Wave Front Sets with respect to the Iterates of an Operator with Constant Coefficients

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We introduce the wave front set $WF^*_P(\mu)$ with respect to the iterates of a hypoelliptic linear partial differential operator with constant coefficients of a classical distribution $\mu \in \mathcal{D}'(\Omega)$ in an open set $\Omega$ in the setting of ultradifferentiable classes of Braun, Meise, and Taylor. We state a version of the microlocal regularity theorem of Hörmander for this new type of wave front set and give some examples and applications of the former result.

1. Introduction

In the 1960s Komatsu characterized in [1] analytic functions $f$ in terms of the behaviour not of the derivatives $D^\alpha f$, but of successive iterates $P(D)^j f$ of a partial differential elliptic operator $P(D)$ with constant coefficients, proving that a $C^\infty$ function $f$ is real analytic in $\Omega$ if and only if for every compact set $K \subset \subset \Omega$ there is a constant $C > 0$ such that

$$\|P(D)^j f\|_{L^2(K)} \leq C^j (j!)^m,$$  \hspace{1cm} (1)

where $m$ is the order of the operator and $\cdot \cdot \cdot \|_{L^2,K}$ is the $L^2$ norm on $K$.

This result was generalized for elliptic operators with variable analytic coefficients by Kotake and Narasimhan [2, Theorem 1]. Later, this result was extended to the setting of Gevrey functions by Newberger and Zielezny [3] and completely characterized by Métivier [4] (see also [5]). Spaces of Gevrey type given by the iterates of a differential operator are called generalized Gevrey classes and were used by Langenbruch [6–9] for different purposes. We mention modern contributions like [10–13] also. More recently, Juan-Huguet [14] extended the results of Komatsu [1], Newberger and Zielezny [3], and Métivier [4] to the setting of nonquasianalytic classes in the sense of Braun et al. [15]. In [14], Juan-Huguet introduced the generalized spaces of ultradifferentiable functions $\mathcal{E}_P^m(\Omega)$ on an open subset $\Omega$ of $\mathbb{R}^n$ for a fixed linear partial differential operator $P$ with constant coefficients and proved that these spaces are complete if and only if $P$ is hypoelliptic. Moreover, Juan-Huguet showed that, in this case, the spaces are nuclear. Later, the same author in [16] established a Paley-Wiener theorem for the classes $\mathcal{E}_P^m(\Omega)$ again under the hypothesis of the hypoellipticity of $P$.

The microlocal version of the problem of iterates was considered by Bolley et al. [17] to extend the microlocal regularity theorem of Hörmander [18, Theorem 5.4]. Bolley and Camus [19] generalized the microlocal version of the problem of iterates in [17] for some classes of hypoelliptic operators with analytic coefficients. We mention [20, 21] for investigations of the same problem for anisotropic and multianisotropic Gevrey classes. On the other hand, a version of the microlocal regularity theorem of Hörmander in the setting of [15] can be found in [22, 23] by one of the authors, which continues the study begun in [24].

Here, we continue in a natural way the previous work in [14] and study the microlocal version of the problem of iterates for generalized ultradifferentiable classes in the sense of Braun et al. [15]. We begin in Section 2 with some notation and preliminaries. In Section 3, we fix a hypoelliptic linear
partial differential operator with constant coefficients $P$ and introduce the wave front set $WF^*_n(u)$ with respect to the iterates of $P$ of a distribution $u \in \mathcal{D}'(\Omega)$ (Definition 7). To do this, we describe carefully the singular support in this setting (Proposition 6). We also prove that the new wave front set gives a more precise information for the study of the propagation of singularities than previous ones in Proposition 9, Theorem 13, and Example 15 (improving the previous works [22, 23] by one of the authors for operators with constant coefficients). More precisely, we clarify in Theorem 13 the necessity of the hypoellipticity of $P$ with a new version of the microlocal regularity theorem of Hörmander for an operator with constant coefficients. In Section 4, we prove that the product of a function in a suitable Gevrey class and a function in $B^r_j(\Omega)$ is still in $B^r_j(\Omega)$ (Proposition 17). This fact is used to give a more involved example, inspired in [25, Theorem 8.1.4], in which we construct a classical distribution with prescribed wave front set (Theorem 18).}

2. Notation and Preliminaries

Let us recall from [15] the definitions of weight functions $\omega$ and of the spaces of ultradifferentiable functions of Beurling and Roumieu type.

**Definition 1.** A nonquasianalytic weight function is a continuous increasing function $\omega : [0, +\infty) \to [0, +\infty]$ with the following properties:

- (a) $\exists L > 0$ s.t. $\omega(2t) \leq L(\omega(t) + 1)$ for all $t \geq 0$,
- (b) $\int_1^{+\infty} (\omega(t)/t^2) dt < +\infty$,
- (c) $\omega(t) = o(\omega(t))$ as $t \to +\infty$,
- (d) $\omega(|t|) = \omega(|t|)$.

Normally, we will denote $\omega$ simply by $\omega$.

For a weight function $\omega$, we define $\overline{\omega} : \mathbb{C}^n \to [0, +\infty]$ by $\overline{\omega}(z) := \omega(|z|)$ and again we denote this function by $\omega$.

The Young conjugate $\omega^* : [0, +\infty) \to [0, +\infty]$ is defined by

$$\omega^*(s) := \sup_{t>0} \left\{ st - \omega(t) \right\}. \quad (2)$$

There is no loss of generality to assume that $\omega$ vanishes on $[0, 1]$. Then $\omega^*$ has only nonnegative values, it is convex, $\omega^*(t)/t$ is increasing and tends to $\infty$ as $t \to +\infty$, and $\omega^* \circ \omega = \omega$.

**Example 2.** The following functions are, after a change in some interval $[0, M]$, examples of weight functions:

- (i) $\omega(t) = t^d$ for $0 < d < 1$.
- (ii) $\omega(t) = (\log 1 + t)^s$, $s > 1$.
- (iii) $\omega(t) = t^{-(\log(\log(1 + t)))^\beta}$, $\beta > 1$.
- (iv) $\omega(t) = \exp(\beta(\log(1 + t))^\alpha)$, $0 < \alpha < 1$.

In what follows, $\Omega$ denotes an arbitrary subset of $\mathbb{R}^n$ and $K \subset \subset \Omega$ means that $K$ is a compact subset in $\Omega$.

**Definition 3.** Let $\omega$ be a weight function.

(a) For a compact subset $K$ in $\mathbb{R}^n$ which coincides with the closure of its interior and $\lambda > 0$, we define the seminorm

$$p_{K, \lambda}(f) := \sup_{x \in K, \alpha \in \mathbb{N}^n} \left| f^{(\alpha)}(x) \right| \exp\left(-\lambda \phi^* \left(\frac{|\alpha|}{\lambda}\right)\right), \quad (3)$$

where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and set

$$\mathcal{E}_\omega^r(K) := \{ f \in C^{\infty}(K) : p_{K, \lambda}(f) < \infty \}, \quad (4)$$

which is a Banach space endowed with the $p_{K, \lambda}(\cdot)$-topology.

(b) For an open subset $\Omega$ in $\mathbb{R}^n$, the class of $\omega$-ultradifferentiable functions of Beurling type is defined by

$$\mathcal{E}_\omega(\Omega) := \{ f \in C^{\infty}(\Omega) : p_{K, \lambda}(f) < \infty, \quad (5)$$

for every $K \subset \subset \Omega$ and every $\lambda > 0$.

The topology of this space is

$$\mathcal{E}_\omega(\Omega) = \prod_{K \subset \subset \Omega} \mathcal{E}_\omega^1(K), \quad (6)$$

and one can show that $\mathcal{E}_\omega(\Omega)$ is a Fréchet space.

(c) For a compact subset $K$ in $\mathbb{R}^n$ which coincides with the closure of its interior and $\lambda > 0$, set

$$\mathcal{E}_\omega(K) := \{ f \in C^{\infty}(K) : \text{there exists } m \in \mathbb{N} \text{ such that } p_{K, \lambda m}(f) < \infty \}. \quad (7)$$

This space is the strong dual of a nuclear Fréchet space (i.e., a (DFN) space) if it is endowed with its natural inductive limit topology; that is,

$$\mathcal{E}_\omega(K) = \text{ind lim}_{m \in \mathbb{N}} \mathcal{E}_\omega(K). \quad (8)$$

(d) For an open subset $\Omega$ in $\mathbb{R}^n$, the class of $\omega$-ultradifferentiable functions of Roumieu type is defined by

$$\mathcal{E}_\omega^r(\Omega) := \{ f \in C^{\infty}(\Omega) : \forall K \subset \subset \Omega, \exists \lambda > 0$$

such that $p_{K, \lambda}(f) < \infty \}. \quad (9)$$

Its topology is the following:

$$\mathcal{E}_\omega(\Omega) = \prod_{K \subset \subset \Omega} \mathcal{E}_\omega^1(K); \quad (10)$$

that is, it is endowed with the topology of the projective limit of the spaces $\mathcal{E}_\omega^r(K)$ when $K$ runs the compact subsets of $\Omega$. This is a complete PLS-space, that is, a complete space which is a projective limit of LB-spaces (i.e., a countable inductive limit of Banach spaces) with compact linking maps in the (LB) steps. Moreover, $\mathcal{E}_\omega^r(\Omega)$ is also a nuclear and reflexive locally convex space. In particular, $\mathcal{E}_\omega(\Omega)$ is an ultrabornological (hence barrelled and bornological) space.

The elements of $\mathcal{E}_\omega(\Omega)$ (resp., $\mathcal{E}_\omega^r(\Omega)$) are called ultradifferentiable functions of Beurling type (resp., Roumieu type) in $\Omega$. 

In the case that \( \omega(t) := t^d \) \((0 < d < 1)\), the corresponding Roumieu class is the Gevrey class with exponent \(1/d\). In the limit case \(d = 1\), not included in our setting, the corresponding Roumieu class \( \mathcal{E}_\omega^k(\Omega) \) is the space of real analytic functions on \( \Omega \), whereas the Beurling class \( \mathcal{E}_\omega^k(\mathbb{R}^n) \) gives the entire functions.

If a statement holds in the Beurling and the Roumieu case, then we will use the notation \( \mathcal{E}_\omega^k(\Omega) \). It means that in all cases, * can be replaced either by \( \omega \) or \( \omega \).

For a compact set \( K \) in \( \mathbb{R}^n \), define
\[
\mathcal{D}_\omega(K) := \{ f \in \mathcal{E}_\omega(\mathbb{R}^n) : \text{supp } f \subset K \},
\]
endowed with the induced topology. For an open set \( \Omega \) in \( \mathbb{R}^n \), define
\[
\mathcal{D}_\omega(\Omega) := \text{ind}_{K \subset \subset \Omega} \mathcal{D}_\omega(K).
\]

Following [14], we consider smooth functions in an open set \( \Omega \) such that there exists \( C > 0 \) verifying for each \( j \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \),
\[
\|P^j(D)f\|_{L^2K} \leq C \exp \left( \lambda \varphi^* \left( \frac{jm}{\lambda} \right) \right),
\]
where \( K \) is a compact subset in \( \Omega \), \( \| \cdot \|_{L^2K} \) denotes the \( L^2 \)-norm on \( K \), and \( P^j(D) \) is the \( j \)th iterate of the partial differential operator \( P(D) \) of order \( m \); that is,
\[
P^j(D) = P(D) \circ \cdots \circ P(D).
\]

If \( j = 0 \), then \( P^0(D)f = f \).

Given a polynomial \( P \in \mathbb{C}[z_1, \ldots, z_n] \) with degree \( m \), \( P(z) = \sum_{|\alpha| \leq m} a_\alpha z^\alpha \), the partial differential operator \( P(D) \) is the following:
\[
P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha,
\]
where \( D = (1/i\partial) \).

