Karhunen-Loève Expansion for the Second Order Detrended Brownian Motion

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1. Introduction

Let \( X = \{ X(t), 0 \leq t \leq 1 \} \) be a centered and continuous Gaussian process on \([0, 1]\) with covariance function

\[
K_X(t, s) = EX(t)X(s).
\]

The Karhunen-Loève expansion of \( X \) is given by the (convergent in mean squares) series

\[
X(t) = \sum_{k=1}^{\infty} \eta_k \sqrt{\lambda_k} f_k(t),
\]

where \( \{ \eta_k, k \geq 1 \} \) is a sequence of i.i.d. \( N(0, 1) \) random variables and \( \{ \lambda_k, k \geq 1 \} \) is at most the countable set of eigenvalues of Fredholm integral operator

\[
T_Xf(t) = \int_0^1 K_X(t, s)f(s)ds
\]

and forms an orthogonal sequence in \( L^2[0, 1] \) and

\[\int_0^1 K_X(t, t)dt < \infty.\]

Deheuvels et al. in [1–4] provided the Karhunen-Loève expansions for the processes that are related with Brownian motion. The Karhunen-Loève expansion for detrended Brownian motion has been studied by Ai et al. [5]. Note that the detrended Brownian motion in [5] can be viewed as projection to a constant function subspace in \( L^2[0, 1] \). That is,

\[
\int_0^1 \tilde{W}_1(t)^2 dt = \min_{c_1, c_2} \int_0^1 (W(t) - c_1 - c_2 t)^2 dt.
\]

To generalize the projection idea into nonlinear detrended process, now we consider

\[
\min_{c_1, c_2, c_3} \int_0^1 (W(t) - c_1 - c_2 t - c_3 t^2)^2 dt
\]

and the optimal constant \( c_j \) satisfy

\[
\frac{\partial}{\partial c_j} \int_0^1 (W(t) - c_1 - c_2 t - c_3 t^2)^2 dt = 0, \quad j = 1, 2, 3.
\]

It is easy to obtain

\[
c_1 = 9 \int_0^1 W(s) ds - 36 \int_0^1 W(s)s ds + 30 \int_0^1 W(s)s^2 ds,
\]

\[
c_2 = -36 \int_0^1 W(s) ds + 192 \int_0^1 W(s)s ds.
\]
\[-180 \int_0^1 W(s) s^2 ds,
\]
\[c_3 = 30 \int_0^1 W(s) ds - 180 \int_0^1 W(s) s ds + 180 \int_0^1 W(s) s^2 ds. \quad (7)\]

Let
\[A = (a_{ij})_{3 \times 3} = \begin{pmatrix} 9 & -36 & 30 \\ -36 & 192 & -180 \\ 30 & -180 & 180 \end{pmatrix}; \quad (8)\]

we have
\[c_j = \sum_{i=1}^3 a_{ij} \int_0^1 s^{j-1} W(s) ds, \quad j = 1, 2, 3. \quad (9)\]

Now we can define the second order detrended process
\[\tilde{W}_2(t) = W(t) - \sum_{j=1}^3 c_j t^{j-1} \]
\[= W(t) + \left( -9 + 36t - 30t^2 \right) \int_0^1 W(s) ds \]
\[+ \left( 36 - 192t + 180t^2 \right) \int_0^1 W(s) s ds \]
\[+ \left( -30 + 180t - 180t^2 \right) \int_0^1 W(s) s^2 ds. \quad (10)\]

2. Main Results

We give the following lemma that provides the explicit covariance function.

