Research Article

Radius Constants for Functions with the Prescribed Coefficient Bounds

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For an analytic univalent function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in the unit disk, it is well-known that $|a_n| \leq n$ for $n \geq 2$. But the inequality $|a_n| \leq n$ does not imply the univalence of $f$. This motivated several authors to determine various radii constants associated with the analytic functions having prescribed coefficient bounds. In this paper, a survey of the related work is presented for analytic and harmonic mappings. In addition, we establish a coefficient inequality for sense-preserving harmonic functions to compute the bounds for the radius of univalence, radius of full starlikeness/convexity of order $\alpha$ ($0 \leq \alpha < 1$) for functions with prescribed coefficient bound on the analytic part.

1. Introduction

Let $A$ denote the class of all analytic functions $f$ defined in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ normalized by $f(0) = 0 = f'(0) - 1$. For functions $f$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1)

belonging to the subclass $\mathcal{S}$ of $A$ consisting of univalent functions, de Branges [1] proved the famous Bieberbach conjecture that $|a_n| \leq n$ for $n \geq 2$. However, the inequality $|a_n| \leq n$ for $n \geq 2$ does not imply that $f$ is univalent. A function $f$ given by (1) whose coefficients satisfy $|a_n| \leq n$ for $n \geq 2$ is necessarily analytic in $\mathbb{D}$ by the usual comparison test and hence a member of $\mathcal{S}$. But it need not be univalent. For example, the function

$$f(z) = z - 2z^2 - 3z^3 - \cdots = 2z - \frac{z}{(1-z)^2}$$

(2)

satisfies the inequality $|a_n| \leq n$ for $n \geq 2$ but its derivative vanishes inside $\mathbb{D}$ and so the function $f$ is not univalent in $\mathbb{D}$. It is therefore of interest to determine the largest subdisk $|z| < \rho < 1$ in which the functions $f$ satisfying the inequality $|a_n| \leq n$ are univalent. Motivated by this problem, various radii problems associated with analytic as well as harmonic functions having prescribed coefficient bounds have been studied and we present a brief review of the research on this topic. Recall that given two subsets $\mathcal{F}$ and $\mathcal{G}$ of $\mathcal{A}$, the $\mathcal{G}$-radius in $\mathcal{F}$ is the largest $R$ such that, for every $f \in \mathcal{F}$, $r^{-1} f(rz) \in \mathcal{G}$ for each $r \leq R$.

1.1. Analytic Case. Most of the classes in univalent function theory are characterized by the quantities $zf'(z)/f(z)$ or $1 + zf''(z)/f'(z)$ lying in a given domain in the right half-plane. For instance, the subclasses $\mathcal{S}^\ast(\alpha)$ and $\mathcal{K}(\alpha)$ ($0 \leq \alpha < 1$) of $\mathcal{S}$ consisting of starlike functions of order $\alpha$ and convex functions of order $\alpha$, respectively, are defined analytically by the equivalences

$$f \in \mathcal{S}^\ast(\alpha) \iff \Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha,$$

(3)

$$f \in \mathcal{K}(\alpha) \iff \Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha.$$
These classes were introduced by Robertson [2]. The classes $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{K} := \mathcal{K}(0)$ are the familiar classes of starlike and convex functions, respectively. Goodman [3] introduced the class $\mathcal{U} \mathcal{C} \mathcal{V} \mathcal{F}$ of uniformly convex functions $f \in \mathcal{A}$, which map every circular arc $\gamma$ contained in $\mathbb{D}$ with center $\xi \in \mathbb{D}$ onto a convex arc. For $f \in \mathcal{A}$, Renning [4] and Ma and Minda [5] independently proved that

$$f \in \mathcal{U} \mathcal{C} \mathcal{V} \mathcal{F} \iff \Re \left( \frac{zf''(z)}{f'(z)} + 1 \right) > \left| \frac{zf'(z)}{f(z)} \right| \quad (z \in \mathbb{D}).$$

(Close)\textmd{ly related to the class $\mathcal{U} \mathcal{C} \mathcal{V}$ is the class $\mathcal{S}_p$ of parabolic starlike functions, introduced by Rønning [4] consisting of functions $f = zg^2$ where $g \in \mathcal{U} \mathcal{C} \mathcal{V}$. That is, a function $f \in \mathcal{S}_p$ satisfies

$$\Re \left( \frac{zf'(z)}{f(z)} \right) > \left| \frac{zf'(z)}{f(z)} - 1 \right| \quad (z \in \mathbb{D}).$$

In 1970, Gavrilov [6] showed that the radius of univalence for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq n$ $(n \geq 2)$ is the real root $r_0 = 0.164$ of the equation $2(1-r)^3 - (1+r) = 0$. In 1982, Yamashita [7] showed that the radius of univalence obtained by Gavrilov [6] is also the radius of starlikeness for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq n$. Yamashita [7] also proved that the radius of convexity for functions $f \in \mathcal{A}$ satisfying $|a_n| \leq n$ $(n \geq 2)$ is the real root $r_0 = 0.090$ of the equation $1 - \alpha + 4r + (1 + \alpha)r^2 = 0$. Similarly, Reade [10] proved that a close-to-star function $f \in \mathcal{A}$ satisfies $|a_n| \leq n^2$ for $n \geq 2$. However, the converse in both the cases is not true, in general. Recently, Mendiratta et al. [11] obtained sharp radii of starlikeness of order $\alpha$ $(0 \leq \alpha < 1)$, convexity of order $\alpha$ $(0 \leq \alpha < 1)$, parabolic starlikeness and uniform convexity for the class $\mathcal{A}_b$ when $|a_n| \leq M/n^2$ or $|a_n| \leq Mn^2$ $(M > 0)$ for $n \geq 3$. All et al. [12] also worked in the similar direction and obtained similar radii constants.

1.2. Harmonic Case. In a simply connected domain $\Omega \subset \mathbb{C}$, a complex-valued harmonic function $f$ has the representation $f = h + \overline{g}$, where $h$ and $g$ are analytic in $\Omega$. We call the functions $h$ and $g$ the analytic and the coanalytic parts of $f$, respectively. Let $\mathcal{H}$ denote the class of all harmonic functions $f = h + \overline{g}$ in $\Omega$ normalized so that $h$ and $g$ take the form

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

$$g(z) = z + \sum_{n=1}^{\infty} b_n z^n.$$

Since the Jacobian of $f$ is given by $J_f = |h'|^2 - |g'|^2$, by a theorem of Lewy [13], $f$ is sense-preserving if and only if $|g'|^2 < |h'|^2$, or equivalently if $H(z) \neq 0$ and the second dilatation $w_f = g'/H$ satisfies $|w_f(z)| < 1$ in $\Delta$. Let $\mathcal{H}_p$ be the subclass of $\mathcal{H}$ consisting of sense-preserving functions. Then it is easy to see that $|H| < 1$ for functions in the class $\mathcal{H}_p$. Set $\mathcal{H}_s := \{ f \in \mathcal{H} : b_1 = 0 \}$ and $\mathcal{H}_s := \mathcal{H}_p \cap \mathcal{H}_s$. Finally, let $\mathcal{S}_H$ and $\mathcal{S}_H^0$ be subclasses of $\mathcal{H}_p$ and $\mathcal{H}_p$, respectively, consisting of univalent functions.

One of the important questions in the study of class $\mathcal{S}_H^0$ and its subclasses is related to coefficient bounds. In 1984,
Clunie and Sheil-Small [14] conjectured that the Taylor coefficients of the series of $h$ and $g$ satisfy the inequality
\[ |a_n| \leq \frac{n+1}{2}(2n+1)(n+1), \]
and it is still open. These researchers proposed this coefficient conjecture because the harmonic Koebe function $K = H + G$ where
\[ H(z) = \frac{z - (1/2)z^2 + (1/6)z^3}{(1 - z)^3} \]
\[ = z + \sum_{n=2}^{\infty} \frac{(n+1)(2n+1)z^n}{6n^2}, \]
\[ G(z) = \frac{(1/2)z^2 + (1/6)z^3}{(1 - z)^3} \]
\[ = \sum_{n=0}^{\infty} \frac{(n-1)(2n-1)z^n}{6n^2}, \]
is expected to play the extremal role in the class $\delta_H^0$. However, this conjecture is proved for all functions $f \in \delta_H^0$ with real coefficients and all functions $f \in \delta_H^0$ for which either $f(D)$ is starlike with respect to the origin, close-to-convex, or convex in one direction (see [14–16]).