The spaces of ultradifferentiable functions with respect to the successive iterates of \( P \) are defined as follows.

Let \( \omega \) be a weight function. Given a polynomial \( P \), an open set \( \Omega \subset \mathbb{R}^n \), a compact subset \( K \subset \subset \Omega \), and \( \lambda > 0 \), we define the seminorm
\[
\|f\|_{K,\lambda} := \sup_{j \in \mathbb{N}_0} \|P^j(D)f\|_{L^2K} \exp \left( -\lambda \varphi^* \left( \frac{jm}{\lambda} \right) \right),
\]
and set
\[
\mathcal{E}_{\omega P,\lambda}(K) := \left\{ f \in C^\infty(\Omega) : \|f\|_{K,\lambda} < +\infty \right\}.
\]

It is a Banach space endowed with the \( \| \cdot \|_{K,\lambda} \)-norm (it can be proved by the same arguments used for the standard class \( \mathcal{E}_\omega^k(K) \) in the sense of Braun et al.; see [15]).

The space of ultradifferentiable functions of Beurling type with respect to the iterates of \( P \) is
\[
\mathcal{E}_{\omega P}^k(\Omega) := \left\{ f \in C^\infty(\Omega) : \|f\|_{K,\lambda} < +\infty \right\},
\]
for each \( K \subset \subset \Omega, \lambda > 0 \),

endowed with the topology given by
\[
\mathcal{E}_{\omega P}^k(\Omega) := \text{proj} \text{ind}_{K \subset \subset \Omega} \mathcal{E}_{\omega P,\lambda}(K).
\]

If \( \{K_n\}_{n \in \mathbb{N}} \) is a compact exhaustion of \( \Omega \), we have
\[
\mathcal{E}_{\omega P}^k(\Omega) = \text{proj} \text{ind}_{K \subset \subset \Omega} \mathcal{E}_{\omega P,\lambda}(K).
\]

This metrizable locally convex topology is defined by the fundamental system of seminorms \( \| \cdot \|_{K,\lambda} \) \( \forall \lambda \in \mathbb{N} \).

The space of ultradifferentiable functions of Roumieu type with respect to the iterates of \( P \) is defined by
\[
\mathcal{E}_{\omega}^k(\Omega) := \left\{ f \in C^\infty(\Omega) : \forall K \subset \subset \Omega \exists \lambda > 0 \text{ such that } \|f\|_{K,\lambda} < +\infty \right\}.
\]

Its topology is defined by
\[
\mathcal{E}_{\omega}^k(\Omega) := \text{proj} \text{ind}_{K \subset \subset \Omega} \mathcal{E}_{\omega P,\lambda}(K).
\]

As in the Gevrey case, we call these classes generalized nonquasianalytic classes. We observe that in comparison with the spaces of generalized quasianalytic classes as defined in [14] we add here \( m \) as a factor inside \( \varphi^* \) in (15), where \( m \) is the order of the operator \( P \), which does not change the properties of the classes and will simplify the notation in the following.

The inclusion map \( \mathcal{E}_\omega(\Omega) \hookrightarrow \mathcal{E}_{\omega}^k(\Omega) \) is continuous (see [14, Theorem 4.11]). The space \( \mathcal{E}_\omega(\Omega) \) is complete if and only if \( P \) is hypoeelliptic (see [14, Theorem 3.3]). Moreover, under a mild condition on \( \omega \) introduced by Bonet et al. [26], \( \mathcal{E}_\omega^k(\Omega) \) coincides with the class of ultradifferentiable functions \( \mathcal{E}_\omega^k(\Omega) \) if and only if \( P \) is elliptic (see [14, Theorem 4.12]).

Denoting by
\[
\tilde{\mathbf{F}}(\xi) := \int \mathbf{e}^{-i\langle x, \xi \rangle} f(x) dx
\]
the classical Fourier transform of \( f \in \mathcal{E}_\omega^k(\Omega) \), we recall from [22, Proposition 3.3] the following characterization of the *-singular support in the sense of Braun et al. [15].

**Proposition 4.** Let \( \Omega \subset \mathbb{R}^n \) be an open set, \( u \in \mathcal{D}'(\Omega) \), and \( x_0 \in \Omega \).

(a) Then \( u \) is a \( \mathcal{E}_\omega^k \)-function in some neighborhood of \( x_0 \) if and only if for some neighborhood \( U \) of \( x_0 \) there exists a bounded sequence \( u_N \in \mathcal{E}_\omega^k(\Omega) \) which is equal to \( u \) in \( U \) and satisfies, for some \( C > 0 \) and \( k \in \mathbb{N} \), the estimates
\[
|\xi|^N \|\tilde{u}_N(\xi)\| \leq C e^{(1/k)\varphi^*(|\xi| N K)}, \quad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n.
\]

(b) Then \( u \) is a \( \mathcal{E}_\omega^k \)-function in some neighborhood of \( x_0 \) if and only if for some neighborhood \( U \) of \( x_0 \) there exists a bounded sequence \( u_N \in \mathcal{E}_\omega^k(\Omega) \) which is equal to \( u \) in \( U \) and such that for every \( k \in \mathbb{N} \) there exists a constant \( C_k > 0 \) satisfying
\[
|\xi|^N \|\tilde{u}_N(\xi)\| \leq C_k e^{k \varphi^*(|\xi| N K)}, \quad \forall N \in \mathbb{N}, \xi \in \mathbb{R}^n.
\]
This led, in [22, Definition 3.4], to the following definition of wave front set WFₙ(𝑢) in the sense of Braun et al. [15].

**Definition 5.** Let Ω be an open subset of ℝⁿ and u ∈ D'(Ω). The [ω]-wave front set WFω(u), resp. (ω)-wave front set WFω(𝑢), of u is the complement in Ω × ((ℝⁿ \ 0) of the set of points (x₀, 𝜉₀) such that there exist an open neighborhood U of x₀ in Ω, a conic neighborhood Γ of 𝜉₀, and a bounded sequence uₙ ∈ BΓ(Ω) (the set of classical distributions with compact support in Ω) equal to u in U such that there are k ∈ ℕ and C > 0 with the property

$$
|\epsilon|^{N} |\tilde{\omega}(\epsilon)| ≤ C\epsilon^{(1/k)p^* (KN)}, \quad N = 1, 2, \ldots, \epsilon ∈ Γ
$$

Resp., which satisfies that for every k ∈ ℕ there is Ck > 0 with the property

$$
|\epsilon|^{N} |\tilde{\omega}(\epsilon)| ≤ C\epsilon^{k p^* (N/k)}, \quad N = 1, 2, \ldots, \epsilon ∈ Γ.
$$

3. Wave Front Sets with respect to the Iterates of an Operator

Now, we assume that A is a bounded open set in ℝⁿ and we use the following notation:

$$
A_s := \{x ∈ A : d(x, ∂A) > s\},
$$

where d(x, ∂A) is the distance of x to the boundary of A. Given a linear partial differential operator P(D), we denote by P(α)(D) the operator corresponding to the polynomial P(α)( Español). If P(D) is hypoelliptic, by [27, Theorem 4.1] and the argument used in the proof of [3, Theorem 1], there are constants C > 0 and γ > 0 such that for every s ≥ 0 and t > 0 we have

$$
\|P(α)(D)f\|_{L^2(A_s)} ≤ C \left(\epsilon^{N} \|P(D)f\|_{L^2(A_s)} + \epsilon^{|α|−γ} \|f\|_{L^2(A_s)}\right),
\quad f ∈ C^{∞}(A).
$$

We observe also that if P(D) has constant coefficients, its formal adjoint is P(−D) and, if P(D) is hypoelliptic, P(−D) is also hypoelliptic (because of the behavior of the associated polynomial P(−ξ)). Moreover, any power P(D)² or P(−D)², with ℓ ∈ ℕ, of P(D) or P(−D), is also hypoelliptic.

We now want to generalize the notion of *-singular support of Proposition 4, using the iterates of a hypoelliptic linear partial differential operator P with constant coefficients. The idea is to substitute the sequence uₙ which satisfies an estimate of the form (23) or (24) by the sequence fₙ = P(D)²u whose Fourier transform satisfies the following estimates (29) and (30).

**Proposition 6.** Let P(D) be a linear partial differential operator of order m with constant coefficients which is hypoelliptic. Let Ω be an open subset of ℝⁿ, u ∈ D'(Ω), x₀ ∈ Ω and consider the following three conditions:

(i) \( f^N = P(D)^N u, \)

(ii) (Roumieu) \( ∃ k ∈ \mathbb{N}, ∀ M ∈ \mathbb{R}, ∃ C_M > 0, ∀ N ∈ \mathbb{N}, \) and \( ξ ∈ \mathbb{R}^m, \) we have

\[
\left| \tilde{f}_N (ξ) \right| ≤ C_M e^{(1/k)p^* (kNm)} (1 + |ξ|^m)^M,
\] (29)

(iii) (Beurling) \( ∀ k ∈ \mathbb{N} \) and \( M ∈ \mathbb{R}, ∃ C_{k,M} > 0, ∀ N ∈ \mathbb{N}, \) and \( ξ ∈ \mathbb{R}^m, \) we have

\[
\left| \tilde{f}_N (ξ) \right| ≤ C_{k,M} e^{k p^* (Nm/k)} (1 + |ξ|^M).
\] (30)

Then, the distribution \( u ∈ \mathcal{D}'_ω(u) \) (u ∈ \( \mathcal{D}'_ω(U) \)), where \( U \) is some neighborhood of \( x_0 \), if and only if there exist a neighborhood \( V \) of \( x_0 \) and a sequence \( \{f_n\} \) in \( \mathcal{D}'(Ω) \) that satisfies (i) and (ii) in \( V \) (that satisfies (i) and (iii) in \( V \)).

**Proof.**

Sufficiency (Roumieu case). Let u ∈ \( \mathcal{D}'_ω(U) \) with U = \( B_{3r}(x_0) \), the ball in ℝⁿ of center \( x_0 \) and radius 3r, \( r > 0 \). We choose \( χ ∈ \mathcal{D}(Ω) \) such that \( χ = 1 \) in \( B_r(x_0) \) and \( χ = 0 \) in \( (B_{2r}(x_0))^c \). We set \( f_n = χ P(D)^n u \). Then, \( f_n = P(D)^n u \) in \( B_r(x_0) \).