**Lemma 1.** For convenience, we add $K_X(s,t)$ into formula (11), that is
\[K_X(s,t) = E(\tilde{W}_2(t) \tilde{W}_2(s)) \]
\[= t \wedge s - \sum_{p,q=1}^3 a_{pq} \left( \frac{t}{p} - \frac{t^{p+1}}{p(p+1)} \right) s^{q-1} \]
\[\sum_{i,j=1}^3 a_{ij} \frac{s^{j+1}}{i(i+1)} t^{i-1}, \]
\[+ \sum_{p,q=1}^3 \sum_{i,j=1}^3 a_{ij} a_{pq} \frac{p + i + 2}{(p + i + 1)(i + 1)} t^{j-1} s^{q-1}, \quad (11)\]

where $a_{ij}, a_{pq}, i, j, p, q = 1, 2, 3$ is given in (8).

**Proof.** Consider
\[\tilde{W}_2(t) = W(t) - \sum_{j=1}^3 c_j t^{j-1}, \quad 0 \leq t \leq 1 \quad (12)\]
and $\tilde{W}_2(t)$ is a mean zero Gaussian process; we obtain
\[E(\tilde{W}_2(t) \tilde{W}_2(s)) \]
\[= E\left( W(t) - \sum_{j=1}^3 c_j t^{j-1} \right) \left( W(s) - \sum_{q=1}^3 c_q s^{q-1} \right) \]
\[= E\left( W(t) - \sum_{i,j=1}^3 a_{ij} \left( \int_0^1 u^{i-1} W(u) du \right) t^{j-1} \right) \]
\[\cdot E\left( W(s) - \sum_{p,q=1}^3 a_{pq} \left( \int_0^1 v^{p-1} W(v) dv \right) s^{q-1} \right). \quad (13)\]

We notice that
\[E(W(t)W(s)) = t \wedge s, \quad (14)\]
\[E\left( \int_0^1 v^{p-1} W(v) dv \right) \int_0^1 W(t) W(v) dv \]
\[= E\left( \int_0^1 W(t) W(v) v^{p-1} dv \right) \]
\[= \int_0^1 (t \wedge v) v^{p-1} dv \]
\[= \int_0^1 v^p dv + \int_0^1 t v^{p-1} dv \]
\[= \frac{t^p - t^{p+1}}{p}, \]
\[E\left( \int_0^1 u^{i-1} W(u) du \right) \left( \int_0^1 v^{p-1} W(v) dv \right) \]
\[= \int_0^1 u^{i-1} E\left( W(u) \int_0^1 v^{p-1} W(v) dv \right) du \]
\[= \int_0^1 u^{i-1} \left( \frac{u}{p} - \frac{u^{p+1}}{p(p+1)} \right) du \]
\[= \frac{p + i + 2}{(p + i + 1)(i + 1)}. \quad (16)\]

Substituting (16), (17), and (19) into (15), we derive
\[E(\tilde{W}_2(t) \tilde{W}_2(s)) \]
\[= t \wedge s - \sum_{p,q=1}^3 a_{pq} \left( \frac{t}{p} - \frac{t^{p+1}}{p(p+1)} \right) s^{q-1} \]
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\[ -\sum_{i,j=1}^{3} a_{ij} \left( \frac{s^i - s^{i+1}}{i(i+1)} \right) t^{j-1} \]

\[ + \sum_{p,q=1}^{3} \sum_{i,j=1}^{3} a_{i,p} a_{j,q} \frac{p+i+2}{(p+1)(p+i+1)(i+1)} t^{j-1} s^{p+1} \]

(17)

Lemma 2 (see [3]). If \( t \in [0, 1] \), \( \zeta_j(t) = \sum_{k=1}^{\infty} \omega_k \sqrt{\lambda_{k,j}} e_{k,j}(t) \), \( j = 1, 2, \ldots \), then the condition

\[ \int_{[0,1]}^2 \zeta_1^2(t) dt \overset{law}{=} \int_{[0,1]}^2 \zeta_2^2(t) dt \]

(18)

is equivalent to the identity

\[ \lambda_{k,1} = \lambda_{k,2} \quad \forall k \geq 1. \]

(19)