If $f \in \delta_H^0$ for which $f(D)$ is convex, Clunie and Sheil-Small [14] proved that the Taylor coefficients of $h$ and $g$ satisfy the inequalities
\[ |a_n| \leq \frac{n+1}{2}, \quad |b_n| \leq \frac{n-1}{2}, \quad \forall n \geq 2, \] (12)
and equality occurs for the harmonic half-plane mapping
\[ L(z) = M(z) + N(z), \]
\[ M(z) := \frac{z - (1/2)z^2}{(1 - z)^2}, \]
\[ N(z) := \frac{- (1/2)z^2}{(1 - z)^2}. \] (13)

Let $\mathcal{H}_0$ and $\mathcal{H}_{\infty}$ be subclasses of $\delta_H^0$ consisting of functions $f$ for which $f(D)$ is convex and $f(D)$ is starlike with respect to origin, respectively. Recall that convexity and starlikeness are not hereditary properties for univalent harmonic mappings (see [17–19]). Chuaqui et al. [19] introduced the notion of fully starlike and fully convex harmonic functions that do inherit the properties of starlikeness and convexity, respectively. The last two authors [18] generalized this concept to fully starlike functions of order $\alpha$ and fully convex harmonic functions of order $\alpha$ for $0 \leq \alpha < 1$. Let $\mathcal{F}\delta_H^0(\alpha)$ and $\mathcal{F}\mathcal{H}_0(\alpha)$ ($0 \leq \alpha < 1$) be subclasses of $\delta_H^0$ consisting of fully starlike functions of order $\alpha$ and fully convex functions of order $\alpha$, with $\mathcal{F}\delta_H^0 := \mathcal{F}\delta_H^0(0)$ and $\mathcal{F}\mathcal{H}_0 := \mathcal{F}\mathcal{H}_0(0)$. The functions in the classes $\mathcal{F}\delta_H^0(\alpha)$ and $\mathcal{F}\mathcal{H}_0(\alpha)$ are characterized by the conditions $(\partial/\partial \theta) \arg f(re^{i\theta}) > \alpha$ and $(\partial/\partial \theta)(\arg((\partial/\partial \theta)f(re^{i\theta}))) > \alpha$ for every circle $|z| = r$, $z = re^{i\theta}$, respectively, where $0 \leq \theta < 2\pi$, $0 < r < 1$.

The radius of full convexity of the class $\mathcal{H}_0$ is $\sqrt{2} - 1$ while the radius of full starlikeness of functions $f = h + g$ whose coefficients satisfy the conditions (10) and (12). This, in turn, provides a bound for the radius of full starlikeness for convex and starlike mappings in $\delta_H^0$. These results are generalized in context of fully starlike and fully convex functions of order $\alpha$ ($0 \leq \alpha < 1$) in [18]. The authors [18] proved the following result.

**Theorem 2** (see [18]). Let $h$ and $g$ have the form (9) with $b_1 = g'(0) = 0$ and $0 \leq \alpha < 1$. Then we have the following.

(a) If the coefficients of the series satisfy the conditions (10), then $f = h + g$ is univalent and fully starlike of order $\alpha$ in the disk $|z| < r_0$, where $r_0 = r_0(\alpha)$ is the real root in $(0, 1)$ of the equation $2(1 - \alpha)(1 - r)^3 + \alpha(1 - r)^2 - (1 + r)^2 = 0$.

(b) If the coefficients of the series satisfy the conditions (12), then $f = h + g$ is univalent and fully starlike of order $\alpha$ in the disk $|z| < r_0$, where $r_0 = r_0(\alpha)$ is the real root in $(0, 1)$ of the equation $(2 - \alpha)(1 - r)^3 + \alpha(1 - r)^2 - 1 - r = 0$.

Moreover, the results are sharp for each $\alpha \in [0, 1)$.

**Theorem 2** gives the bounds for the radius of full starlikeness of order $\alpha$ ($0 \leq \alpha < 1$) for the classes $\delta_H^0$ and $\mathcal{H}_0$. In addition, the authors in [18] also determined the bounds for the radius of full convexity of order $\alpha$ ($0 < \alpha < 1$) for these classes.

The analytic part of harmonic mappings plays a vital role in shaping their geometric properties. For instance, if $f = h + g \in \mathcal{H}_0$ and $h$ is convex univalent, then $f \in \mathcal{H}_0$ and maps $D$ onto a close-to-convex domain (see [14, Theorem 5.17, p. 20]). However, if $f = h + g \in \mathcal{H}_0$ where $h$ and $g$ are given by (9) and $|a_n| \leq 1$ for $n \geq 2$, then $f$ need not be even univalent; for example, the function
\[ f(z) = z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots, \quad z \in D \] (14)
belongs to $\mathcal{H}_0$ but is not univalent in $D$ since $f(z_0) = f(z_0) = 3/4$ where $z_0 = (3 + \sqrt{3}i)/4 \in D$. Note that a convex univalent function $z + a_2z^2 + a_3z^3 + \cdots$ satisfies $|a_n| \leq 1$ for $n = 2, 3, \ldots$. This paper aims to determine the coefficient inequalities and radius constants for certain subfamilies of $\mathcal{H}_0$ with the prescribed coefficient bound on the analytic part.
A coefficient inequality for functions in the class $H^0_{sp}$ is obtained in Section 2 which, in particular, improves the coefficient inequality proved by Polatoğlu et al. [22] for perturbed harmonic mappings. Using this inequality, the bounds for the radius of univalence, full starlikeness, and full convexity of order $\alpha (0 \leq \alpha < 1)$ are obtained for functions $f = h + \overline{g} \in H^0_{sp}$ where the coefficients of the analytic part $h$ satisfy one of the conditions $|a_n| \leq n$, $|a_n| \leq 1$, or $|a_n| \leq 1/n$ for $n \geq 2$. In addition, we will also discuss a case under which these bounds can be improved.

In the third section, sharp bounds on $\beta$ (depending upon $\alpha$ and $|a_0|$) are determined for a function $f = h + \overline{g} \in H$, where $h$ and $g$ are given by (9), satisfying either of the following two conditions:

$$\sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n^2|b_n| \leq \beta \quad \text{or} \quad \sum_{n=2}^{\infty} n|a_n| + \sum_{n=1}^{\infty} n^2|b_n| \leq \beta,$$

(15)

to be either fully starlike of order $\alpha$ or fully convex of order $\alpha$.

The obtained results are applied to hypergeometric functions in Section 4.

2. A Coefficient Inequality and Radius Constants

Firstly, we will obtain a coefficient inequality for functions in the class $H_{sp}$.

Theorem 3. Let $f = h + \overline{g} \in H_{sp}$, where $h$ and $g$ are given by (9). Then

$$|b_n| \leq |b_1| |a_n| + \frac{(1 - |b_1|^2)^{n-1}}{n} \sum_{k=1}^{n-1} k|a_k|,$$

(16)

for $n \geq 2$, with $a_1 = 1$. In particular, one has

$$|b_n| \leq \frac{n!}{(n-k)!} k|a_k|, \quad n = 2, 3, \ldots$$

(17)

Proof. Since $f \in H_{sp}$, the function $w(z) = g'(z)/h'(z) = \sum_{n=0}^{\infty} w_n z^n$ is analytic in $D$ and $|w(z)| < 1$ in $D$. On equating the coefficients of $z^{n+1}$ in $g'(z) = w(z) h'(z)$, we obtain

$$nb_n = a_1 w_{n-1} + 2w_{n-2}a_2 + 3w_{n-3}a_3 + \cdots + (n-1)w_1 a_{n-1} + nw_0 a_n,$$

(18)

where $a_1 = 1$. Since $|w_n| \leq 1 - |w_0|^2$ (see [23, p. 172]), it immediately follows that

$$|b_n| \leq (1 - |w_0|^2) \sum_{k=1}^{n-1} k|a_k| + n|w_0| |a_n|, \quad (a_1 = 1).$$

(19)

Since $w_0 = g'(0)/h'(0) = b_1$, the desired result follows.

For specific choices of the analytic part $h$ in a harmonic function $f = h + \overline{g} \in H_{sp}$, Theorem 3 yields the following result.

**Corollary 4.** Let $f = h + \overline{g} \in H_{sp}$, where $h$ and $g$ are given by (9). Then we have the following.

(i) If $|a_n| \leq n$ or, in particular, $h$ is univalent, then $|b_n| \leq (2n + 1)(n + 1)/6, n = 2, 3, \ldots$

(ii) If $|a_n| \leq 1$ or, in particular, $h$ is convex univalent, then $|b_n| \leq (n + 1)/2, n = 2, 3, \ldots$

(iii) If $|a_n| \leq 1/n$ or, in particular, $\Re h'(z) > 0$, then $|b_n| \leq 1, n = 2, 3, \ldots$

**Remark 5.** Polatoğlu et al. [22] determined the coefficient inequality for harmonic functions in a subclass of $H_{sp}$, for which the analytic part is a univalent function in $D$. They proved that if $f = h + \overline{g} \in H_{sp}$, where $h$ and $g$ are given by (9) and if $h$ is univalent in $D$, then

$$|b_n| \leq \frac{1}{n} (2n^2 - n^2 - 4n - 6), \quad n = 1, 2, \ldots$$

(20)

It is evident that Corollary 4(i) improves this bound.