Now, fix \( ℓ ∈ \mathbb{N} \). From the hypoellipticity of \( P(D) \), there are constants \( D, d > 0 \) such that, for \( |ξ| \) large enough, \( |P(ξ)| ≥ D|ξ|^d \). Then, from the definition of \( f_n \), we obtain, for \( |ξ| \) large enough,

\[
D^n |ξ|^d \left| \tilde{f}_N (ξ) \right| ≤ |P(ξ)|^n \left| \tilde{f}_N (ξ) \right| = \left| \int_{\mathbb{R}^n} χ(x) P(D)^n u(x) e^{-i(x,ξ)} dx \right|.
\] (31)

We integrate by parts in the integral above, which will be equal to

\[
\left| \int_{\mathbb{R}^n} P(D)^{n/2} \left( χ(x) P(D)^n u(x) \right) e^{-i(x,ξ)} dx \right|.
\] (32)

From the generalized Leibniz rule, we can write (here \( m \) is the order of \( P(D) \))

\[
P(D)^{ℓ/2} \left( χ(x) P(D)^n u(x) \right) = \sum_{|α| ≤ m} \frac{1}{α!} |D^α χ(x)| \cdot (P^{(α)})(D) \left( P(D)^n u(x) \right).
\] (33)

Since \( P(D)^{ℓ} \) is hypoelliptic and \( P(D)^N u \) is a \( C^{∞} \)-function in the bounded set \( B_{3r}(x_0) \), we can apply formula (28) to the operator \( P(D)^{ℓ} \) with \( t = ℓ \), for \( 0 < ℓ < r \), \( A_{s+ℓ} = B_{2r}(x_0) \),

\[
\]
and $f = P(D)^N u$ (and $A_j = B_{2^{r+T}}(x_0)$) to obtain constants $C_{\ell', \gamma} > 0$ (which do not depend on $N$) such that

$$
\left\| (P^\ell)^{(\alpha)} (D) \left( P(D)^N u \right) \right\|_{L^2(B_2(x_0))} \leq C_{\ell'} \left( e^{\left| \alpha \right|} \left\| P(D)^{N+\ell} u \right\|_{L^2(B_2(x_0))} + e^{\left| \alpha \right| - \gamma} \left\| P(D)^N u \right\|_{L^2(B_2(x_0))} \right),
$$

(34)

Now, as $u \in \mathcal{S}^P_{\omega_0}(U)$, there are constants $k \in \mathbb{N}$ and $C > 0$ such that (we use the convexity of $\varphi$)

$$
\left\| \hat{f} \right\|_{L^2(U)} \leq C \left( e^{\left( k M(1+\left| \xi \right|) \right) - \ell} \right),
$$

(35)

and $C_{\ell, \gamma} > 0$ (which do not depend on $N$) such that, by Parseval's formula,

$$
\left\| P(D)^N u \right\|_{L^2(K)} = \left\| f \right\|_{L^2(K)} \leq \left\| f \right\|_{L^2(R^n)}
\leq \frac{1}{(2\pi)^n} \left( \int_{R^n} (1 + \left| \xi \right|)^{-2M} \left( \int_{R^n} (1 + \left| \xi \right|)^2 d\xi \right)^{1/2} \right)
\leq C e^{(1/k)\varphi(2kM)} \left( \int_{R^n} (1 + \left| \xi \right|)^{-2M} d\xi \right)^{1/2}.
$$

(39)

In a similar way, using the Fourier transform, we can see that the distributions $D^\alpha u$ satisfy analogous estimates for each multi-index $\alpha$ on $K$. By the hypoellipticity of $P(D)$ we conclude that $u \in C^\infty(\Omega)$, and this finishes the proof in the Roumieu case.

As above, in the Beurling case we can argue in a similar way.

In the rest of the paper, it is assumed that the operator $P(D)$ is hypoelliptic, but not elliptic. Hypoellipticity is not only useful for Proposition 6, but also because it gives some good properties of the space $\mathcal{B}_P^\alpha(\Omega)$, such as completeness (cf. [14]). On the contrary, the elliptic case is not really interesting here since $\mathcal{B}_P^\alpha(\Omega) = \mathcal{B}_\alpha(\Omega)$ if and only if $P$ is elliptic, as we have already mentioned at the end of Section 2.

Proposition 6 leads us to define the wave front set with respect to the iterates of an operator.

**Definition 7.** Let $\Omega$ be an open subset of $\mathbb{R}^n$, $u \in \mathcal{D}'(\Omega)$, and $P(D)$ a linear partial differential hypoelliptic operator of order $m$ with constant coefficients. We say that a point $\xi_0 \in \mathbb{R}^n \setminus \{0\}$ is not in the $\omega_0$-wave front set with respect to the iterates of $P$, $WF_{\omega_0}^P(u)$, if there is a neighborhood $U$ of $\xi_0$, an open conic neighborhood $\Gamma$ of $\xi_0$, and a sequence $\{f_j\}_{j \in \mathbb{N}_0} \subset \mathcal{B}_P(\Omega)$ such that (i) and (ii) of the following conditions hold (i) and (iii) of the following conditions hold:

(i) For every $N \in \mathbb{N}$, $f_N = P(D)^N u$ in $U$.

(ii) Roumieu:

(a) there are constants $k \in \mathbb{N}$, $M > 0$, and $C > 0$, such that

$$
\left| f_N (\xi) \right| \leq C_N (e^{(1/k)\varphi(\frac{N m}{k})} + \left| \xi \right|)^M, \quad N \in \mathbb{N}, \quad \xi \in \mathbb{R}^n;
$$

(40)

(b) there is a constant $k \in \mathbb{N}$ such that for all $\ell \in \mathbb{N}_0$, there is $C_{\ell} > 0$ with the property

$$
\left| f_N (\xi) \right| \leq C_{\ell} \left( 1 + \left| \xi \right| \right)^{-\ell}, \quad N \in \mathbb{N}, \quad \xi \in \Gamma.
$$

(41)
(iii) Beurling:

(a) there are $M, C > 0$ such that for all $k \in \mathbb{N}$, there is $C_k > 0$ such that
\[
|\hat{f}_N(\xi)| \leq C_k C^N (e^{[(kN)^p]}(Nk))^{N_m} (1 + |\xi|)^M,
\]
\[N \in \mathbb{N}, \quad \xi \in \mathbb{R}^n;\] (42)

(b) for all $\ell \in \mathbb{N}_0$ and $k \in \mathbb{N}$ there is $C_{k,\ell} > 0$ such that
\[
|\hat{f}_N(\xi)| \leq C_{k,\ell} e^{[(k\ell N^2)]} (1 + |\xi|)^\ell,
\]
\[N \in \mathbb{N}, \quad \xi \in \Gamma.\] (43)

We select $p \in \mathbb{N}$ as in Lemma 8 (or bigger if necessary) and set $f_N = \chi_{Nmp}(P(D))^N u$. We first observe that, as $u = u_s$ in $U$ for all $N \in \mathbb{N}$ and $\chi_{Nmp}(P(D))^N u_s$ for every $s \in \mathbb{N}$, we want to prove (i), (ii)(a), and (ii)(b) in Definition 7. By the choice of $\chi_N$, condition (i) is fulfilled in the neighborhood $V$. To see (ii)(a), we observe that from Lemma 8 there is $\tilde{C} > 0$ such that
\[
|\hat{f}_N(\xi)| \leq \tilde{C}^N (mN + |\xi|)^{N_m} (1 + |\xi|)^M,
\]
\[\xi \in \mathbb{R}^n, \quad N = 0, 1, 2, \ldots,\] (49)

for some constant $M > 0$. Since the weight function $\omega$ satisfies $\omega(t) = o(t)$ as $t$ tends to infinity, from [22, Remark 2.4(b)], for every $k \in \mathbb{N}$ there is $C_k > 0$ such that
\[
Nm \leq (C_k)^{(1/Nm^p)}(Nk)^{(Nmk)}, \quad N \in \mathbb{N}.
\]
(50)

In particular, for $k = 1$, we obtain
\[
|\hat{f}_N(\xi)| \leq C_1 C^N \left(e^{[(Nmp)]}(Nm) + |\xi|\right)^{N_m} (1 + |\xi|)^M,
\]
\[\xi \in \mathbb{R}^n, \quad N = 0, 1, 2, \ldots,\] (51)

which proves (ii)(a).

We prove now (ii)(b). We fix $\ell \in \mathbb{N}$ and set, for $f_N = \chi_{Nmp}(P(D))^N u_{Nmp} + \epsilon$,
\[
(1 + |\xi|^\ell) |\hat{f}_N(\xi)| \leq (1 + |\xi|^\ell) \int |\chi_{Nmp}(\eta)| |P(\xi - \eta)| d\eta
\]
\[
= J_1(\xi) + J_2(\xi),\]
(52)

where $J_1(\xi)$ is the integral when $|\eta| \leq c|\xi|$, for $c > 0$ to be chosen, and $J_2(\xi)$ is the integral when $|\eta| \geq c|\xi|$, both considered with the factor $(1 + |\xi|)^\ell$. In $J_2(\xi)$, we have
\[
|\xi - \eta| \leq |\xi| + |\eta| \leq (1 + c^{-1}) |\eta|.
\]
(53)

Since $u_{Nmp}$ is a bounded sequence in $\mathcal{S}'(\Omega)$, there is $M > 0$ such that $|\hat{u}_{Nmp}(\xi)| \leq C_2 (1 + |\xi|)^M$ for all $\xi \in \mathbb{R}^n$ and $N \in \mathbb{N}$.

From (48), we can differentiate $\chi_{Nmp}$ up to the order $N m$ to obtain constants $C_{2,\ell}$ that depend on $n, \ell$, and $M$ such that (see [22, Lemma 3.5])
\[
|\chi_{Nmp}(\eta)| \leq C_2 e^{-N^{m+1}}
\]
\[
\times \left(\frac{e^{[1]}(Nk)^p(Nkm)}{(1 + |\eta|)^{N_m(1 + |\xi|)^M}}\right)_{\eta \in \mathbb{R}^n}.
\]
(54)

As $P(D)$ has order $m$, we also have $|P(\xi)|^N \leq C(1 + |\xi|)^{Nm}$ for some constant $C > 0$ and each $\xi \in \mathbb{R}^n$ and $N \in \mathbb{N}$.
Moreover, in \(J_2(\xi), (1 + |\xi|)^{\ell} \leq (1 + c^{-1})^{\ell}(1 + |\eta|)^{\ell}\) and
\[(1 + |\xi - \eta|)^{N_m + M} \leq (1 + c^{-1})^{N_m + M}(1 + |\eta|)^{N_m + M}.
\]
Therefore, from (54), we obtain
\[
|J_2(\xi)|
\leq D_C \left(1 + c^{-1}\right)^{M + N_m + \ell} 
\times \int_{|\eta| > |\xi|} \left(1 + |\eta|\right)^{N_m + \ell}(1 + |\eta|)^{M} |\hat{\chi}_{Nmp}(\eta)| \, d\eta
\leq D'_C C_2^{N_m + 1}(1 + c^{-1})^{M + N_m + \ell} e^{(1/k)\rho^*(Nmk)}
\]
for some \(D, D' > 0.\)

On the other hand, if we consider the estimate \((1 + |\xi|)^{\ell} \leq (1 + |\xi - \eta|)^{\ell}(1 + |\eta|)^{\ell},\) we obtain
\[
|J_1(\xi)| \leq \left(\int (1 + |\eta|)^{\ell} |\hat{u}_{Nmp}(\eta)| \, d\eta\right)
\cdot \sup_{|\eta| < \xi} |\hat{u}_{Nmp}(\xi - \eta)|
\cdot (1 + |\xi - \eta|)^{\ell} \cdot |P(\xi - \eta)|^{Nm}.
\]

We observe that the integral is less than or equal to \(C_C A^N\) for some constant \(C_C > 0\) that depends on \(\ell\) and the support of \(\chi_{Nmp}\) and some constant \(A > 0.\) Now, we write \(\xi = \xi - \eta.\) If \(\Gamma\) is a conic neighborhood of \(\xi,\) such that \(\Gamma \subset F,\) we can select \(0 < c < 1\) such that for \(\xi \in \Gamma\) and \(|\xi - \xi| \leq c|\xi|,\) we have \(\xi \in F.\) Consequently, we obtain, by assumption on \(\hat{u}_{Nmp}\) (see (47)), and by the estimate \(|P(\xi)|^{N} \leq C_N (1 + |\xi|)^{N}\) for some positive constant \(C,\) for \(\xi \in \Gamma,\)
\[
|J_1(\xi)| \leq C_C N^{\ell} \sup_{|\xi - \xi| < c|\xi|} |\hat{u}_{Nmp}(\xi)| \cdot (1 + |\xi|)^{\ell} \cdot |P(\xi)|^{Nm}
\leq \bar{C}_C e^{C_N + 1} e^{(1/k)\rho^*(Nmk + k\ell)}
\]
for some \(\bar{C} > 0.\) We conclude, using the convexity of \(\varphi^*,\) that there are constants \(D_C > 0\) and \(E > 0\) such that
\[
(1 + |\xi|)^{\ell} |f_N(\xi)| \leq |J_1(\xi)| + |J_2(\xi)|
\leq D_C e^{C_N + 1} e^{(1/k)\rho^*(Nmk + k\ell)}, \quad \xi \in \Gamma.
\]

**Beurling Case.** Let us assume now that \((x_0, \xi_0) \notin W^F_{\omega}(u).\) From Definition 5, there exist a neighborhood \(U\) of \(x_0,\) an open conic neighborhood \(F\) of \(\xi_0,\) and a bounded sequence \(\{u_N\}_{N \in \mathbb{N}} \subset \mathcal{B}(\Omega)\) such that \(u_N \equiv u\) in \(U\) for every \(N \in \mathbb{N}\) and for every \(k \in \mathbb{N}\) there is \(C_k > 0,\) such that
\[
|\xi|^N |\hat{u}_N(\xi)| \leq C_k e^{k\rho^*(N/k)}, \quad \xi \in F, \ N \in \mathbb{N}.
\]

We take \(\chi_N\) and \(f_N\) as in the Roumieu case. From (50), for any \(k \in \mathbb{N},\) there is \(D_k > 0\) satisfying
\[
|f_N(\xi)| \leq D_k e^{C_N} \left(e^{k/Nm\rho^*(Nmk/k)} + |\xi|\right)^{Nm}(1 + |\xi|^M),
\]
\(\xi \in \mathbb{R}^n, \ N = 0, 1, 2, \ldots,\)

which proves (iii)a).