In the following, we will give some preliminaries, notions, and facts that are needed in Theorem 3. For \( \nu > -1 \), \( J_\nu(\cdot) \) is Bessel function [6] with index \( \nu \) and the positive zeros of \( J_\nu(\cdot) \) are infinite sequence \( 0 < z_{\nu,1} < z_{\nu,2} < \cdots \). When \( \nu = 3/2 \), \( \nu = 5/2 \), the positive zeros of \( J_{3/2,k}, J_{5/2,k} \) are \( z_{3/2,k}, z_{5/2,k}, k = 1, 2, \ldots \), and they are in such a way that

\[ 0 < z_{3/2,1} < z_{5/2,1} < z_{3/2,2} < \cdots \]

(20)

Now we can state one of the main results of this paper.

Theorem 3. For the second order detrended Brownian motion \( W_2(t) \) and a generalized Brownian bridge \( B_2(t) \) with \( n = 2 \) in [7],

\[ B_2(t) = B(t) - \frac{1}{36} t (60t^2 + 18t + 67) B(1) \]

\[ - t (60t^2 - 96t + 11) \int_0^1 B(s) ds \]

\[ + 10t (12t^2 - 18t + 1) \int_0^1 B(s) ds \]

(21)

One has the distribution identities

\[ \int_0^1 W_2(t)^2 dt \overset{law}{=} \int_0^1 B_2(t)^2 dt \]

\[ \overset{law}{=} \sum_{k=1}^{\infty} \frac{\eta_k^2}{4z_{3/2,k}^2} + \sum_{k=1}^{\infty} \frac{\eta_k^2}{4z_{5/2,k}^2} \]

(22)

where \( \{\eta_k, k \geq 1\} \) and \( \{\eta_k, k \geq 1\} \) denote two independent sequences of independently and identically distributed \( N(0, 1) \) random variables.

Proof. By straightforward induction based on the equation and splitting the integration range from \( t \), we get

\[ \lambda f(t) = \int_0^t sf(s) ds + t \int_t^1 f(s) ds \]

\[ - 3 \sum_{p,q=1}^{3} a_{i,p} \frac{t}{p} - \frac{t^{p+1}}{p(p+1)} \int_0^1 s^{p+1} f(s) ds \]

\[ - 3 \sum_{i,j=1}^{3} a_{i,j} t^{j-1} \int_0^1 \left( \frac{s}{i} - \frac{s^{i+1}}{i(i+1)} \right) f(s) ds \]

\[ + \sum_{p,q=1}^{3} \sum_{i,j=1}^{3} a_{i,p} a_{j,q} \frac{p+i+2}{(p+1)(p+i+1)(i+1)} t^{j-1} \]

\[ \times \int_0^1 s^{p+1} f(s) ds. \]

(23)

By differentiation of both sides of (23) with respect to \( t \), we have

\[ \lambda f'(t) = \int_0^1 f(s) ds - \sum_{p,q=1}^{3} a_{i,p} \frac{1}{p} \int_0^1 s^{p-1} f(s) ds \]

\[ - 3 \sum_{i,j=1}^{3} a_{i,j} t^{j-2} \int_0^1 \left( \frac{s}{i} - \frac{s^{i+1}}{i(i+1)} \right) f(s) ds \]

\[ + \sum_{p,q=1}^{3} \sum_{i,j=1}^{3} a_{i,p} a_{j,q} \frac{(p+i+2)(j-1)}{(p+1)(p+i+1)(i+1)} t^{j-2} \]

\[ \times \int_0^1 s^{p+1} f(s) ds. \]

(24)