Now, we will determine the radius of univalence, radius of full starlikeness/full convexity of order $\alpha (0 \leq \alpha < 1)$ for the class $H_{sp}$ with specific choices of the coefficient bound on the analytic part. We will make use of the following sufficient coefficient conditions for a harmonic function to be in the classes $\mathcal{F}^*_{H}(\alpha)$ and $\mathcal{F} H_{H}(\alpha)$ ($0 \leq \alpha < 1$) that directly follow from the corresponding results in [24, 25].

**Lemma 6** (see [24, 25]). Let $f = h + \overline{g}$, where $h$ and $g$ are given by (9) and let $0 \leq \alpha < 1$. Then we have the following.

(i) If

$$\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| \leq 1,$$

then $f \in \mathcal{F}^*_{H}(\alpha)$.

(ii) If

$$\sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \leq 1,$$

then $f \in \mathcal{F} H_{H}(\alpha)$.

**Theorem 7.** Let $f = h + \overline{g} \in H^0_{sp}$, where $h$ and $g$ are given by (9) with $b_1 = g'(0) = 0$ and $0 \leq \alpha < 1$. Then we have the following.

(i) If $|a_n| \leq n$ or, in particular, $h$ is univalent and fully starlike of order $\alpha$ in the disk $|z| < r_1$ where $r_1 = r_1(\alpha)$ is the real root of the equation

$$12 (1 - \alpha) r^4 + (49\alpha - 48) r^3 + 8 (9 - 8\alpha) r^2 + 3 (11\alpha - 18) r + 6 (1 - \alpha) = 0$$

(23)

in the interval $(0, 1)$. 

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(ii) If $|a_n| \leq 1$ or, in particular, $h$ is convex univalent, then $f$ is univalent and fully starlike of order $\alpha$ in the disk $|z| < r_2$ where $r_2 = r_2(\alpha)$ is the real root of the equation

\[
4(1-\alpha)r^3 + (9\alpha - 12)r^2 + (12 - 7\alpha)r \\
- 2(1-\alpha) = 0
\tag{24}
\]

in the interval $(0,1)$.

(iii) If $|a_n| \leq 1/n$ or, in particular, $R\,h'(z) > 0$, then $f$ is univalent and fully starlike of order $\alpha$ in the disk $|z| < r_3$ where $r_3 = r_3(\alpha)$ is the real root of the equation

\[
2(1-\alpha)r^3 + (5\alpha - 4)r^2 + (1 - 3\alpha)r \\
- 2\alpha(1-r)^2 \log(1-r) = 0
\tag{25}
\]

in the interval $(0,1)$.

Proof. Since $f = h + \overline{g} \in \mathcal{R}_0$, we obtain

\[
|b_n| \leq \frac{1}{n} \sum_{k=1}^{n-1} |a_k|, \quad (n \geq 2; \; a_1 = 1),
\tag{26}
\]

by applying Theorem 3. We will make use of (26) to obtain the coefficient bounds for $b_n$ in three different cases specified in the theorem. For $r \in (0,1)$, let $f_r : \mathbb{D} \to \mathbb{C}$ be defined by

\[
f_r(z) := \frac{f(rz)}{r} = z + \sum_{n=2}^{\infty} a_n r^{n-1}z^n + \sum_{n=2}^{\infty} b_n r^{n-1}z^n.
\tag{27}
\]

We will show that $f_r \in \mathcal{F}_\alpha$. In view of Lemma 6(i), it suffices to show that the sum

\[
S = \sum_{n=2}^{\infty} n \frac{n-\alpha}{n-1} |a_n| r^{n-1} + \sum_{n=2}^{\infty} \frac{n+\alpha}{n-1} |b_n| r^{n-1}
\tag{28}
\]

is bounded above by 1 for $0 \leq r < r_i$ for $i = 1, 2, 3$.

(i) Since $|a_n| \leq n$, it is easy to deduce that $|b_n| \leq (n - 1)(2n - 1)/6$ by (26). Using these coefficient bounds in (28) and simplifying, we have

\[
S \leq \frac{1}{6(1-\alpha)} \left[ 2 \sum_{n=2}^{\infty} n^3 r^{n-1} + (3 + 2\alpha) \sum_{n=2}^{\infty} n^2 r^{n-1} \\
+ (1 - 9\alpha) \sum_{n=2}^{\infty} nr^{n-1} + \frac{\alpha r}{1-r} \right].
\tag{29}
\]

Thus $S \leq 1$ if $r$ satisfy the inequality

\[
2 \sum_{n=2}^{\infty} n^3 r^{n-1} + (2\alpha + 3) \sum_{n=2}^{\infty} n^2 r^{n-1} \\
+ (1 - 9\alpha) \sum_{n=2}^{\infty} nr^{n-1} + \frac{\alpha r}{1-r} \leq 6(1-\alpha).
\tag{30}
\]

By using the identities

\[
\frac{r}{1-r)^2} = \sum_{n=1}^{\infty} n^2 r^n,
\tag{31}
\]

the last inequality reduces to

\[
2 \left( r^2 + 4r + 1 \right) + (2\alpha + 3) \left( 1 - r^2 \right) \\
+ \frac{1 - 9\alpha}{1-r} + \frac{\alpha}{1-r} \leq 12(1-\alpha)
\tag{32}
\]

or equivalently

\[
2 \left( r^2 + 4r + 1 \right) + (2\alpha + 3) \left( 1 - r^2 \right) \\
+ (1 - 9\alpha) (1-r)^2 + \alpha(1-r)^3 \leq 12(1-\alpha) (1-r)^4.
\tag{33}
\]

This gives

\[
12(1-\alpha) r^4 + (49\alpha - 48) r^3 + 8(9 - 8\alpha) r^2 \\
+ 3(11\alpha - 18) r + 6(1-\alpha) \geq 0.
\tag{34}
\]

Thus by Lemma 6(i), $f_r \in \mathcal{F}_\alpha$ for $r \leq r_1$ where $r_1$ is the real root of (23) in $(0,1)$. In particular, $f$ is univalent and fully starlike of order $\alpha$ in $|z| < r_1$.

(ii) If $|a_n| \leq 1$ then (26) gives $|b_n| \leq (n - 1)/2$. These coefficient bounds lead to the following inequality for the sum (28):

\[
S \leq \frac{1}{2(1-\alpha)} \left[ \sum_{n=2}^{\infty} n^2 r^{n-1} + (1 + \alpha) \sum_{n=2}^{\infty} nr^{n-1} - \frac{3\alpha r}{1-r} \right].
\tag{35}
\]

Therefore it follows that $S \leq 1$ if $r$ satisfy the inequality

\[
\sum_{n=2}^{\infty} n^2 r^{n-1} + (1 + \alpha) \sum_{n=2}^{\infty} nr^{n-1} - \frac{3\alpha r}{1-r} \leq 2(1-\alpha).
\tag{36}
\]

Making use of identities (31) in the last inequality, we obtain

\[
\frac{1 + r}{(1-r)^2} + \frac{1 + \alpha}{(1-r)^2} - \frac{3\alpha}{1-r} \leq 4(1-\alpha),
\tag{37}
\]

which simplifies to

\[
2(1-\alpha) + (7\alpha - 12) r + (12 - 9\alpha) r^2 - 4(1-\alpha) r^3 \geq 0.
\tag{38}
\]
Lemma 6(i) shows that \( f_r \in \mathcal{F}\delta_H^r (\alpha) \) for \( r \leq r_2 \) where \( r_2 \) is the real root of (24) in (0, 1). In particular, \( f \) is univalent and fully starlike of order \( \alpha \) in \( |z| < r_2 \).