To prove (iii)b, fix \(\ell \in \mathbb{N}\) and consider now the estimate (use (48) and (50))
\[
|\hat{\chi}_{Nmp}(\eta)| \leq C_C e^{C_N + 1} \frac{C_k e^{k\rho^*(Nmk/k)}}{|\eta| + e^{(k/Nm)\rho^*(Nmk/k)}} e^{(1/k)\rho^*(Nmk/k)}
\times (1 + |\eta|)^{N_m - M - \ell}, \quad \eta \in \mathbb{R}^n.
\]

Here,
\[
(1 + |\xi|)^{\ell} |f_N(\xi)| \leq (1 + |\xi|)^{\ell}
\times \left(\int |\hat{\chi}_{Nmp}(\eta)| |P(\xi - \eta)|^{N_m} \right.
\times \left|\hat{u}_{Nmp}(\xi - \eta)| \, d\eta\right) =: J_1(\xi) + J_2(\xi),
\]
where \(J_1(\xi)\) is the integral when \(|\eta| \leq c|\xi|,\) for \(c > 0\) to be chosen, and \(J_2(\xi)\) is the integral when \(|\eta| \geq c|\xi|\). In this case, we use (60) and obtain a constant \(C_C > 0\) which depends on \(\ell\) and \((m, n)\) and a constant \(E > 0\) with the property that for every \(k \in \mathbb{N}\) there is a constant \(C_k > 0\) such that for any \(\xi \in \Gamma\) and \(N \in \mathbb{N},\)
\[
(1 + |\xi|)^{\ell} |f_N(\xi)| \leq C_k e^{N + 1} C_k e^{k\rho^*(Nmk/k)},
\]

This concludes the Beurling case. \(\square\)

**Corollary 10.** Let \(u \in \mathcal{D}'(\Omega),\) and let \(K\) be a compact subset of \(\Omega\) and \(F\) a closed cone in \(\mathbb{R}^n.\) Let \(\omega\) be a weight function. Suppose that \(\{\chi_N\} \subset \mathcal{D}(K)\) is like in (48). Then, we have the following:

(a) If \(W^F_{\omega}(u) \cap (K \times F) = 0,\) then the sequence \(g_N = \chi_{Nmp} P(D)^N u,\) for \(p \in \mathbb{N}\) large enough independent of \(N,\) satisfies that there is \(k \in \mathbb{N}\) such that for every \(\ell \in \mathbb{N},\) there is \(C_\ell > 0\) with
\[
|g_N(\xi)| \leq C_\ell e^{(1/k)\rho^*(Nmk/k)} (1 + |\xi|)^{-\ell}, \quad \xi \in F, \ N \in \mathbb{N}.
\]

(b) If \(W^F_{\omega}(u) \cap (K \times F) = 0,\) then the sequence \(g_N = \chi_{Nmp} P(D)^N u,\) for \(p \in \mathbb{N}\) large enough independent of \(N,\) satisfies that for every \(k, \ell \in \mathbb{N}\) there is \(C_{k, \ell} > 0\) with
\[
|g_N(\xi)| \leq C_{k, \ell} e^{k\rho^*(Nmk/k)} (1 + |\xi|)^{-\ell}, \quad \xi \in F, \ N \in \mathbb{N}.
\]
Proof. We make a sketch of proof of (a) only. Let $x_0 \in K$, $\xi_0 \in F \setminus \{0\}$ and choose $U$ and $\Gamma$, with $\Gamma$ a conic subset of $F$ and $f_N$ according to Definition 7. If the support of $\chi_N$ is in $U$, we have $\chi_{N\eta}(P(D)^N u) = \chi_{N\eta}(f_N)$. Now, the proof is like (ii)(b) of Proposition 9 for the set $\Gamma$ and $f_N$ instead of $(P(D)^{N\eta})$. To obtain a uniform estimate in $F$, we can proceed as in [22, Lemma 3.5] at the end of the proof of (a). See also the proof of [25, Lemma 8.4.4].

The singular support of a classical distribution $u \in \mathcal{D}'(\Omega)$ with respect to the class $\mathcal{E}_s^P$ is the complement in $\Omega$ of the biggest open set $U$, where $u|_U \in \mathcal{E}_s^P(U)$. As a consequence of Propositions 6 and 9 and Corollary 10, we obtain the following result.

**Corollary 11.** The projection in $\Omega$ of $WF^P_s(u)$ is the singular support with respect to the class $\mathcal{E}_s^P(\Omega)$ if $u \in \mathcal{D}'(\Omega)$.

**Proof.** Follow the lines of the proofs of [22, Theorem 3.6] and [25, Theorem 8.4.5].

**Remark 12.** We observe that from the definition it is obvious that if $P(\Omega)$ is hypoelliptic, then for $* = (\omega)$ or $\omega$

$$WF^P_s(u) = WF^P_s(Pu).$$

Then, by Proposition 9, the following inclusions hold:

$$WF^P_s(u) \subseteq WF^P_s(Pu) \subseteq WF^P_s(Pu) \subset WF^P_s(u).$$

Now, we can state an improvement of [22, Theorem 4.8] for operators with constant coefficients.

**Theorem 13.** Let $P(D) = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$, $a_\alpha \in \mathbb{C}$, be a hypoelliptic linear partial differential operator with constant coefficients and order $m$ and let $\Omega$ be an open subset of $\mathbb{R}^n$. Let $P_m$ be the principal part of $P$ and $\Sigma = \{(x, \xi) \in \Omega \times \mathbb{R}^n \setminus \{0\} : P_m(\xi) = 0\}$ the characteristic set of $P(D)$. Then, for any distribution $u \in \mathcal{D}'(\Omega)$

$$WF^P_s(u) \subseteq WF^P_s(Pu) \subseteq \Sigma.$$

**Proof.** Let $(x_0, \xi_0) \notin WF^P_s(u)$ such that $P_m(\xi_0) \neq 0$. Then, there are a neighborhood $U$ of $\chi_N$, a conic neighborhood $\Gamma$ of $\xi_0$, and a sequence $\{f_n\}_{N \in \mathbb{N}} \in \mathcal{E}_s(\Omega)$ that verify (i), (ii)(a)-(ii)(b) in the Roumieu case, and (iii)(a)-(iii)(b) in the Beurling case of Definition 7. We take $F \subseteq \Gamma$ such that $P_m(\xi) \neq 0$ for $\xi \in F$. We take a compact neighborhood $K \subset U$ of $x_0$ and consider a sequence $\{\chi_N\}_{N \in \mathbb{N}} \in \mathcal{E}_s(U)$ satisfying (48) such that $\chi_N \equiv 1$ on $K$.

We set now $u_N = \chi_{3m^2 \Gamma} u$. We want to estimate

$$\tilde{u}_N(\xi) = \int u(x) \chi_{3m^2 \Gamma}(x) e^{-i(x,\xi)} dx.$$

To estimate $|\tilde{u}_N(\xi)|$ in $F$, we will solve in an approximate way the following equation:

$$iP(D)^N v(x) = \chi_{3m^2 \Gamma}(x) e^{-i(x,\xi)}.$$

As in [17], we put $v(x) = e^{-i(x,\xi)} w(x,\xi)/P_m(\xi)^N$. For $(x, \xi) \in K \times F$, we have

$$iP(D)^N e^{-i(x,\xi)} P_m^{-1}(\xi) w = \sum_{k \leq m} (-1)^k a_k D^k e^{-i(x,\xi)} P_m^{-1}(\xi) w \tag{72}$$

$$\times \sum_{\beta=0}^{|\alpha|} \alpha \beta e^{-i(x,\xi)} \frac{\alpha \beta}{P_m(\xi)^N}.$$
the coefficients of $R^s$ are not zero. Therefore, (we write $\chi$ for $\chi_{3m^2N}$ for simplicity)

\[
(I - R)^N w_N
\]

\[
= \chi + \sum_{h=1}^{N} \sum_{j=mN+1-h}^{mN} \binom{N}{j} (-1)^{h+j} R^{h+j} \chi
\]

\[
= \chi - e_N
\]

for

\[
e_N := \sum_{h=1}^{N} \sum_{j=mN+1-h}^{mN} \binom{N}{j} (-1)^{h+j+1} R^{h+j} \chi.
\]

Then,

\[
t \cdot P(D)^N \left( e^{-i(x \cdot \xi)} p_m^{-N} w_N \right) = e^{-i(x \cdot \xi)} (I - R)^N w_N
\]

\[
= e^{-i(x \cdot \xi)} (\chi - e_N).
\]

If we apply these identities to $u$, we obtain

\[
\tilde{u}_N(\xi) = \int e^{-i(x \cdot \xi)} p_m^{-N} u(x) \, dx
\]

\[
= \int e^{-i(x \cdot \xi)} e_N(x, \xi) u(x) \, dx
\]

\[
+ \int t \cdot P(D)^N \left( e^{-i(x \cdot \xi)} p_m^{-N} w_N \right) \cdot u(x) \, dx
\]

\[
= e^{-i(x \cdot \xi)} e_N(x, \xi) u(x) \, dx
\]

\[
+ \int e^{-i(x \cdot \xi)} p_m^{-N} (N \cdot \xi) w_N(x, \xi) P(D)^N u(x) \, dx
\]

\[
=: H_1(\xi) + H_2(\xi),
\]

where the integrals denote action of distributions.