By differentiation of both sides of (24) with respect to \( t \), we have

\[ \lambda f''(t) + f(t) \]

\[ = \sum_{i,p,q=1}^{3} a_{i,p} a_{j,q} \frac{2(p+i+2)}{(p+1)(p+i+1)(i+1)} \int_0^1 s^{p-1} f(s) ds \]

\[ - 2 \sum_{i=1}^{3} a_{i,1} \int_0^1 \left( \frac{s}{i} - \frac{s^{i+1}}{i(i+1)} \right) f(s) ds \]

\[ + \sum_{q=1}^{3} a_{1,q} \int_0^1 s^{q-1} f(s) ds \]

\[ + \left( \sum_{q=1}^{3} a_{1,q} \int_0^1 s^{q-1} f(s) ds \right) t \]

\[ + \sum_{q=1}^{3} a_{1,q} \int_0^1 s^{q-1} f(s) ds \]

\[ \times \left( \sum_{q=1}^{3} a_{1,q} \int_0^1 s^{q-1} f(s) ds \right) t^2. \]

(25)
We can simplify this equation to
\[ \lambda f''(t) + f(t) + b_1 + b_2 t + b_3 t^2 = 0, \]  \hspace{1cm} (26)
where
\[ b_1 = -\sum_{i,p,q=1}^3 a_{ij} a_{pq} \frac{2 (p + i + 2)}{(p + 1) (p + i + 1) (i + 1)} \int_0^1 s^{q-1} f(s) \, ds + 2 \sum_{q=1}^3 a_{ij} \int_0^1 \left( \frac{s}{i} - \frac{s^i+1}{i (i+1)} \right) f(s) \, ds - \sum_{q=1}^3 a_{ij} \int_0^1 s^{q-1} f(s) \, ds, \]  \hspace{1cm} (27)
\[ b_2 = -\sum_{q=1}^3 a_{ij} \int_0^1 s^{q-1} f(s) \, ds, \]  \hspace{1cm} (28)
\[ b_3 = -\sum_{q=1}^3 a_{ij} \int_0^1 s^{q-1} f(s) \, ds. \]  \hspace{1cm} (29)

We solve the inhomogeneous second differential equation to obtain
\[ f(t) = c_1 \frac{t}{\sqrt{\lambda}} + c_2 \frac{\sin \frac{t}{\sqrt{\lambda}}}{\sqrt{\lambda}} + 2 \lambda b_2 - b_1 - b_2 t - b_3 t^2. \]  \hspace{1cm} (30)

We substitute \( f(t) \) into (28) and (29) to obtain
\[ \left( \sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} + 6 \lambda \cos \frac{1}{\sqrt{\lambda}} - 12 \lambda \sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} + 6 \lambda \right) c_1 \]
\[ + \left( -\sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} + 6 \lambda \sin \frac{1}{\sqrt{\lambda}} \right) \]
\[ + 12 \lambda \sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} - 12 \lambda \sqrt{\lambda} + \sqrt{\lambda} \) \) \] \[ = 0, \]
\[ \left( -2 \sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} + 14 \lambda \cos \frac{1}{\sqrt{\lambda}} + 30 \lambda \sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} - 16 \lambda \right) c_1 \]
\[ + \left( 2 \sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} - 14 \lambda \sin \frac{1}{\sqrt{\lambda}} \right) \]
\[ - 30 \lambda \sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} + 30 \lambda \sqrt{\lambda} - 3 \sqrt{\lambda} \) \] \] \[ = 0. \]  \hspace{1cm} (31)

In order that there are nonzero choices for \( c_1, c_2, \) the determinant of the above two equations has to be zero, which can be written as
\[ D_{11} D_{22} - D_{12} D_{21} = 0, \]  \hspace{1cm} (32)
where
\[ D_{11} = \sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} + 6 \lambda \cos \frac{1}{\sqrt{\lambda}} - 12 \lambda \sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} + 6 \lambda, \]
\[ D_{12} = -\sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} + 6 \lambda \sin \frac{1}{\sqrt{\lambda}} + 12 \lambda \sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} - 12 \lambda \sqrt{\lambda} + \sqrt{\lambda}, \]
\[ D_{21} = -2 \sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} - 14 \lambda \cos \frac{1}{\sqrt{\lambda}} + 12 \lambda \sqrt{\lambda} \sin \frac{1}{\sqrt{\lambda}} - 16 \lambda, \]
\[ D_{22} = 2 \sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} - 14 \lambda \sin \frac{1}{\sqrt{\lambda}} - 30 \lambda \sqrt{\lambda} \cos \frac{1}{\sqrt{\lambda}} + 30 \lambda \sqrt{\lambda} - 3 \sqrt{\lambda}. \]  \hspace{1cm} (33)

