma}^2 \longrightarrow 0,\\ &\qquad\qquad \text{as } k \longrightarrow \infty.\end{aligned} \tag{47}$$

Thus, for k large enough, we have

$$\begin{aligned}J\left(t_k u^{(k)}\right) &\ge J\left(\overline{\xi}^{(k)}\right)\\ &\ge \frac{\delta}{2}\left\|\overline{\xi}^{(k)}\right\|_{l^2}^2 - \sum_{n\in Z^m}\gamma_n F_n\left(\overline{\xi}_n^{(k)}\right)\\ &\ge 2\delta M - \sum_{n\in Z^m}\gamma_n F_n\left(\overline{\xi}_n^{(k)}\right).\end{aligned} \tag{48}$$

By (47), (48), and $M > c/2\delta$, we have

$$\lim_{k\to\infty} J\left(t_k u^{(k)}\right) \ge 2\delta M > \theta c. \tag{49}$$

Noting that $J(0) = 0$ and $J(u^{(k)}) \to c$, as $k \to \infty$, then $0 < t_k < 1$ when k is big enough. Thus, $\langle J'(t_k u^{(k)}), t_k u^{(k)}\rangle = 0$. In view of $(H3)$, it follows that

$$\begin{aligned}J\left(t_k u^{(k)}\right) &= J\left(t_k u^{(k)}\right) - \frac{1}{2}\left\langle J'\left(t_k u^{(k)}\right), t_k u^{(k)}\right\rangle\\ &= \sum_{n\in Z^m}\gamma_n\left(\frac{1}{2}f_n\left(t_k u_n^{(k)}\right)t_k u_n^{(k)} - F_n\left(t_k u_n^{(k)}\right)\right)\\ &\le \theta\sum_{n\in Z^m}\gamma_n\left(\frac{1}{2}f_n\left(u_n^{(k)}\right)u_n^{(k)} - F_n\left(u_n^{(k)}\right)\right)\\ &= \theta\left(J\left(u^{(k)}\right) - \frac{1}{2}\left\langle J'\left(u^{(k)}\right), u^{(k)}\right\rangle\right).\end{aligned} \tag{50}$$

By (50), we have

$$\lim_{k\to\infty} J\left(t_k u^{(k)}\right) \le \theta c. \tag{51}$$

This contradicts (49), so $\{u^{(k)}\}$ is bounded in l^2.

Second, we show that there exists a convergent subsequence of $\{u^{(k)}\}$. In fact, there exists a subsequence, still denoted by the same notation, such that

$$u^{(k)} \rightharpoonup u, \quad \text{in } l^2. \tag{52}$$

By Lemma 2, we have

$$u^{(k)} \longrightarrow u, \quad \text{in } l^2_\gamma. \tag{53}$$

By direct calculation, we obtain

$$\begin{aligned}
&\left\|u^{(k)}-u\right\|_{l^2_\omega}^2 \\
&\quad=\left\langle J'\left(u^{(k)}\right)-J'(u), u^{(k)}-u\right\rangle \\
&\qquad+\sum_{n\in Z^m}\gamma_n\left(f_n\left(u_n^{(k)}\right)-f_n\left(u_n\right)\right)\left(u_n^{(k)}-u_n\right) \\
&\quad\le\left\langle J'\left(u^{(k)}\right)-J'(u), u^{(k)}-u\right\rangle \\
&\qquad+\sum_{n\in Z^m}\gamma_n\left(\epsilon\left(\left|u_n^{(k)}\right|+\left|u_n\right|\right)+c_1\left(\left|u_n^{(k)}\right|^{p-1}+\left|u_n\right|^{p-1}\right)\right) \\
&\qquad\times\left(u_n^{(k)}-u_n\right)\le\left\langle J'\left(u^{(k)}\right)-J'(u), u^{(k)}-u\right\rangle \\
&\qquad\qquad+\overline{\gamma}^{1/2}\left(\epsilon\left(\left\|u^{(k)}\right\|_{l^2}+\|u\|_{l^2}\right)\right. \\
&\qquad\qquad\left.+c_1\left(\left\|u^{(k)}\right\|_{l^{2(p-1)}}^{p-1}+\|u\|_{l^{2(p-1)}}^{p-1}\right)\right) \\
&\qquad\qquad\times\left\|u^{(k)}-u\right\|_{l^2_\gamma}.
\end{aligned}\tag{54}$$

Therefore, combining (11), (24), (52), (53), (54), and the boundedness of $\{u^{(k)}\}$, it is clear that

$$\lim_{k\to\infty}\left\|u^{(k)}-u\right\|_{l^2}=0\tag{55}$$

and this means J satisfies $(C)_c$ condition. The proof is complete. □

Now, we are ready to prove Theorem 5.

Proof of Theorem 5. Let a, ρ, and $e\in l^2$ be obtained in Lemma 10.

Since $J(te)\to-\infty$ as $|t|\to\infty$, there exists a real number t_0 such that

$$\left\|t_0e\right\|_{l^2}>\rho,\qquad J\left(t_0e\right)<0.\tag{56}$$

Immediately, we obtain

$$\max\left\{J(0), J\left(t_0e\right)\right\}=0<a\le\inf_{\|u\|=\rho}J(u).\tag{57}$$

Now that we have verified all assumptions of Lemma 4, we know J possesses a $(C)_c$ sequence $\{u_j\}\subset l^2$ for the mountain pass level $c\ge a$ with

$$c=\inf_{h\in\Gamma}\max_{s\in[0,1]}J(h(s)),\tag{58}$$

where

$$\Gamma=\left\{h\in C\left([0,1], l^2\right): h(0)=0,\ h(1)=t_0e\right\}.\tag{59}$$

By Lemma 11, $\{u_j\}$ has a convergent subsequence $\{u_{j_m}\}$ such that $u_{j_m}\to u$ as $j_m\to+\infty$ for some bounded $u\in l^2$. Since $J\in C^1(l^2,R)$, we have

$$\begin{gathered}
J\left(u_{j_m}\right)\longrightarrow J(u), \\
\left(1+\left\|u_{j_m}\right\|_{l^2}\right)J'\left(u_{j_m}\right)\longrightarrow\left(1+\|u\|_{l^2}\right)J'(u),
\end{gathered}\tag{60}$$

as $j_m\to+\infty$. By the uniqueness of limit and the fact that u is bounded, we obtain that u is a nontrivial critical point of J as the corresponding critical value $c\ge a>0$. Hence, (1) has at least one nontrivial solution u in l^2.

Finally, we show that $u=\{u_n\}$ satisfies (28). In fact, similar to [39], for $n\in Z$, let

$$w_n=\begin{cases}-\dfrac{\gamma_nf_n(u_n)}{u_n}, & u_n\ne0,\\ 0, & u_n=0;\end{cases}\tag{61}$$

then

$$\overline{L}u_n=\omega u_n,\tag{62}$$

where

$$\overline{L}u_n=Lu_n+w_nu_n.\tag{63}$$

Clearly, $\lim_{|n|\to\infty}w_n=0$. Thus, the multiplication by w_n is a compact operator in l^2, which implies that

$$\sigma_{\text{ess}}\left(\overline{L}\right)=\sigma_{\text{ess}}(L),\tag{64}$$

where σ_{ess} stands for the essential spectrum. Equation (62) means that $u=\{u_n\}$ is an eigenfunction that corresponds to the eigenvalue of finite multiplicity $\omega\notin\sigma_{\text{ess}}(\overline{L})$ of the operator $\overline{L}$. Equation (28) follows from the standard theorem on exponential decay for such eigenfunctions [1]. Now the proof of Theorem 5 is complete. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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