We suppose now that $u$ has order $M > 0$ in a neighborhood of $K$. Since $H_1(\xi) = \langle u, e_N e^{-i(x \cdot \xi)} \rangle$, we have

\[
|H_1(\xi)| \leq C \sum_{|\beta| \leq M} \left| D_x^\beta \left( e_N(x, \xi) e^{-i(x \cdot \xi)} \right) \right|
\]

\[
\leq C \sum_{|\beta| \leq M} \sum_{\alpha \geq 0} \binom{\beta}{\alpha} \left| D_x^\alpha e_N(x, \xi) \right|
\]

\[
\leq C' \sum_{|\beta| \leq M} \left( 1 + |\xi| \right)^{M-|\beta|} \left| D_x e_N(x, \xi) \right|.
\]

In order to estimate this expression, first we estimate

\[
|D_x^a e_N| \leq \sum_{j=1}^{N} \sum_{j=mN+1-h}^{mN} \binom{N}{j} \left| D_x \left( R^{j+h} \chi_{3m^2N} \right) \right|
\]

\[
\leq \sum_{j=1}^{N} \sum_{j=mN+1-h}^{mN} \binom{N}{j} \left| D_x \left( R^{j+h} \chi_{3m^2N} \right) \right|
\]

The number of terms in $e_N$ depends on

\[
\leq \sum_{j=1}^{N} \sum_{j=mN+1-h}^{mN} \binom{N + mN - 1}{j} \leq 2^{Nm}.
\]

Now, since $\sum_{h=0}^{N} \binom{N}{h} = 2^N$ and in the sum of the expression of $e_N$, $mN < s = h + j \leq mN + N$, we obtain (we recall that $R = R_1 + \cdots + R_m$)

\[
|D_x^s e_N| \leq 2^{(m+2)N} \sum_{s=mN+1}^{mN+2N} \left| D_x^s \left( R^s \chi_{3m^2N} \right) \right|
\]

\[
\leq C N \sum_{s=mN+1}^{mN+2N} \sum_{j=mN+1}^{mN+2N} \frac{s!}{j!} \left| R^s \chi_{3m^2N} \right|
\]

\[
\leq B N \left( \sum_{s=0}^{mN} \left( 3m^2N \right)^{N-s} \right) \left| \xi \right|^{-N-1}.
\]

Therefore, we obtain a new constant $C > 0$ such that

\[
\left| H_1(\xi) \right| \leq C \left( 1 + |\xi| \right)^M \left( N+M \right) \left| \xi \right|^{-N-1}, \quad \forall |\xi| > N.
\]

We study now

\[
H_2(\xi) = \int e^{-i(x \cdot \xi)} p_m^{-N}(\xi) w_N(x, \xi) P(D)^N u(x) \, dx
\]

\[
= p_m^{-N}(\xi) \int e^{-i(x \cdot \xi)} w_N(x, \xi) f_N(x) \, dx
\]

\[
= p_m^{-N}(\xi) \cdot F \left( w_N f_N(\xi) \right)
\]

\[
= p_m^{-N}(\xi) \cdot \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \tilde{w}_N(\eta) \cdot \tilde{f}_N(\xi - \eta) \, d\eta := S_1(\xi) + S_2(\xi),
\]

where we have splitted $H_2(\xi)$ in the sum of $S_1(\xi)$ and $S_2(\xi)$, the first when $|\eta| \leq c|\xi|$ and the second when $|\eta| \geq c|\xi|$, for a constant $c > 0$ to be chosen.
First, we estimate \( w_N \) defined in formula (76). Proceeding in a similar way as before with the expression of \( e_N \), if we take \( |\xi| > mN \) and \( |\beta| \leq 2m^2N \) and estimate the binomials as in (85), we find a constant \( A > 0 \) such that
\[
|D_x^\beta w_N| \leq \sum_{j=0}^N \left( \frac{-N}{j} \right) (-1)^j D_x^\beta \left( R^j x \chi_{m^2N} \right)
\]
\[
\leq \sum_{j=0}^N \left( \frac{-N}{j} \right) \sum_{j_1 + \cdots + j_m = j} \frac{j!}{j_1! \cdots j_m!} |\eta|^{jm} (3m^2N)^m
\]
\[
\leq C^{N+1} (3m^2N)^m \sum_{j=0}^N \left( \frac{-N}{j} \right) |\eta| jm (3m^2N)^m
\]
\[
\leq A^N (mN)^{|\beta|}.
\]  

(90)

At this point, we have to separate Beurling and Roumieu cases.

**Roumieu Case.** From Definition 7(ii)(a), we have
\[
|\tilde{f}_N(\xi)| \leq C_N (e^{(1/2Nm\kappa \psi^*}(Nk\mu) + |\xi|)^N (1 + |\xi|)^M, \quad N \in \mathbb{N}, \quad \xi \in \mathbb{R}^n,
\]
for some constants \( C > 0, M > 0 \), and \( k \in \mathbb{N} \). Now, as \( \omega_N \in \mathcal{D}(U) \), by (90), we have, as in [22, Lemma 3.5],
\[
|\tilde{w}_N(\eta)| \leq C^{N+1} e^{(1/2Nm\kappa \psi^*}(Nk\mu) + |\eta|)^N (1 + |\eta|)^{-n - 1 - M}, \quad \eta \in \mathbb{R}^n.
\]  

(92)

We proceed now as in the proof of (ii)(b) of Proposition 9 in order to estimate \( H_2(\xi) = S_1(\xi) + S_2(\xi) \). In \( S_2(\xi) \), we have \( |\xi - \eta| \leq (1 + c^{-1})|\eta| \) and, by (92), we deduce
\[
|S_2(\xi)| \leq (2n)^{-n} |P_m(\xi)|^{-N}
\]
\[
\times \int_{|\eta| \leq |\xi|} |\tilde{w}_N(\eta)\tilde{f}_N(\xi - \eta)| d\eta
\]
\[
\leq D N^{-n} (1 + c^{-1})^{Nn + M}
\]
\[
\times \int_{|\eta| \leq |\xi|} |\tilde{w}_N(\eta)| e^{(1/2Nm\kappa \psi^*}(Nk\mu) + |\eta|)^N (1 + |\eta|)^{M} d\eta
\]
\[
\leq B^N e^{(1/2Nm\kappa \psi^*}(Nk\mu) \xi^{-n} - Nm,
\]  

(93)

for some constants \( D, B > 0 \).

For \( S_1(\xi) \) we have
\[
|S_1(\xi)| \leq |P_m(\xi)|^{-N} \sup_{|\eta| \leq |\xi|} \left| \tilde{f}_N(\xi - \eta) \right|.
\]  

(94)

As in the proof of Proposition 9, we can estimate \( S_1(\xi) \), in the Rougieu case, with the use of (ii)(b) of Definition 7 in the following way: we select \( c > 0 \) for which there are \( C > 0 \) and \( k \in \mathbb{N} \) such that for \( \xi \) in some neighborhood \( 1' \) of \( \xi_0 \) (see the argument before inequality (58)),
\[
\sup_{|\eta| \leq |\xi|} \left| \tilde{f}_N(\xi - \eta) \right|
\]
\[
\leq C^{N+1} e^{(1/2Nm\kappa \psi^*}(Nk\mu) (1 + |\xi - \eta|)^M.
\]  

(95)

Consequently, since \( \|\tilde{w}_N\|_{L_1} \leq A^N \) for some constant \( A > 0 \),
\[
|S_1(\xi)| \leq D^{N+1} e^{(1/2Nm\kappa \psi^*}(Nk\mu) |\xi|^{M} |P_m(\xi)|^{-N}
\]
\[
\leq E^{N+1} e^{(1/2Nm\kappa \psi^*}(Nk\mu) |\xi|^{-N - m}.
\]  

(96)

Therefore, if we combine (96) and (93), we have two constants \( C > 0 \) and \( h \in \mathbb{N} \) such that for \( \xi \) in some conic neighborhood of \( \xi_0 \) and \( |\xi| \geq e^{(1/2Nm\kappa \psi^*}(Nk\mu) (2N(m-1)h) \), by (89),
\[
|H_2(\xi)| \leq C^{N+1} e^{(1/2h \psi^*}(2N(m-1)h) |\xi|^{M - N - m}
\]
\[
\leq C^{N+1} e^{(1/2h \psi^*}(2N(m-1)h) |\xi|^{M - N - m}.
\]  

(97)

As in (50), we have \( N^{-N} \leq Ae^{\psi^*(N)} \) for some constant \( A > 0 \) and every \( N \in \mathbb{N} \). Then, from (88), we deduce a similar estimate to the one of \( |H_2(\xi)| \) for \( |H_1(\xi)| \). Now, from the bounds for \( H_1(\xi) \) and \( H_2(\xi) \), there are constants \( C, h > 0 \) such that, for \( \xi \) in some conic neighborhood of \( \xi_0 \) and \( |\xi| \geq e^{(1/2Nm\kappa \psi^*}(Nk\mu) (2N(m-1)h) \),
\[
|\tilde{u}_N(\xi)| \leq C^{-N} (1 + |\xi|)^M e^{(1/2h \psi^*}(hN) |\xi|^{-N}.
\]  

(98)

We have a similar estimate when \( |\xi| \leq e^{(1/2Nm\kappa \psi^*}(Nk\mu) (2N(m-1)h) \). In fact, since the sequence \( u_N \) is bounded in \( \mathcal{B}^{(h)} \), there are constants \( D > 0 \) and \( M' > 0 \) which satisfy
\[
|\tilde{u}_N(\xi)| \leq D (1 + |\xi|)^{M'}, \quad \xi \in \mathbb{R}^n.
\]  

(99)

Then, we have
\[
|\tilde{u}_N(\xi)|
\]
\[
\leq D (1 + |\xi|)^{M'} \leq C \left( e^{(1/2N(m-1)h) \psi^*}(2N(m-1)h) \right)^{M'+N} |\xi|^{-N}
\]
\[
\leq C \left( e^{(1/2N(m-1)h) \psi^*}(2N(m-1)h) \right)^{M'+N} |\xi|^{-N}
\]
\[
\leq D' e^{(1/2h' \psi^*}(h') \xi^{-N}.
\]  

(100)
Beurling Case. In this setting we will proceed in a similar way. We can select $0 < c < 1$ and apply now (iii)(b) of Definition 7 to obtain, for every $k \in \mathbb{N}$, a constant $C_k > 0$ such that, for all $\xi$ in some neighborhood of $\xi_0$,

$$|S_2(\xi)| \leq C_k C^N e^{k \Phi^*(N(\eta)/k)} \delta^{M-Nm}, \quad N \in \mathbb{N}, |\xi| > N.$$  

(103)

Therefore, from (101) and (103), we have $C > 0$ and for a fixed $k \in \mathbb{N}$ a constant $C_k > 0$ such that for $\xi$ in some conic neighborhood of $\xi_0$ and $|\xi| \geq e^{k(N(m-1))} (N(m-1)/k)$, we have

$$|H_2(\xi)| \leq C_k C^N e^{k \Phi^*(N(\eta)/k)} \delta^{M-Nm} \leq C_k C^N e^{k \Phi^*(N(\eta)/k)} \delta^{M-N}.$$  

(104)

As in the Roumieu case, we deduce a similar estimate for $|H_1(\xi)|$. Then, the bounds for $H_1(\xi)$ and $H_2(\xi)$ give a constant $C > 0$ and, for every $k \in \mathbb{N}$, a constant $C_k > 0$ such that for $\xi$ in some conic neighborhood of $\xi_0$ and $|\xi| \geq e^{k(N(m-1))} (N(m-1)/k)$, we have

$$|\bar{u}_N(\xi)| \leq C_k C^N e^{k \Phi^*(N(\eta)/k)} \delta^{M-N}.$$  

(105)

Finally, we also have a similar estimate when $|\xi| \geq e^{k(N(m-1))} (N(m-1)/k)$, which concludes the proof of the theorem.

Remark 14. If $P(D)$ is elliptic, then $\Sigma = \emptyset$ and Theorem 13 and Remark 12 imply that

$$WF_\ast(u) = WF_\ast^P(u).$$  

(106)

Example 15. We show that the inclusions

$$WF_\ast(u) < WF_\ast(u),$$  

$$WF_\ast(u) < WF_\ast(Pu)$$

of Remark 12 are strict. As in [14] (see [26]), we consider a nonquasianalytic weight function $\omega$ satisfying the following condition: there exists a constant $H \geq 1$ such that for all $t \geq 0$,

$$2\omega(t) \leq \omega(\Delta t) + H.$$  

(108)

For example, if $\omega$ is a Gevrey weight, then it satisfies such a property. We consider now a polynomial $P$ with constant complex coefficients such that it is hypoelliptic but not elliptic (for instance, the heat operator). Then by [14, Theorem 4.12], there is $u \in \mathcal{E}_\omega^P(\Omega) \setminus \mathcal{E}_\omega^P(\Omega) \ast (\Omega)$ (for some open subset $\Omega$ of $\mathbb{R}^n$). Then, $WF_{\omega}^P(u) = \emptyset$ but $WF_{\omega}(u) \neq \emptyset$, which implies that the inclusion

$$WF_{\omega}^P(u) \subseteq WF_{\omega}(u)$$  

is strict.