(iii) Using (26), it is easily seen that \( |b_0| \leq (n-1)/n \). Using the coefficient bounds for \( |a_n| \) and \( |b_n| \) in (28), it follows that

\[
S \leq \frac{1}{1-\alpha} \left[ \sum_{n=2}^{\infty} n r^{n-1} - 2 \alpha \sum_{n=2}^{\infty} \frac{1}{n} r^{n-1} + \frac{\alpha r}{1-r} \right].
\]

The sum \( S \leq 1 \) if \( r \) satisfy the inequality

\[
\sum_{n=2}^{\infty} n r^{n-1} - 2 \alpha \sum_{n=2}^{\infty} \frac{1}{n} r^{n-1} + \frac{\alpha r}{1-r} \leq 1 - \alpha.
\]

Using (31) and the identity \(-\log(1-r) = r + r^2/2 + r^3/3 + \cdots\), the last inequality reduces to

\[
\frac{1}{(1-r)^2} + \frac{2\alpha}{r} \log(1-r) + \frac{\alpha}{1-r} \leq 2(1-\alpha),
\]

which is equivalent to

\[
2(1-\alpha)^3 + (5\alpha - 4) r^2 + (1-3\alpha) r
- 2\alpha(1-r)^2 \log(1-r) \geq 0.
\]

By applying Lemma 6(i), \( f_r \in \mathcal{F}\delta_H^r (\alpha) \) for \( r \leq r_3 \) where \( r_3 \) is the real root of (25) in (0, 1). In particular, \( f \) is univalent and fully starlike of order \( \alpha \) in \( |z| < r_3 \). This completes the proof of the theorem.

Remark 8. By (26), it follows that \( |b_1| \leq 1/2 \) for all functions \( f \in \mathcal{H}_\mathcal{F}^0 \). The bound 1/2 is sharp for the function \( f_0(z) = z + 2z^2/2 \in \mathcal{H}_\mathcal{F}^0 \). Since \( f_0 \) is univalent in \( \mathbb{D} \), the coefficient inequality \( |b_0| \leq 1/2 \) remains sharp in the subclass \( \delta_H^0 \). Clunie and Sheil-Small [14] were the first to observe the sharp inequality \( |b_1| \leq 1/2 \) for functions in the class \( \delta_H^0 \).

Remark 9. Let \( f = h + \overline{g} \in \mathcal{H}_\mathcal{F}^0 \), where \( h \) and \( g \) are given by (9). In the proof of part (i) of Theorem 7, we noticed that if \( |a_n| \leq n \) then \( |b_n| \leq (n-1)/(2n-1)/6 \). The bound for \( |b_n| \) coincides with conjectured bound for \( |b_n| \) when \( f \in \delta_H^0 \) proposed by Clunie and Sheil-Small [14].

The next theorem calculates the radius of full convexity of order \( \alpha \) \((0 \leq \alpha < 1)\) for the class \( \mathcal{H}_\mathcal{F}^0 \) under certain choices of the coefficient bound on the analytic part.

Theorem 10. Let \( f = h + \overline{g} \in \mathcal{H}_\mathcal{F}^0 \), where \( h \) and \( g \) are given by (9) with \( b_1 = g'(0) = 0 \) and \( 0 \leq \alpha < 1 \). Then we have the following.

(a) If \( |a_n| \leq n \) or, in particular, \( h \) is univalent, then \( f \) is fully convex of order \( \alpha \) in the disk \( |z| < s_1 \) where \( s_1 = s_1(\alpha) \) is the real root of the equation

\[
2(1-\alpha)^2 - 10(1-\alpha)r^4 + 2(10 - 11\alpha) r^3 + (7\alpha - 6) r^2 + (15 - 8\alpha) r - (1-\alpha) = 0
\]

in the interval (0, 1).

(b) If \( |a_n| \leq 1 \) or, in particular, \( h \) is convex univalent, then \( f \) is fully convex of order \( \alpha \) in the disk \( |z| < s_2 \) where \( s_2 = s_2(\alpha) \) is the real root of the equation

\[
2(1-\alpha)^4 - 8(1-\alpha)r^3 + 2(6 - 5\alpha) r^2 - 5(1-\alpha) r + (1-\alpha) = 0
\]

in the interval (0, 1).

(c) If \( |a_n| \leq 1/n \) or, in particular, \( \text{Re}(h'(z)) > 0 \), then \( f \) is fully convex of order \( \alpha \) in the disk \( |z| < s_3 \) where \( s_3 = s_3(\alpha) \) is the real root of the equation

\[
2(1-\alpha)r^3 - 2(3-2\alpha)r^2 + (7 - 3\alpha) r - (1-\alpha) = 0
\]

in the interval (0, 1).

Proof. Following the method of the proof of Theorem 7, it suffices to show that the function \( f_r \) defined by (27) belongs to \( \mathcal{H}_\mathcal{F}^r (\alpha) \). Using the coefficient bounds \( |a_n| \leq n \) and \( |b_n| \leq (n-1)/(2n-1)/6 \), we deduce that

\[
S' = \sum_{n=2}^{\infty} \frac{n}{1-\alpha} |a_n| r^{n-1} + \sum_{n=2}^{\infty} \frac{n}{1-\alpha} |b_n| r^{n-1}
\leq \frac{1}{6(1-\alpha)} \left[ \sum_{n=2}^{\infty} n r^{n-1} + (2\alpha + 3) \sum_{n=2}^{\infty} n^2 r^{n-1} + (1-9\alpha) \sum_{n=2}^{\infty} n^3 r^{n-1} + \alpha \sum_{n=2}^{\infty} n r^{n-1} \right].
\]

According to Lemma 6(ii), we need to show that \( S' \leq 1 \), or equivalently

\[
2 \sum_{n=2}^{\infty} n r^{n-1} + (2\alpha + 3) \sum_{n=2}^{\infty} n^3 r^{n-1} + (1-9\alpha) \sum_{n=2}^{\infty} n^2 r^{n-1} + \alpha \sum_{n=2}^{\infty} n r^{n-1} \leq 6(1-\alpha).
\]

Using (31) and the identity \( \sum_{n=1}^{\infty} n^r r^n = r(1+r)(1+10r + r^3)/(1-r)^3 \), the last inequality reduces to

\[
(1-\alpha) + (8\alpha - 15) r + (18 - 21\alpha) r^2 + (22\alpha - 20) r^3 + 10(1-\alpha)r^4 - 2(1-\alpha)r^5 \geq 0.
\]

Lemma 6(ii) shows that \( f_r \in \mathcal{H}_\mathcal{F}^r (\alpha) \) for \( r \leq s_1 \) where \( s_1 \) is the real root of (43) in (0, 1). In particular, \( f \) is fully convex of order \( \alpha \) in \( |z| < s_1 \). This proves (a). The proof of (b) and (c) follows on similar lines.

The sharpness of the radii constants for the class \( \mathcal{H}_\mathcal{F}^0 \) obtained in Theorems 7 and 10 is still unresolved. However, these constants can be shown to be sharp for certain subclasses of \( \mathcal{H}_\mathcal{F}^0 \) as seen by the following theorem.
Theorem 11. Let \( A_n, B_n \geq 0 \) \((n = 2, 3, \ldots)\) and let \( \mathcal{F} \) be a family of harmonic functions \( f = h + g \in \mathcal{F}^h \) where \( h \) and \( g \), given by (9) with \( b_i = g_i(0) = 0 \), satisfy \( |a_n| \leq A_n \) and \( |b_n| \leq B_n \) for \( n = 2, 3, \ldots \). Furthermore, if \( r(\partial \varphi / \partial \theta), r(\partial \varphi / \partial \theta^\alpha) \), \( \mathcal{F} \), and \( r(\partial \varphi / \partial \theta^\alpha) \) denote, respectively, the radii of univalence, full starlikeness of order \( \alpha \) \((0 \leq \alpha < 1)\), and full convexity of order \( \alpha \) \((0 \leq \alpha < 1)\) in \( \mathcal{F} \), then we have the following.

1. If \( A_n = n \) and \( B_n = (n - 1)(2n - 1)/6 \), then
   \[ r(\partial \varphi / \partial \theta) = r(\partial \varphi / \partial \theta^\alpha) = r_1(\alpha) \]
   \[ r(\partial \varphi / \partial \theta^\alpha) = r_2(\alpha) \]
   where \( r_1(\alpha) \) and \( s_1(\alpha) \) are the real roots of \((23)\) and \((43)\), respectively, in \((0, 1)\).

2. If \( A_n = 1 \) and \( B_n = (n - 1)/2 \), then
   \[ r(\partial \varphi / \partial \theta) = r_3(0) = 1 - 1/2^{1/3} = 0.292893 \]
   \[ r(\partial \varphi / \partial \theta^\alpha) = r_2(\alpha) \]
   where \( r_2(\alpha) \) is the real root of \((24)\) and \((44)\), respectively, in \((0, 1)\).

3. If \( A_n = 1/n \) and \( B_n = (n - 1)/n \), then
   \[ r(\partial \varphi / \partial \theta) = r_3(0) = 1 - 1/\sqrt{2} = 0.292893 \]
   \[ r(\partial \varphi / \partial \theta^\alpha) = r_2(\alpha) \]
   where \( r_2(\alpha) \) is the real root of \((25)\) and \((45)\), respectively, in \((0, 1)\).

Proof. Note that the roots of \((23)\) in \((0, 1)\) are decreasing as functions of \( \alpha \in [0, 1) \). Consequently, \( r_1(\alpha) \leq r_1(0) \). A similar remark holds for \((24), (25), (43)-(45)\). This observation together with Theorems 7 and 10 gives \( r(\partial \varphi / \partial \theta) \geq r_1(0), r(\partial \varphi / \partial \theta^\alpha) \geq r_1(\alpha) \), and \( r(\partial \varphi / \partial \theta^\alpha) \geq s_1(\alpha) \) for \( i = 1, 2, 3 \) in the respective three cases specified in the theorem. Therefore it is enough to show that these radii constants are best possible.

1. For sharpness of the numbers \( r_1(\alpha) \), let \( f_5 : \mathbb{D} \to \mathbb{C} \) be defined by
   \[ f_5(z) = 2z - \frac{z}{1 - z} + \frac{3z^2 + z^3}{6(1 - z)^3} \]
   \[ = z - \sum_{n=2}^{\infty} n z^n + \frac{1}{6} \sum_{n=2}^{\infty} (n - 1)(2n - 1) z^n. \]
   As \( f_5 \) has real coefficients, for \( r \in (0, 1) \), the Jacobian of \( f_5 \) takes the form
   \[ f_5'(r) = \frac{(1 - 7r + 14r^2 - 8r^3 + 2r^4)(1 - 9r + 12r^2 - 8r^3 + 2r^4)}{(1 - r)^8}. \]
   Since \( f_5'(r_1(0)) = 0 \) the function \( f_5 \) is not univalent in \(|z| < r \) if \( r > r_1(0) \). Also, since
   \[ \frac{\partial}{\partial \theta} \arg f_5(re^{i\theta}) \bigg|_{r = \alpha} = \frac{6(1 - 9r_1 + 12r_1^2 - 8r_1^3 + 2r_1^4)}{6(1 - 9r_1 + 12r_1^2 - 8r_1^3 + 2r_1^4)} = \alpha \]
   it follows that \( f_5 \) is not fully starlike of order \( \alpha \) in \(|z| < r \) if \( r > r_1 \), where \( r_1 = r_1(\alpha) \) is the real root of \((23)\) in \((0, 1)\).

For sharpness of the numbers \( s_1(\alpha) \), consider the function
   \[ f_C(z) = 2z - \frac{z}{1 - z} - \frac{3z^2 + z^3}{6(1 - z)^3} \]
   \[ = z - \sum_{n=2}^{\infty} n z^n - \frac{1}{6} \sum_{n=2}^{\infty} (n - 1)(2n - 1) z^n, \]
   and observe that
   \[ \frac{\partial}{\partial \theta} \left( \arg f_C(re^{i\theta}) \right) \bigg|_{r = 0, r = r_2} = \frac{1 - 15s_1 + 18s_1^2 - 20s_1^3 + 10s_1^4 - 2s_1^5}{(1 - s_1)(1 - 7s_1 + 14s_1^2 - 8s_1^3 + 2s_1^4)} = \alpha. \]
   This shows that \( f_C \) is not fully convex of order \( \alpha \) in \(|z| < r \) if \( r > s_1 \), where \( s_1 = s_1(\alpha) \) is the real root of \((43)\) in \((0, 1)\).

2. The Jacobian of the function \( f_5 : \mathbb{D} \to \mathbb{C} \) defined by
   \[ f_5(z) = 2z - \frac{z}{1 - z} + \frac{z^2}{2(1 - z)^2} \]
   \[ = z - \sum_{n=2}^{\infty} n z^n + \frac{1}{2} \sum_{n=2}^{\infty} (n - 1) z^n \]
   vanishes at \( r = r_2(0) \) and
   \[ \frac{\partial}{\partial \theta} \arg f_5(re^{i\theta}) \bigg|_{r = 0, r = r_2} = \frac{2(1 - 6r_2 + 6r_2^2 - 2r_2^3)}{2 - 7r_2 + 9r_2^2 - 4r_2^3} = \alpha. \]

These two observations imply that the numbers \( r_2(\alpha) \) are sharp, where \( r_2 = r_2(\alpha) \) is the real root of \((24)\) in \((0, 1)\). For sharpness of the constants \( s_2(\alpha) \), observe that the function
   \[ f_C(z) = 2z - \frac{z}{1 - z} + \frac{z^2}{2(1 - z)^2} \]
   \[ = z - \sum_{n=2}^{\infty} n z^n - \frac{1}{2} \sum_{n=2}^{\infty} (n - 1) z^n \]
   satisfies
   \[ \frac{\partial}{\partial \theta} \left( \arg f_C(re^{i\theta}) \right) \bigg|_{r = 0, r = r_2} = \frac{1 - 10s_2 + 12s_2^2 - 8s_2^3 + 2s_2^4}{1 - 5s_2 + 10s_2^2 - 8s_2^3 + 2s_2^4} = \alpha, \]
   where \( s_2 = s_2(\alpha) \) is the real root of \((44)\) in \((0, 1)\).

3. The function \( f_5 : \mathbb{D} \to \mathbb{C} \) defined by
   \[ f_5(z) = 2z + \log(1 - z) + \frac{z}{1 - z} + \log(1 - z) \]
   \[ = z - \sum_{n=2}^{\infty} n z^n + \frac{1}{n} \sum_{n=2}^{\infty} n z^n \]
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satisfies $f(r_3(0)) = 0$ and

$$\frac{\partial}{\partial \theta} \arg f_S(re^{i\theta}) \bigg|_{\theta = 0, r = r_3} = \frac{r_3 (1 - 4r_3 + 2r_3^2)}{(1 - r_3) [r_3 (3 - 2r_3) + 2 (1 - r_3) \log (1 - r_3)]} = \alpha,$$  

(59)

where $r_3 = r_3(\alpha)$ is the real root of (25) in $(0, 1)$. If $s_3 = s_3(\alpha)$ is the real root of (45) in $(0, 1)$, then

$$\frac{\partial}{\partial \theta} \left( \arg \left( \frac{\partial}{\partial \theta} f_C(re^{i\theta}) \right) \right) \bigg|_{\theta = 0, r = s_3} = \frac{1 - 7s_3 + 6s_3^2 - 2s_3^3}{1 - 3s_3 + 4s_3^2 - 2s_3^3} = \alpha,$$  

(60)

where $f_C : \mathbb{D} \to \mathbb{C}$ is defined by

$$f_C(z) = 2z + \log (1 - z) - \left( \frac{z}{1 - z} + \log (1 - z) \right) = z - \sum_{n=2}^{\infty} \frac{n-1}{n} z^n. $$  

(61)

Now, we will discuss a particular case under which the results of Theorems 7 and 10 can be further improved.

**Theorem 12.** Let $f = h + \overline{g} \in \mathcal{H}_K^0$, where $h$ and $g$ are given by (9) with $b_1 = g'(0) = 0$. Further, suppose that the dilatation $w(z) = g'(z)/h'(z) = z$ for all $z \in \mathbb{D}$. Then we have the following:

(i) If $|a_n| \leq n$ or, in particular, $h$ is univalent, then $f$ is univalent and fully starlike in the disk $|z| < R_1$ where $R_1 = 0.135918$ is the real root of the equation $2r^3 - 5r^2 + 8r - 1 = 0$ in the interval $(0, 1)$. Moreover, $f$ is fully convex in $|z| < S_1$ where $S_1 = 0.073935$ in the real root of the equation $2r^3 - 8r^2 + 7r^2 - 14r + 1 = 0$ in the interval $(0, 1)$.

(ii) If $|a_n| \leq 1$ or, in particular, $h$ is convex univalent, then $f$ is univalent and fully starlike in the disk $|z| < R_2$ where $R_2 = (5 - \sqrt{17})/4 = 0.219224$. Also, $f$ is fully convex in $|z| < S_2$ where $S_2 = 0.120385$ in the real root of the equation $2r^3 - 6r^2 + 9r - 1 = 0$ in the interval $(0, 1)$.

(iii) If $|a_n| \leq 1/n$ or, in particular, $\text{Re} h'(z) > 0$, then $f$ is univalent and fully starlike in the disk $|z| < R_3$ where $R_3 = 1/3 = 0.333333$. And $f$ is fully convex in $|z| < S_3$ where $S_3 = (3 - \sqrt{6})/3 = 0.183503$.