On the other hand, if we consider now a $\omega$-hypoelliptic polynomial $P$ which is not elliptic (e.g., the heat operator in $\mathbb{R}^n$ for $\omega(t) = t^{1/3}$), then as before there will be $u \in \mathcal{E}_\omega^P(\Omega) \setminus \mathcal{E}_\omega^P(\Omega)$. In particular, $WF_{\omega}^P(u) = \emptyset$. Now, if $WF_{\omega}(Pu) = \emptyset$, we will have $Pu \in \mathcal{E}_\omega^P(\Omega)$ and since $P$ is $\omega$-hypoelliptic, $u \in \mathcal{E}_\omega^P(\Omega)$, which is a contradiction. Therefore, $WF_{\omega}(Pu) \neq \emptyset$ and we conclude that the inclusion

$$WF_{\omega}^P(u) \subseteq WF_{\omega}(Pu)$$  

is strict.

Let us also remark that for the heat operator $Q(D) = \partial_1 - \Delta_x$, we can explicitly write its characteristic set $\Sigma$, so that the previous considerations give, for $u \in \mathcal{E}_\omega^P(\Omega) \setminus \mathcal{E}_\omega^P(\Omega)$, the following information on $WF_{\omega}(u)$, because of Theorem 13:

$$\emptyset \neq WF_{\omega}(u) \subset WF_{\omega}^P(u) \cup \Sigma \quad \Sigma = \{(t, x, \tau, 0) \in \Omega \times \mathbb{R}^{m_1} : \tau \neq 0\}.$$  

(111)

In the Beurling setting, we can proceed in a similar way. Let us finally notice that the inclusion

$$WF_\ast(Pu) \subseteq WF_\ast(u)$$  

(112)

of Remark 12 is strict in general.

4. Distributions with Prescribed Wave Front Set

The proof of the following lemma is straightforward.

Lemma 16. Let $\omega$ be a weight function. Then, for every $a > 0$ and $m \in \mathbb{N}$

$$\begin{align*}
(i) & t^a e^{-a \omega(t)} \leq e^{a \Phi^*(m/a)} \forall t \geq 1; \\
(ii) & \inf_{t \in [0,1]} t^a e^{-a \omega(t)} \leq t^a e^{-a \omega(t)} \forall t \geq 1.
\end{align*}$$

Now, we show that the product of a Gevrey function with a function in $\mathcal{E}_\omega^P(\Omega)$ belongs to the last space.

Proposition 17. Let $\omega$ be a nonquasianalytic weight function such that $\omega(t^s) = o(\sigma(t))$ as $t \to \infty$, where $\gamma > 0$ is the constant in (28) and $\sigma(t) = t^{1/3}$ is a Gevrey weight, with $s > 1$. If $\chi \in \mathcal{E}_\omega^P(\Omega)$ and $u \in \mathcal{E}_\omega^P(\Omega)$, where $\ast = \{\omega\}$ or $\omega$, then the multiplication $\chi u \in \mathcal{E}_\omega^P(\Omega)$. 


Proof. We will analyse the $L^2$-norms of $P(D)^j(\chi u)$ on a compact set $K$ in $\Omega$. First, we observe that, by the generalized Leibniz rule over $P(D)$ applied $j$ times,

$$P(D)^j(\chi u) = P(D) \left[ P(D)^{\frac{j-1}{j}} P(D) (\chi u) \right]$$

$$= \sum_{|\alpha_1|=\ldots=|\alpha_j|=m} \frac{1}{\alpha_1! \ldots \alpha_j!} D^{\alpha_1+\cdots+\alpha_j} \chi$$

$$\cdot P(\alpha_j) (P(\alpha_j) \cdots (P(\alpha_j) (D) u)),$$

We fix now a compact set $K$ in $\Omega$ such that $dist(K, \partial \Omega) \geq r > 0$. We apply $L^2$-norms in the compact set $K$

$$\| P(D)^j(\chi u) \|_{L^2_K} \leq \sum_{|\alpha_1|=\ldots=|\alpha_j|=m} \frac{1}{\alpha_1! \ldots \alpha_j!} \| D^{\alpha_1+\cdots+\alpha_j} \chi \|_{L^2_K} \times \| P(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K}.$$ 

(113)

Since $\chi \in C^0(\Omega)$, there is a constant $A > 0$ such that, for each $\alpha \in \mathbb{N}_0^m$ and $x \in K$ we have

$$|D^\alpha \chi (x)| \leq A^{|\alpha|} |x|^{s|\alpha|}.$$ 

(115)

Consequently,

$$\sup_{x \in K} \| D^{\alpha_1+\cdots+\alpha_j} \chi (x) \| \leq A^{\sum_{i=1}^j |\alpha_i|} \leq A^m (jm)^{s(|\alpha_1|+\cdots+|\alpha_j|)}.$$ 

(117)

Therefore,

$$\| P(D)^j(\chi u) \|_{L^2_K} \leq \sum_{|\alpha_1|=\ldots=|\alpha_j|=m} \frac{1}{\alpha_1! \ldots \alpha_j!} \| D^{\alpha_1+\cdots+\alpha_j} \chi \|_{L^2_K} \times \| P(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K}.$$ 

(116)

$$\| P(D)^j(\chi u) \|_{L^2_K} \leq \sum_{|\alpha_1|=\ldots=|\alpha_j|=m} \frac{A^{jm} (jm)^{s(|\alpha_1|+\cdots+|\alpha_j|)}}{|\alpha_1! \ldots \alpha_j!|} \times \| P(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K}.$$ 

(118)

Now, we apply (28) $j$ times to the factor $\| P(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K}$.

We will use the notation $K(\varepsilon) = K + B(0, \varepsilon)$, for $\varepsilon > 0$. In the first step,

$$\| P(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K} \leq C \left( \varepsilon_1^{a_1} \| p(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K} \right) + \varepsilon_1^{a_1} \| p(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K}.$$ 

(119)

In the second step, $K(e_1)$ is replaced by $K(e_1 + e_2)$ and so on in the next steps. Therefore, to avoid that, after $j$ steps, the set $K(e_1 + \cdots + e_j)$ leaves $\Omega$ and to keep it bounded for all $j$, we may take $e_k$ depending on $k$ for all $1 \leq k \leq j$. We take $e_k = Bk^{-\gamma}$ with $B > 0$ a constant such that

$$e_1 + \cdots + e_j = B \left( 1 + \frac{1}{2^2} + \cdots + \frac{1}{j^\gamma} \right) < \frac{r}{2}$$

(120)

for all $j$. It is obvious that $e_k \gamma < e_{k+1} \gamma$ for all $1 \leq k \leq j - 1$. Moreover, we can assume that $e_k < 1$ for all $1 \leq k \leq j$.

After $j$ steps we get

$$\| p(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K} \leq C j^2 e_1^{a_1} \cdots e_j^{a_j} \left( \| p(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K} \right) + \varepsilon_1 \| p(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K} + \cdots + \varepsilon_j \| p(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K}.$$ 

(121)

With our selection of $e_k$, for $1 \leq k \leq j$, we have

$$e_k = \frac{B^{a_k} \cdots a_j}{2^{|a_k|+\cdots+|a_j|}}.$$ 

(122)

Summing up, we obtain

$$\| P(D)^j(\chi u) \|_{L^2_K} \leq \left( \| p(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K} \right) + \left( \frac{j^\gamma}{B^\gamma} \right) \| p(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K} \cdots + \left( \frac{j^\gamma}{B^\gamma} \right) \| p(\alpha_j) \cdots (P(\alpha_j) (D) u) \|_{L^2_K}.$$ 

(123)
If we use the multinomial theorem,
\[
\sum_{|\alpha| \leq m} \frac{m^{(\alpha)}}{\alpha!} \leq \sum_{|\alpha| = 0}^{\infty} \frac{m^{(\alpha)}}{\alpha!} \leq e^{n^n},
\]
(124)
where \( n \) is the dimension of the multi-index \(|\alpha|\) or |\(\alpha\)|. Then, it is clear that
\[
\sum_{|\alpha| \leq m} \frac{(2C/B)^m A^{m_j} m^{(\alpha_1 + \cdots + \alpha_j)}}{\alpha_1! \cdots \alpha_j!} \leq E^l
\]
(125)
for some constant \( E > 0 \) that depends on \( P(D) \), \( \chi \), and the compact set \( K(r/2) \).

Now, we control the sequence \((j(j-1) \cdots (j-k+1))\) for \( k = 1, \ldots, j \), which is the factor of \( \|P^{j-k} u\|_{2,K(r/2)} \) and less than or equal to
\[
\binom{j}{k} (k!)^w \leq 2^{jw} k!^w.
\]
(126)

For \( * = (\omega) \), since \( \omega(t^w) = o(t^{1/w}) \) as \( t \to +\infty \), there is a constant \( F > 0 \) such that
\[
(k!)^w \leq Fe^{-w}(k!), \quad k \in \mathbb{N}.
\]

Since \( \varphi^*(x)/x \to \infty \) as \( t \to \infty \), for any constant \( h \in \mathbb{N} \),
\[
(k!)^w \leq Fe^{1/h}(kh) \leq Fe^{1/h}(kh(hm))
\]
(128)
on the other hand, since \( u \in \mathcal{B}_0^{(\omega)}(\Omega) \), there are constants \( G > 0 \) and \( h \in \mathbb{N} \) that depend on \( K(r/2) \) such that
\[
\|P^{j-k} u\|_{2,K(r/2)} \leq Ge^{1/h}(kh(hm)), \quad k = 0, 1, \ldots, j, \ j \in \mathbb{N}.
\]
(129)

Then, from the convexity of \( \varphi^* \),
\[
\|P(D)^j(\chi u)\|_{2,K} \leq E^{jw} \left( \|P^{j} u\|_{2,K(r/2)} + Fe^{1/h}(mh)\|P^{j-1} u\|_{2,K(r/2)} \right.
\]
\[
+ Fe^{1/h}(2mh)\|P^{j-2} u\|_{2,K(r/2)} + \cdots + Fe^{1/h}(jn h)\|P^{j-n} u\|_{2,K(r/2)}
\]
\[
\leq (j+1) 2^{jw} E^{jw} F Ge^{1/h}(mh)
\]
(130)

If \( * = (\omega) \), since \( \omega(t^w) = o(t^{1/w}) \) as \( t \to +\infty \) for every \( \ell \in \mathbb{N} \), there is \( D_\ell > 0 \) such that
\[
(k!)^w \leq D_\ell e^{\alpha_\ell}(k/\ell), \quad k \in \mathbb{N}.
\]
(131)

Moreover, if \( u \in \mathcal{E}_0^{(\omega)}(\Omega) \) for each \( \ell \in \mathbb{N} \), there is \( C_\ell > 0 \) such that
\[
\|P^{j-k} u\|_{2,K(r/2)} \leq C_\ell e^{\alpha_\ell}(j-k)/\ell), \quad k = 0, 1, \ldots, j, \ j \in \mathbb{N}.
\]
(132)

Now, we can proceed as in the Roumieu case to obtain
\[
\|P(D)^j(\chi u)\|_{2,K(r/2)} \leq (j+1) 2^{jw} E^{jw} C D_\ell e^{\alpha_\ell}(j/\ell), \quad j \in \mathbb{N},
\]
(133)
which concludes the proof.