**Proof.** Setting $w_1 = 1$ and $w_n = 0$ for $n \neq 1$ in (18), we obtain

$$|b_n| \leq \frac{n-1}{n} |a_{n-1}| \quad (n \geq 2; \ a_1 = 1).$$  

(62)

Let $f_r$ be defined by (27). For the proof of (i), note that since $|a_n| \leq n$, it is easily seen that $|b_n| \leq (n-1)^2/n$ using (62). Using these coefficient bounds, we have

$$S = \sum_{n=2}^{\infty} n |a_n| r^{n-1} + \sum_{n=2}^{\infty} n |b_n| r^{n-1}$$

$$\leq \sum_{n=2}^{\infty} n^2 r^{n-1} + \sum_{n=2}^{\infty} n(n-1)^2 r^{n-1}$$

$$= \frac{(1 + r)^2}{(1 - r)^3} - 1,$$

using the identities (31). Thus $S \leq 1$ if $r$ satisfy the inequality $2(1 - r)^3 \geq (1 + r)^2$ or $1 - 8r + 5r^2 - 2r^3 \geq 0$. By Lemma 6(i), it follows that $f_r \in \mathcal{H}_R^s$ for $r \leq R_1$ where $R_1$ is the real root of $2r^3 - 5r^2 + 8r - 1 = 0$ in $(0, 1)$. In particular, $f$ is univalent and fully starlike in $|z| < R_1$. For full convexity, observe that

$$S' = \sum_{n=2}^{\infty} n^2 |a_n| r^{n-1} + \sum_{n=2}^{\infty} n(n-1)^2 r^{n-1}$$

$$= 5r^2 + 6r + 1$$

$$= \frac{(1 - r)^3}{(1 - r)^3} - 1.$$  

The sum $S' \leq 1$ if $r$ satisfy the inequality $2r^4 - 8r^3 + 7r^2 - 14r + 1 \geq 0$. Thus Lemma 6(ii) shows that $f_r \in \mathcal{H}_R^s$ for $r \leq S_1$ where $S_1$ is the real root of $2r^3 - 8r^2 + 7r^2 - 14r + 1 = 0$ in $(0, 1)$. In particular, $f$ is fully convex in $|z| < S_1$. This proves (i). The other two parts of the theorem are similar and hence their proofs are omitted.

**Remark 13.** Observe that $r_i(0) < R_i$ $(i = 1, 2, 3)$ and $s_i(0) < S_i$ $(i = 1, 2, 3)$. Here $r_i(0), s_i(0), R_i,$ and $S_i$ are as defined in Theorems 7, 10, and 12.

**Remark 14.** If $f = h + \overline{g} \in \mathcal{H}_K^0$, where $h$ and $g$ are given by (9) with $b_1 = g'(0) = 0$ and the dilatation $w(z) = g'(z)/h'(z)$ is given by $w(z) = z^m$ $(m \geq 1)$, then

$$|b_n| \leq \frac{n-m}{n} |a_{n-m}| \quad (n \geq m + 1; \ a_1 = 1),$$

(63)

by setting $w_m = 1$ and $w_n = 0$ for $n \neq m$ in (18). Radius constants may be obtained in this case by carrying out a similar calculation as in the proof of Theorem 12.

3. **Sufficient Coefficient Estimates for Full Starlikeness and Convexity**

In this section, we determine sufficient coefficient inequalities for functions to be in the classes $\mathcal{F}_S^R(\alpha)$ and $\mathcal{F}_H^R(\alpha)$. As an application, these results are applied to hypergeometric functions in Section 4.
Theorem 15. Let \( f = h + \bar{g} \in \mathcal{H} \), where \( h \) and \( g \) are given by (9). Suppose that \( \lambda \in (0, 1) \). Then one has the following.

(a) If
\[
\sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \leq \lambda, \tag{66}
\]
then \( f \) is fully starlike of order \( 2(1 - \lambda)/(2 + |b_1| + \lambda) \).

(b) If
\[
\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| \leq \lambda, \tag{67}
\]
then \( f \) is fully starlike of order \( 2(2 - \lambda - |b_1|)/(4 + 3|b_1| + \lambda) \). Moreover, \( f \) is fully convex of order \( 2(1 - \lambda)/(2 + |b_1| + \lambda) \).

The results are sharp.

Proof. If we set \( \alpha_0 = 2(1 - \lambda)/(2 + |b_1| + \lambda) \) then \( \alpha_0 \in [0, 1) \) and
\[
\sum_{n=2}^{\infty} \frac{n-\alpha_0}{1-\alpha_0} |a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha_0}{1-\alpha_0} |b_n|
\leq \sum_{n=2}^{\infty} \frac{n+\alpha_0}{1-\alpha_0} |a_n| + \sum_{n=1}^{\infty} \frac{n+\alpha_0}{1-\alpha_0} |b_n|
= \frac{1}{1-\alpha_0} \left( \sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \right)
+ \frac{\alpha_0}{1-\alpha_0} \left( \sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \right)
\leq \frac{\lambda}{1-\alpha_0} + \frac{\alpha_0}{1-\alpha_0} \left( \frac{\sum_{n=2}^{\infty} n |a_n| + |b_n|}{1-\alpha_0} \right)
\leq \frac{\lambda}{1-\alpha_0} + \frac{\alpha_0}{1-\alpha_0} \left( \frac{|b_1|}{2} + |b_1| \right)
= 2\lambda + \alpha_0 \left( \frac{\lambda + |b_1|}{2} \right)
= \frac{2\lambda + \alpha_0 (\lambda + |b_1|)}{2 (1 - \alpha_0)} = 1.
\]
By Lemma 6(i), it follows that \( f \) is fully starlike of order \( 2(1 - \lambda)/(2 + |b_1| + \lambda) \). The harmonic function
\[
f_1(z) = z + |b_1| \bar{z} + \frac{\lambda - |b_1|}{2} \bar{z}^2, \quad |b_1| < \lambda, \tag{69}
\]
satisfies the coefficient inequality (66). Further, for \( z = re^{i\theta} \), we have
\[
\frac{\partial}{\partial \theta} \arg f_1(re^{i\theta}) = \text{Re} \left( \frac{2 \left( z - |b_1| \bar{z} - (\lambda - |b_1|) \bar{z}^2 \right)}{2 \left( z + |b_1| \bar{z} + (\lambda - |b_1|) \bar{z}^2 \right)} \right)
\geq \frac{2 \left( 1 - |b_1| - (\lambda - |b_1|) |z| \right)}{2 \left( 1 + |b_1| + (\lambda - |b_1|) |z| \right)}
> \frac{2 (1 - \lambda)}{2 |b_1| + \lambda}
\]
which shows that the bound for the order of full starlikeness is sharp. This proves (a).

For the proof of (b), observe that
\[
\sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \leq \frac{1}{2} \sum_{n=2}^{\infty} n^2 (|a_n| + |b_n|) + |b_1|
\leq \frac{1}{2} \left( \lambda - |b_1| \right) + |b_1| \tag{71}
= \frac{\lambda + |b_1|}{2} := \mu_0 \text{ (say)},
\]
using (67). Since \( \mu_0 \in (0, 1) \), \( f \) is fully starlike of order \( 2(1 - \mu_0)/(2 + |b_1| + \mu_0) = 2(2 - \lambda - |b_1|)/(4 + 3|b_1| + \lambda) \) by part (a) of the theorem. For the order of full convexity of \( f \), note that
\[
\sum_{n=2}^{\infty} n (n - \alpha_0) |a_n| + \sum_{n=1}^{\infty} n (n + \alpha_0) |b_n|
\leq \sum_{n=2}^{\infty} n (n - \alpha_0) |a_n| + \sum_{n=1}^{\infty} n (n + \alpha_0) |b_n|
= \frac{1}{1-\alpha_0} \left( \sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| \right)
+ \frac{\alpha_0}{1-\alpha_0} \left( \sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \right)
\leq \frac{\lambda}{1-\alpha_0} + \frac{\alpha_0}{1-\alpha_0} \left( \frac{\lambda + |b_1|}{2} \right)
= 2\lambda + \alpha_0 \left( \frac{\lambda + |b_1|}{2} \right)
= \frac{2\lambda + \alpha_0 (\lambda + |b_1|)}{2 (1 - \alpha_0)} = 1,
\]
where \( \alpha_0 \) is as defined in the proof of part (a) of the theorem. By Lemma 6(ii), \( f \) is fully convex of order \( 2(1 - \lambda)/(2 + |b_1| + \lambda) \). In this case, the harmonic function
\[
f_2(z) = z + |b_1| \bar{z} + \frac{\lambda - |b_1|}{4} \bar{z}^2, \quad |b_1| < \lambda, \tag{73}
\]
shows that the result is best possible.

If \( b_1 = 0 \), then Theorem 15 reduces to [26, Theorem 3.6 and Corollary 3.7]. Also, Theorem 15 gives the following two corollaries.

Corollary 16. Let \( f = h + \bar{g} \in \mathcal{H} \), where \( h \) and \( g \) are given by (9) and \( 0 \leq \alpha < 2/(2 + |b_1|) \). Then we have the following.

(i) If
\[
\sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \leq \frac{2 - (2 + |b_1|) \alpha}{2 + \alpha}, \tag{74}
\]
then \( f \in \mathcal{F} \mathcal{D}^x_{\mathcal{H}}(\alpha) \).

(ii) If
\[
\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=1}^{\infty} n^2 |b_n| \leq \frac{2 - (2 + |b_1|) \alpha}{2 + \alpha}, \tag{75}
\]
then \( f \in \mathcal{F} \mathcal{H}^x_{\mathcal{H}}(\alpha) \).

All these results are sharp.
Proof. First, we will prove (i). Setting \( \lambda_0 = \frac{(2 - (2 + |b_1|)\alpha)}{(2 + \alpha)} \) we see that \( \lambda_0 \in (0, 1) \) and the coefficient inequality \( (66) \) is satisfied for \( \lambda = \lambda_0 \). Hence by Theorem 15(a), \( f \) is fully starlike of order \( 2(1 - \lambda_0)/(2 + |b_1| + \lambda_0) = \alpha \). This proves (i). For part (ii), since inequality \( (67) \) is satisfied for \( \lambda = \lambda_0 \) it follows that \( f \) is fully convex of order \( 2(1 - \lambda_0)/(2 + |b_1| + \lambda_0) = \alpha \) by Theorem 15(b). The functions

\[
f_1(z) = z + |b_1| \frac{1}{z} + \frac{(1 - |b_1|) - (1 + |b_1|)\alpha}{2 + \alpha} z^2,
\]

\[
f_2(z) = z + |b_1| \frac{1}{z} + \frac{(1 - |b_1|) - (1 + |b_1|)\alpha}{2 + \alpha} z^2,
\]

show that the upper bound \( 2(2 - (2 + |b_1|)\alpha)/(2 + \alpha) \) is best possible in (i) and (ii), respectively.

**Corollary 17.** Let \( f = h + g \in \mathcal{K} \), where \( h \) and \( g \) are given by \( (9) \) and \( \alpha \in \mathbb{R} \) satisfies

\[
2 \frac{(1 - |b_1|)}{5 + 3 |b_1|} \leq \alpha < 2 \frac{(2 - |b_1|)}{4 + 3 |b_1|}.
\]

If

\[
\sum_{n=2}^{\infty} n^2 |a_n| + \sum_{n=2}^{\infty} n^2 |b_n| \leq 2 \frac{(2 - |b_1|) - \alpha(4 + 3 |b_1|)}{2 + \alpha},
\]

then \( f \in \mathcal{F}^{*}_{H}(\alpha) \). The function

\[
f_0(z) = z + |b_1| \frac{1}{z} + \frac{(1 - |b_1|) - (1 + |b_1|)\alpha}{2 + \alpha} z^2
\]

shows that the bound \( 2(2 - |b_1|)(\alpha(4 + 3 |b_1|))/(2 + \alpha) \) is best possible.

Proof. If we set \( v_0 = (2(2 - |b_1|) - \alpha(4 + 3 |b_1|))/(2 + \alpha) \), then \( v_0 \in (0, 1) \) and the coefficient inequality \( (67) \) is satisfied for \( \lambda = v_0 \) using the hypothesis. By Theorem 15(b), \( f \) is fully starlike of order \( 2(2 - v_0 - |b_1|)/(4 + 3 |b_1| + v_0) = \alpha \) as desired.

If \( b_1 = 0 \), then Corollaries 16 and 17 reduce to the following theorem.

**Theorem 18.** Let \( f = h + \overline{g} \in \mathcal{K} \), where \( h \) and \( g \) are given by \( (9) \) with \( b_1 = g'(0) = 0 \) and let \( \alpha \in \mathbb{R} \).

(1) If \( \alpha \in (0, 1) \), then the sharp implications hold:

\[
\sum_{n=2}^{\infty} n |a_n| + |b_n| \leq \frac{2(1 - \alpha)}{2 + \alpha} \implies f \in \mathcal{F}^{*}_{H}(\alpha),
\]

\[
\sum_{n=2}^{\infty} n^2 |a_n| + |b_n| \leq \frac{2(1 - \alpha)}{2 + \alpha} \implies f \in \mathcal{F}^{*}_{H}(\alpha).
\]

(2) If \( \alpha \in [2/5, 1) \), then

\[
\sum_{n=2}^{\infty} n^2 |a_n| + |b_n| \leq \frac{4(1 - \alpha)}{2 + \alpha} \implies f \in \mathcal{F}^{*}_{H}(\alpha).
\]

In particular, we have

\[
\sum_{n=2}^{\infty} n^2 |a_n| + |b_n| \leq 1 \implies f \in \mathcal{F}^{*}_{H}(\frac{2}{5}).
\]

**4. Interplay between Hypergeometric Functions and Full Starlikeness and Convexity**

In recent years, there has been a growth of interest in the interplay between hypergeometric functions and harmonic mappings in \( \mathbb{D} \); see [27–31]. Let \( F(\beta, \gamma, \delta; z) \) be the Gaussian hypergeometric function defined by

\[
F(\beta, \gamma, \delta; z) := \sum_{n=0}^{\infty} \left( \frac{\beta}{\delta} \right)_n \frac{\gamma^n}{\delta^n} z^n, \quad z \in \mathbb{D},
\]

where \( \beta, \gamma, \delta \) are complex numbers with \( \delta \neq 0, -1, -2, \ldots \), and \( (\theta)_n \) is the Pochhammer symbol: \( (\theta)_0 = 1 \) and \( (\theta)_n = \theta(\theta + 1) \cdots (\theta + n - 1) \) for \( n = 1, 2, \ldots \). Since the hypergeometric series in \( (83) \) converges absolutely in \( \mathbb{D} \), it follows that \( F(\beta, \gamma, \delta; z) \) defines an analytic function in \( \mathbb{D} \) and plays an important role in the theory of univalent functions.

The first author and Silverman [27] initiated the study of harmonic functions \( \phi_1 \) and \( \phi_2 \) where \( \phi_1(z) \equiv \phi_1(\beta_1, \gamma_1, \delta_1; z) \) and \( \phi_2(z) \equiv \phi_2(\beta_2, \gamma_2, \delta_2; z) \) are the hypergeometric functions defined by

\[
\phi_1(z) := z F(\beta_1, \gamma_1, \delta_1; z), \quad \phi_2(z) := F(\beta_2, \gamma_2, \delta_2; z) - 1.
\]

Making use of Corollaries 16 and 17, we determine the sufficient conditions in terms of hypergeometric inequalities for the function \( \Phi = \phi_1 + \phi_2 \) to be in the classes \( \mathcal{F}^{*}_{H}(\alpha) \) and \( \mathcal{F}^{*}_{H}(\alpha) \). However, we first need the well-known Gauss summation formula

\[
F(\beta, \gamma, \delta; z) := \frac{\Gamma(\delta) \Gamma(\delta - \beta - \gamma)}{\Gamma(\delta - \beta) \Gamma(\delta - \gamma)}, \quad \text{Re}(\delta - \beta - \gamma) > 0
\]

and the following result by the first author [29].

**Lemma 19.** If \( \beta, \gamma, \delta > 0 \), then

(i) \( F(\beta + k, \gamma + k, \delta + k; 1) = ((\delta)_k/(\delta - \beta - \gamma - k)_{k+1})F(\beta, \gamma, \delta; 1) \) for \( k = 0, 1, 2, \ldots \), if \( \delta > \beta + \gamma + k \);

(ii) \( \sum_{n=2}^{\infty} (n - 1)((\beta)_n - (\gamma)_n)(\delta)_{n-1} (\delta - 1)_{n-1} = (\beta/\gamma - \beta - \gamma - 1)F(\beta, \gamma, \delta; 1) \) if \( \delta > \beta + \gamma + 1 \);

(iii) \( \sum_{n=2}^{\infty} (n - 1)^2((\beta)_n - (\gamma)_n)(\delta)_{n-1} (\delta - 1)_{n-1} = (\beta/\gamma - \beta - \gamma - 1)F(\beta, \gamma, \delta; 1) \) if \( \delta > \beta + \gamma + 2 \).