Let us recall that, by Proposition 9 and Theorem 13 if \( \omega \) is a nonquasianalytic weight and \( P(D) \) is elliptic, then
\[
WF_{\omega} P u = WF_{\omega} u \quad \forall u \in \mathcal{D}',
\]
(134)
for * being equal to \( \{\omega\} \) or \( \omega \). Let us then assume \( P(D) \) is not elliptic and prove the following result, which generalizes Theorems 8.1.4 and 8.4.14 of [25].

**Theorem 18.** Let \( \omega \) be a nonquasianalytic weight function such that \( \omega(t^w) = o(\sigma(t)) \) as \( t \to +\infty \), where \( \sigma(t) = t^{1/b} \) is a Gevrey weight function, with \( s > 1 \) and \( b = \max(s, 3/2) \), with \( s \) the constant in (28). Let \( P(D) \) be a linear partial differential operator with constant coefficients which is nonhyperbolic but not elliptic. Given an open subset \( \Omega \) of \( \mathbb{R}^n \) and a closed conic subset \( S \) of \( \Omega \times (\mathbb{R}^n \setminus \{0\}) \), then there is a distribution \( u \in \mathcal{D}'(\Omega) \) with \( \theta \neq WF_{\omega} P u \subset S \) in particular, if \( S = \{(x_0, t\xi_0), t > 0\} \) for some \( x_0 \in \Omega \) and \( \xi_0 \in \mathbb{R}^n \) with \( |\xi_0| = 1 \), we have \( WF_\omega P u = S \).

**Proof.** Let us first remark that it is sufficient to prove the statement when \( \Omega = \mathbb{R}^n \).

Moreover, since \( P \) is hypoelliptic but not elliptic, we can find \( \delta > 0 \) and \( 0 < d < m \) such that
\[
|P(\xi)| \geq \delta |\xi|^d,
\]
(135)
for \( \xi \) big enough. Choose a sequence \((x_\ell, \theta_\ell) \in S\) with \(|\theta_\ell| = 1\) so that every \((x, \theta) \in S\) with \(|\theta| = 1\) is the limit of a subsequence.

Let us now set \( \sigma(t) := \omega(t^{3/2}) \) and separate Beurling and Roumieu cases.

**Roumieu Case.** Take \( \phi \in \mathcal{D}_\omega(\mathbb{R}^n) \) with \( \hat{\phi}(0) = 1 \).

Then, there exist \( c > 0 \) and \( h \in \mathbb{N} \) such that
\[
|\hat{\phi}(\xi)| \leq ce^{-\frac{1}{h}(\alpha_\xi)} \quad \forall \xi \in \mathbb{R}^n.
\]
(136)
Since \( \log t = o(\sigma(t)) \) as \( t \to +\infty \), by definition of weight function, by Lemma 1.7 of [13], there exists a weight function \( \alpha \) such that \( \log t = o(\sigma(t)) \) and \( \alpha(t) = o(\sigma(t)) \) for \( t \to +\infty \).

Note that for every \( \ell \in \mathbb{N} \), there is \( k_\ell \in \mathbb{N} \) such that
\[
\exp \left\{ -\frac{\sigma(k^{d/m})}{\alpha(k^{d/m})} \log k \right\} < k^{-\ell} \quad \forall k \geq k_\ell
\]
(137)
and define then
\[
u(x) = \sum_{k=1}^{+\infty} e^{-\sigma(k^{d/m})/(\alpha(k^{d/m}))} \log k \phi(k (x - x_k)) e^{ik(x, \theta_k)}
\]
(138)
This is a continuous function in \( \mathbb{R}^n \) and we will prove that \( \theta \neq WF_{\omega} P u \subset S \).
To prove first that $\text{WF}_{\omega}^P u \subset S$, we take $(x_0, \xi_0) \not\in S$ and prove that $(x_0, \xi_0) \not\in \text{WF}_{\omega}^P u$. To this aim, we choose an open neighborhood $U$ of $x_0$ and an open conic neighborhood $\Gamma$ of $\xi_0$ such that
\[(U \times \Gamma) \cap S = \emptyset.\] (139)

Write $u = u_1 + u_2$, where $u_1$ is the sum of terms in $U$ with $x_\xi \notin U$ and $u_2$ is the sum of terms with $x_\xi \in U$.

Therefore, there is a neighborhood $U_1$ of $x_0$ with $\overline{U_1} \subset U$ such that $u_1$ is in $\mathcal{E}'(\omega(U_1))$ since all but a finite number of terms vanish in $U_1$. Moreover, every weight function $\omega$ is increasing by definition, so that $\omega \leq \sigma$, $\mathcal{E}'(\sigma) \subset \mathcal{E}'(\omega)$ and hence $u_1 \in \mathcal{E}'(\omega(U_1))$.

Consider then
\[f_N = P(D)^N u_2(x) = \sum_{x_\xi \in U} e^{-\sigma|k^m|/a(k^m)\log k}P(D)^N \times \left[ \phi(k(x-x_\xi)) e^{k^l(x_\xi \theta_\xi)} \right].\] (140)

Note that it is a totally convergent series since
\[\sup_{x_\xi \in U} \left| P(D)^N \left[ \phi(k(x-x_\xi)) e^{k^l(x_\xi \theta_\xi)} \right] \right| \leq C_N k^{mN}\] (141)

for some $C_N > 0$ and because of (137) with $l \geq 3mN + 2$.

Let us then compute the Fourier transform
\[\hat{f}_N(\xi) = \sum_{x_\xi \in U} e^{-\sigma|k^m|/a(k^m)\log k}P(\xi)^N \times \mathcal{F} \left[ \phi(k(x-x_\xi)) e^{k^l(x_\xi \theta_\xi)} \right] = \sum_{x_\xi \in U} e^{-\sigma|k^m|/a(k^m)\log k}k^{-n}P(\xi)^N \times \hat{\phi}(\xi - k^l \theta_\xi) e^{i(x_\xi k^l \theta_\xi - \xi)}\] (142)

with $\theta_\xi \notin \Gamma$ because of (139).

If $\Gamma_1$ is a conic neighborhood of $\xi_0$ with $\overline{\Gamma_1} \subset \Gamma \cup \{0\}$, then $|\xi - \eta| \geq c_0(|\xi| + |\eta|)$ when $\xi \in \Gamma_1$ and $\eta \notin \Gamma$, for some $c_0 > 0$, since this is true when $|\xi| + |\eta| = 1$. Thus,
\[|\xi - k^l \theta_\xi| \geq c_0 \left( |\xi| + k^l \right) \geq c_0 \frac{1}{3} (|\xi| + |\xi| + k^l) \geq c_0 \sqrt{|\xi| \cdot |\xi| \cdot k^l} = c_0 |\xi|^{1/3} k, \quad \xi \in \Gamma_1.\] (143)

It follows from (136) that
\[|\phi\left( \frac{\xi - k^l \theta_\xi}{k^l} \right)| \leq c \exp \left\{ -\frac{1}{h} \left( \frac{\xi - k^l \theta_\xi}{k^l} \right) \right\} \leq ce^{-\left| (1/h)\sigma(\xi^2/k) \right|} \leq c'e^{-\left| (1/h)\omega(\xi) \right|}, \quad \xi \in \Gamma_1,\] (144)

for some $c' > 0$, since $\omega(2t) \leq L(\omega(t) + 1)$ for some $L > 0$ by definition of weight function. Therefore, by (142) and Lemma 16(i), if we fix $\ell \notin \mathbb{N}$, for $\xi \in \Gamma_1$, $|\xi| \geq 1$,
\[(1 + |\xi|)^f |f_N(\xi)| \leq (1 + |\xi|)^f \sum_{x_\xi \in U} e^{-\sigma|k^m|/a(k^m)\log k}k^{-n}P(\xi)^N \times |P(\xi)|^\ell ce^{-\left| (1/h)\sigma(\xi^2/k) \right|} \leq c'' e^{-\left| (1/h)\omega(\xi) \right|},\] (145)

for some $c'' > 0$. Now, from the convexity of $\varphi^*$, it follows easily that condition (ii) of Definition 7 is satisfied. But also condition (ii)(a) of Definition 7 is satisfied
\[|\hat{f}_N(\xi)| \leq \sum_{x_\xi \in U} e^{-\sigma|k^m|/a(k^m)\log k}k^{-n}P(\xi)^N \times k^{-|\xi|^m}|P(\xi)|^\ell ce^{-\left| (1/h)\sigma(\xi^2/k) \right|} \leq c' e^{-\left| (1/h)\varphi^*(mn+h) \right|}, \quad \xi \in \mathbb{R}^n,\] (146)

for some $c' > 0$. This, together with $u_1 \in \mathcal{E}'(\omega(U_1))$, proves that $(x_0, \xi_0) \not\in \text{WF}_{\omega}^P u \neq \emptyset$.

Let us now prove that $\text{WF}_{\omega}^P u \neq \emptyset$. Choose $\chi \in \mathcal{D}(\bar{\sigma}(\mathbb{R}^n))$ equal to 1 near $x_0 \in \Omega$, where $\bar{\sigma}$ is the Gevrey weight of the hypotheses. To prove that $\text{WF}_{\omega}^P u \neq \emptyset$, we proceed by contradiction and assume that the wave front set is empty. Then, $u \in \mathcal{E}'(\omega(U))(\Omega)$.

Set $\phi_k(k(x-x_\xi)) := \chi(x)\phi(k(x-x_\xi))$. (147)

By hypothesis $\sigma = o(\bar{\sigma})$ which implies in particular that $\mathcal{D}(\bar{\sigma}(\mathbb{R}^n)) \subset \mathcal{E}'(\sigma)(\mathbb{R}^n)$. Then, the sequence $\phi_k(y) = \chi(y/k + x_\xi)\phi(y)$ is a bounded set in $\mathcal{D}'(\sigma)(\mathbb{R}^n)$ and, in fact, the supports $\text{supp} \phi_k \subset \text{supp} \phi$ for all $k$. We can use [15, Proposition 3.4] to obtain constants $c, h > 0$ such that
\[|\phi_k(\xi)| \leq c e^{-\left( (1/h)\sigma(\xi) \right)}\] (148)

for all $j \in \mathbb{N}$ and all $\xi \in \mathbb{R}^n$.

The Fourier transform of $P(D)^N (\chi u)$ is a sum of the form (142) with $\phi$ replaced by $\phi_k$. We observe that
\[|k^l \theta_\xi - j^l \theta_\xi| \geq |k^l - j^l| \geq k^l + j^l \geq k_j, \quad \text{if} k \neq j,\] (149)
Moreover, for \( x_k \) close to \( x_0 \) and \( k \) large enough, the equality \( \phi_k = \phi \) is satisfied. Consequently, from (135), we have, for some \( c' > 0 \),

\[
\left| \mathcal{F} \left[ (P(D)^N (\chi u))^N \right] \left( k^3 \theta_k \right) \right| = \left| e^{-\sigma(k^{d/m}/\alpha(k^{d/m}))} \log k - \sum_{j \neq k} e^{-\sigma(j^{d/m}/\alpha(j^{d/m}))} \log j \right| N^{k_3^3} \theta_j \right| \\
\geq \left| P(k^3 \theta_k)^N \right| \left( e^{-\sigma(k^{d/m}/\alpha(k^{d/m}))} \log k - \sum_{j \neq k} e^{-\sigma(j^{d/m}/\alpha(j^{d/m}))} \log j \right)
\]

\[
\geq \delta \left| P(k^3 \theta_k)^N \right| \left( e^{-\sigma(k^{d/m}/\alpha(k^{d/m}))} \log k - c' e^{-1/(h \sigma(k))} \right)
\]

(150)

In fact, for \( k \) large enough

\[
1/h \sigma(k) \geq -\log \left( \frac{1}{2c} \right) + \frac{\sigma \left( k^{d/m} \right)}{\alpha \left( k^{d/m} \right)} \log k + n \log k,
\]

(151)
since, for \( k \to +\infty \), \( \sigma(k) \to +\infty \), \( \sigma(k^{d/m})/\sigma(k) \) is bounded (\( d < m \) in (135)), \( \log k = o(\sigma(k)) \), and \( \log k = o(\sigma(k)) \).

On the other hand, by Proposition 17, the product \( \chi u \in \mathcal{B}_f^p(\Omega) \). We obtain \( C > 0 \) and \( H' \in \mathbb{N} \) such that, for all \( \xi \in \mathbb{R}^n \),

\[
\left| \mathcal{F} \left( (P(D)^N (\chi u))^N \right) \left( \xi \right) \right| = \int e^{-\sigma(k^{d/m})} P(D)^N (\chi u) (x) \, dx \\
\leq D \left\| P(D)^N (\chi u) \right\|_{L^2, \text{supp} \chi} \\
\leq C D e^{(1/h') \gamma^* (Nm h')},
\]

(152)

where \( D > 0 \) is a constant that depends on the Lebesgue measure of \( \text{supp} \chi \). Consequently, from (150), we have

\[
\delta \left| P(k^3 \theta_k)^N \right| \left( e^{-\sigma(k^{d/m}/\alpha(k^{d/m}))} \log k - c' e^{-1/(h \sigma(k))} \right) \\
\leq \delta \left| P(k^3 \theta_k)^N \right| \left( e^{-\sigma(k^{d/m}/\alpha(k^{d/m}))} \log k - c' e^{-1/(h \sigma(k))} \right)
\]

for every \( N \in \mathbb{N} \) and \( k \).

Now, (153) implies, by Lemma 16(ii),

\[
e^{-\sigma(k^{d/m}/\alpha(k^{d/m}))} \log k \leq e^{-\sigma(k^{d/m}/\alpha(k^{d/m}))} \log k \\
\leq 2C \left| P(D)^N (\chi u)^N \right| \left( (\delta^{1/(m k^{d/m})})^{-N m} \right) e^{(1/h') \gamma^* (Nm h')} \}
\]

(154)

But for every fixed \( h' \), there is \( k \) large enough so that

\[
e^{-\sigma(k^{d/m}/\alpha(k^{d/m}))} \log k \leq 1 \left( \delta^{1/(m k^{d/m})} \right) - (n + 3d) \log k - \log (2Cd),
\]

since we can argue as in (151), which is a contradiction. Therefore, \( WF_{(\omega)^P} \neq 0 \).

\( Beurling Case. \) Take \( \phi \in \mathcal{D}(\omega) (\mathbb{R}^n) \) with \( \hat{\phi} (0) = 1 \).

For every \( h \in \mathbb{N} \), there exists then a constant \( c_h > 0 \) such that

\[
\left| \hat{\phi} (\xi) \right| \leq c_h e^{-h \sigma(\xi)} \quad \forall \xi \in \mathbb{R}^n.
\]

(156)

Note that for every fixed \( \ell \in \mathbb{N} \),

\[
\exp \{ -\sigma \left( k^{d/m} \right) \} = \exp \left\{ -\frac{\sigma \left( k^{d/m} \right)}{\log (k^{d/m})} \cdot \frac{m}{2d} \cdot \log k^{\ell} \right\} < k^{-\ell},
\]

(157)

for \( k \) large enough since \( \log k = o(\sigma(k)) \) as \( k \to +\infty \). Define then

\[
u (x) = \sum_{k=1}^{\infty} e^{-\sigma(k^{d/m})} \left( \phi \left( k \left( x - x_k \right) \right) \right) \left( k^3 \theta_k \right)
\]

(158)

This is a continuous function in \( \mathbb{R}^n \) and we will prove that \( \Theta \neq WF_{(\omega) P} \subset S \).

The proof of the inclusion \( WF_{(\omega) P} \subset S \) is similar to that in the Roumieu case. We take \( (x_0, \xi_0) \notin S \), choose an open neighborhood \( U \) of \( x_0 \) and an open conic neighborhood \( \Gamma \) of \( \xi_0 \) such that \( (U \times \Gamma) \cap S \neq \emptyset \), and write \( u = u_1 + u_2 \), where \( u_1 \) is the sum of terms in (158) with \( x_k \notin U \) and \( u_2 \) is the sum of terms with \( x_k \in U \).

We choose a neighborhood \( U_1 \) of \( x_0 \) with \( \overline{U} \subset U \) such that \( u_1 \) is in \( \mathcal{B}_f^p (U) \) and \( \mathcal{B}_f^p (U_1) \) since all but a finite number of terms vanish in \( U_1 \).

Then, we consider the totally convergent series (because of (157) with \( \ell \) large enough)

\[
F_N = P(D)^N u_2 (x) \\
= \sum_{x_k \in \mathcal{U}} e^{-\sigma(k^{d/m}) P(D)^N \left( \phi \left( k \left( x - x_k \right) \right) \right) \left( k^3 \theta_k \right)}
\]

(159)
and compute its Fourier transform
\[
\hat{f}_N(\xi) = \sum_{x_k \in U} e^{-\omega(kd/2m)}k^{-1}P(\xi)^N \phi \left( \frac{\xi - k^3 \theta_k}{k} \right) e^{(x_k,k^3 \theta_k^2)},
\]
with \( \theta_k \notin \Gamma \).

For a conic neighborhood \( \Gamma_1 \) of \( \xi_0 \) with \( \bar{\Gamma}_1 \subset \Gamma \cup \{0\} \), we have that (143) is satisfied and hence, from (156),
\[
\left| \hat{\phi} \left( \frac{\xi - k^3 \theta_k}{k} \right) \right| \leq c_{\phi} \exp \left\{ -h\omega \left( \frac{\xi - k^3 \theta_k}{k} \right) \right\}
\leq c_{\phi} e^{-h\omega(k)} e^{-\omega(k)}, \quad \xi \in \Gamma_1,
\]
for some \( c_{\phi} > 0 \), since \( \omega(2t) \leq L(\omega(t) + 1) \) for some \( L > 0 \). Now, we fix \( \varepsilon \in \mathbb{N} \). By Lemma 16(i),
\[
(1 + |\xi|)^{\ell} \left| \hat{f}_N(\xi) \right| \leq (1 + |\xi|)^{\ell} \sum_{x_k \in U} e^{-\omega(kd/2m)}k^{-1}|P(\xi)|^N c_{\phi} e^{-h\omega(k)}
\leq c_{\phi} e^{-\omega(kd/2m)(\varepsilon + c_{\phi} k)} e^{-\omega(k)} \leq c_{\phi} e^{h\varepsilon^2(2m+\varepsilon) / h}, \quad \xi \in \Gamma_1,
\]
for some \( c_{\phi} > 0 \). From the convexity of \( \phi^* \), we conclude that condition (iii)(b) of Definition 7 is satisfied. But also condition (iii)(a) of Definition 7 is satisfied
\[
\left| \hat{f}_N(\xi) \right| \leq \sum_{x_k \in U} e^{-\omega(kd/2m)}k^{-1}|P(\xi)|^N c_{\phi} \times e^{-h\omega(\xi - k^3 \theta_k)} \leq c_{\phi} e^{-\omega(kd/2m)(\varepsilon + c_{\phi} k)} e^{-\omega(k)} \leq c_{\phi} e^{h\varepsilon^2(2m+\varepsilon) / h}, \quad \xi \in \mathbb{R}^n,
\]
for some \( c_{\phi} > 0 \). This, together with \( u_1 \in \mathcal{B}_{1,\varepsilon}(U_1) \), proves that \( (x_0, \xi_0) \notin W^p_{(\omega)} \) and hence \( W^p_{(\omega)} \) is not equal to \( S \).

Let us prove now that \( W^p_{(\omega)} \) is not empty.

Choose \( \chi \in \mathcal{D}_{(\sigma)}(\mathbb{R}^n) \) equal to 1 near \( x_0 \). We proceed by contradiction and assume that \( W^p_{(\omega)} = \emptyset \). Then, \( u \in \mathcal{B}_{1,\varepsilon}(\Omega) \).

Set \( \phi_k(k(x - x_0)) := \chi(x) \phi(k(x - x_0)) \) as in the Roumieu case. Since \( \sigma = o(\sigma) \), \( \mathcal{D}_{(\sigma)}(\mathbb{R}^n) \subset \mathcal{D}_{(\sigma)}(\mathbb{R}^n) \) [15, Proposition 4.7] Then the sequence \( \{\phi_k\} \) is a bounded set in \( \mathcal{D}_{(\sigma)}(\mathbb{R}^n) \) and \( \text{supp} \phi_k \subset \text{supp} \phi \) for all \( k \), as in the Roumieu case. By [15, Proposition 3.4], for each \( h \in \mathbb{N} \), there is \( c_{\phi} > 0 \) such that for all \( j \in \mathbb{N} \) and \( \xi \in \mathbb{R}^n \),
\[
\left| \hat{\phi}_j(\xi) \right| \leq c_{\phi} e^{-\omega(h)}.
\]
If \( x_k \) is close to \( x_0 \) and \( k \) is large enough, then \( \phi_k = \phi \) and by (149), we have
\[
\left| \mathcal{F} \left( P(D)^N (\chi u) \right)(k^3 \theta_k) \right| = \left| e^{-\omega(k^3 \theta_k)}k^{-1}P(k^3 \theta_k)^N \sum_{j \neq k} e^{-\omega(j \theta_k)}j^{-1}P(k^3 \theta_j)^N \right|
\leq \left| \phi_j \left( k^3 \theta_k - j \theta_j \right) \right| e^{j(x_j,j^2 \theta_k^2)}
\geq \left| P(k^3 \theta_k) \right|^N \left( e^{-\omega(k^3 \theta_k)}k^{-1} - \sum_{j \neq k} e^{-\omega(j \theta_k)}j^{-1} \right)
\geq \delta N^k \frac{3}{k^3} e^{-\omega(k)}
\geq \delta N^k \frac{3}{k^3} \frac{1}{k^3} e^{-\omega(k)}. \tag{165}
\]

On the other hand, by Proposition 17, \( \chi u \in \mathcal{B}_{(\sigma)}(\Omega) \) and proceeding as in the Roumieu case, we obtain that for every \( h \in \mathbb{N} \), there would exist \( c_h > 0 \) such that
\[
\left| \mathcal{F} \left( P(D)^N (\chi u) \right)(k^3 \theta_k) \right| \leq c_h e^{h^2(1/h)} \forall k. \tag{166}
\]

But (166) and (165) give a contradiction since they imply, by Lemma 16(ii), that
\[
e^{-\omega(kd/2m)} = e^{-\omega(kd/2m)}
\leq 2C_h k^h \inf_{N \in \mathbb{N}} \left\{ \left( \sigma^{1/3} \right)^{1/2m} \right\} \frac{1}{e^{h^2(1/h)}} \leq 2C_h \delta k^h \frac{1}{e^{h^2(1/h)}} \tag{167}
\]
must hold for every \( h > 0 \) and \( k \) large enough.

However, since \( \omega(2t) \leq L(\omega(t) + 1) \) for some \( L > 0 \), there exists a constant \( c_1 > 0 \) such that
\[
\omega(\left(\frac{k^3}{k^2} \right)) \leq c_1 \omega(\left(\frac{1}{k^2} \right)), \quad \tag{168}
\]
contradicting (167) for \( k \) large enough. Then \( W^p_{(\omega)} = \emptyset. \]

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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