**Theorem 20.** Let \( \beta_j, \gamma_j \in \mathbb{C} \), and \( \delta_j \in \mathbb{R} \) satisfy \( \delta_j > |\beta_j| + |\gamma_j| + 1 \) for \( j = 1, 2 \). Set \( \eta = \beta_2/\delta_2 \) and let \( 0 \leq \alpha < 2/(2 + |\eta|) \). If

\[
\left( \frac{|\beta_1||\gamma_1|}{|\delta_1| - |\beta_1| - |\gamma_1| - 1} + 1 \right) F(|\beta_1|, |\gamma_1|, \delta_1; 1)
\]

\[
+ \frac{|\beta_1||\gamma_1|}{|\delta_2| - |\beta_2| - |\gamma_2| - 1} F(|\beta_2|, |\gamma_2|, \delta_2; 1)
\]

\[
\leq 4 \frac{1 + |\eta|}{2 + \alpha}
\]

then \( \Phi = \phi_1 + \phi_2 \in \mathcal{F}^{*}_{H}(\alpha) \), where \( \phi_1 \) and \( \phi_2 \) are given by \( (84) \).
Proof. Observe that
\[
\Phi(z) = z + \sum_{n=2}^{\infty} \left( \frac{(\beta_1)_{n-1}(\gamma_1)_{n-1} z^n}{(\delta_1)_{n-1}(1)_{n-1}} + \frac{(\beta_2)_{n}(\gamma_2)_{n}}{(\delta_2)_{n}(1)_{n}} \right) z^n. \tag{87}
\]
Using the fact \(|(\theta)_{n}| \leq |(\beta)_{n}|\), Gauss summation formula given by (85), and Lemma 19, we have
\[
\sum_{n=2}^{\infty} \left| \frac{(\beta_1)_{n-1}(\gamma_1)_{n-1}}{(\delta_1)_{n-1}(1)_{n-1}} \right| + \sum_{n=1}^{\infty} \left| \frac{(\beta_2)_{n}(\gamma_2)_{n}}{(\delta_2)_{n}(1)_{n}} \right|
\leq \sum_{n=2}^{\infty} \frac{|(\beta_1)|_{n-1}(|(\gamma_1)|_{n-1}}{|(\delta_1)|_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{|(\beta_2)|_{n}(|(\gamma_2)|_{n}}{|(\delta_2)|_{n}(1)_{n}}
= \sum_{n=1}^{\infty} (n+1) \left| \frac{|(\beta_1)|_{1}(|(\gamma_1)|_{1}}{|(\delta_1)|_{1}} + \sum_{n=1}^{\infty} \frac{|(\beta_2)|_{n}(|(\gamma_2)|_{n}}{|(\delta_2)|_{n}} \right.
\left. = \left( \frac{|(\beta_1)|_{1}}{|(\delta_1)|_{1}} |(\gamma_1)|_{1} + 1 \right) F(|(\beta_1)|_{1}, |(\gamma_1)|_{1}, |(\delta_1)|_{1}; 1) \right.
\left. + \frac{|(\beta_2)|_{1}}{|(\delta_2)|_{1}} |(\gamma_2)|_{1} \right) F\left(|(\beta_2)|_{1}, |(\gamma_2)|_{1}, |(\delta_2)|_{1}; 1 \right) - 1
\leq \frac{4 - (1 + |\eta|) \alpha}{2 + \alpha} - 1 = \frac{2 - (2 + |\eta|) \alpha}{2 + \alpha}. \tag{88}
\]
By Corollary 16(i), \(\Phi \in \mathcal{F}\delta_{H}^*(\alpha)\).

**Theorem 21.** Let \(\beta_j, \gamma_j \in \mathbb{C}\), and \(\delta_j \in \mathbb{R}\) satisfy \(\delta_j > |\beta_j| + |\gamma_j| + 2\) for \(j = 1, 2\). Set \(\eta = \beta_2 \gamma_2 / \delta_2\). Then one has the following.

(i) If \(0 \leq \alpha < 2(2 + |\eta|)\)
\[
\frac{|(\beta_1)|_{2}(|(\gamma_1)|_{2}}{(\delta_1 - |(\beta_1)| - |(\gamma_1)| - 2)} + \frac{3 |(\beta_1)|}{|(\delta_1)| - |(\gamma_1)| - 1} + 1 \right)
\times F\left(|(\beta_1)|_{1}, |(\gamma_1)|_{1}, |(\delta_1)|_{1}; 1 \right)
\left. + \frac{|(\beta_2)|_{2}(|(\gamma_2)|_{2}}{(\delta_2 - |(\beta_2)| - |(\gamma_2)| - 2)} + \frac{|(\beta_2)|}{|(\delta_2)| - |(\gamma_2)| - 1} \right) \tag{89}
\times F\left(|(\beta_2)|_{1}, |(\gamma_2)|_{1}, |(\delta_2)|_{1}; 1 \right)
\leq \frac{4 - (1 + |\eta|) \alpha}{2 + \alpha}
\]
then \(\Phi = \phi_1 + \phi_2 \in \mathcal{F}\delta_{H}^*(\alpha)\).

(ii) If \(2(1 - |\eta|)/(5 + 3 |\eta|) \leq \alpha < 2(2 - |\eta|)/(4 + 3 |\eta|)\)
\[
\frac{|(\beta_1)|_{2}(|(\gamma_1)|_{2}}{(\delta_1 - |(\beta_1)| - |(\gamma_1)| - 2)} + \frac{3 |(\beta_1)|}{|(\delta_1)| - |(\gamma_1)| - 1} + 1 \right)
\times F\left(|(\beta_1)|_{1}, |(\gamma_1)|_{1}, |(\delta_1)|_{1}; 1 \right)
\left. + \frac{|(\beta_2)|_{2}(|(\gamma_2)|_{2}}{(\delta_2 - |(\beta_2)| - |(\gamma_2)| - 2)} + \frac{|(\beta_2)|}{|(\delta_2)| - |(\gamma_2)| - 1} \right) \tag{90}
\times F\left(|(\beta_2)|_{1}, |(\gamma_2)|_{1}, |(\delta_2)|_{1}; 1 \right)
\leq \frac{3(2 - \alpha) - |\eta| (2 + 3 \alpha)}{2 + \alpha}
\]
then \(\Phi = \phi_1 + \phi_2 \in \mathcal{F}\delta_{H}^*(\alpha)\), where \(\phi_1\) and \(\phi_2\) are given by (84).

**Proof.** Note that
\[
\sum_{n=2}^{\infty} \frac{|(\beta_1)|_{n-1}(|(\gamma_1)|_{n-1}}{|(\delta_1)|_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{|(\beta_2)|_{n}(|(\gamma_2)|_{n}}{|(\delta_2)|_{n}(1)_{n}}
\leq \sum_{n=2}^{\infty} \frac{3 |(\beta_1)|_{n-1}(|(\gamma_1)|_{n-1}}{|(\delta_1)|_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{|(\beta_1)|}{|(\delta_1)| - |(\gamma_1)| - 1} + 1 \right)
\times F\left(|(\beta_1)|_{1}, |(\gamma_1)|_{1}, |(\delta_1)|_{1}; 1 \right)
\left. + \frac{|(\beta_2)|_{n}(|(\gamma_2)|_{n}}{|(\delta_2)|_{n}(1)_{n}} \right)
\leq \frac{3(2 - \alpha) - |\eta| (2 + 3 \alpha)}{2 + \alpha} \tag{91}
\]
Under the hypothesis of part (i), it is easy to see that
\[
\sum_{n=2}^{\infty} \frac{|(\beta_1)|_{n-1}(|(\gamma_1)|_{n-1}}{|(\delta_1)|_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{|(\beta_1)|}{|(\delta_1)| - |(\gamma_1)| - 1} - 1 \right)
\leq \frac{2 - (2 + |\eta|) \alpha}{2 + \alpha} \tag{92}
\]
By Corollary 16(ii), it follows that \(\Phi = \phi_1 + \phi_2 \in \mathcal{F}\delta_{H}^*(\alpha)\). Hypothesis of part (ii) shows
\[
\sum_{n=2}^{\infty} \frac{|(\beta_1)|_{n-1}(|(\gamma_1)|_{n-1}}{|(\delta_1)|_{n-1}(1)_{n-1}} + \sum_{n=1}^{\infty} \frac{|(\beta_1)|}{|(\delta_1)| - |(\gamma_1)| - 1} + 1 \right)
\times F\left(|(\beta_1)|_{1}, |(\gamma_1)|_{1}, |(\delta_1)|_{1}; 1 \right)
\leq \frac{3(2 - \alpha) - |\eta| (2 + 3 \alpha)}{2 + \alpha} \tag{93}
\]
Hence \(\Phi \in \mathcal{F}\delta_{H}^*(\alpha)\) by Corollary 17.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.
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