We obtain, after some simplification,
\[ 24 \lambda^2 \sqrt{\lambda} + 4 \lambda \sqrt{\lambda} \]
\[ = \left( 24 \lambda^2 - \lambda \right) \sin \frac{1}{\sqrt{\lambda}} + \left( 24 \lambda^2 \sqrt{\lambda} - 8 \lambda \sqrt{\lambda} \right) \cos \frac{1}{\sqrt{\lambda}} \]  \hspace{1cm} (34)

Then \( \lambda \neq 0 \) is an eigenvalue if and only if (34) holds. We therefore obtain
\[ D(\lambda) = -720 \left( \left( 24 \lambda^{-7/2} - \lambda^{-5/2} \right) \sin \lambda^{1/2} + \left( 24 \lambda^{-4} - 8 \lambda^{-3} \right) \cos \lambda^{1/2} - 24 \lambda^{-4} - 4 \lambda^{-3} \right), \]  \hspace{1cm} (35)
with \( D(0) = 1. \)

According to the trigonometric function formula
\[ \sin \frac{1}{\sqrt{\lambda}} = 2 \sin \frac{1}{2 \sqrt{\lambda}} \cos \frac{1}{2 \sqrt{\lambda}}, \]
\[ \cos \frac{1}{\sqrt{\lambda}} = 2 \cos^2 \frac{1}{2 \sqrt{\lambda}} - 1 = 1 - 2 \sin^2 \frac{1}{2 \sqrt{\lambda}}, \]  \hspace{1cm} (36)
we can observe that
\[ D_{11} D_{22} - D_{12} D_{21} = -12 \pi \sqrt{\lambda} J_{5/2} \left( \frac{1}{2 \sqrt{\lambda}} \right) J_{5/2} \left( \frac{1}{2 \sqrt{\lambda}} \right) = 0, \]  \hspace{1cm} (37)
where \( J_{5/2}(z), J_{5/2}(z) \) are Bessel functions as follows:
\[ J_{5/2}(z) = \frac{\sqrt{2 \pi} z}{\pi} \left( \frac{\sin \frac{z}{z} - \cos \frac{z}{z}}{z^2} \right), \]  \hspace{1cm} (38)
\[ J_{5/2}(z) = \frac{\sqrt{2 \pi} z}{\pi} \left( \left( -\frac{1}{z} + \frac{3}{z^3} \right) \sin \frac{z}{z} - \frac{3}{z^2} \cos \frac{z}{z} \right), \]
which gives two sequences of eigenvalues of (37), namely, \( (2 \zeta_{5/2,k})^{-2} \) and \( (2 \zeta_{5/2,k})^{-2} \).
Similarly, we can obtain the two eigenvalues \((2z^{3/2}, k)^{-2}, (2z^{5/2}, k)^{-2}\) corresponding to those of integral operator of a generalized Brownian bridge \(B_s(t)\). Note that the integral operator is

\[
\int_0^1 K_2(s, t) f(s) \, ds.
\]

(39)

Actually, in Lemma 2, we have the distribution identities

\[
\int_0^1 \hat{W}_2(t)^2 \, dt \overset{law}{=} \int_0^1 B_2(t)^2 \, dt = \sum_{k \geq 1} \frac{\eta_k^2}{4z^{3/2}_k} + \sum_{k \geq 1} \frac{\eta_k^*^2}{4z^{5/2}_k},
\]

(40)

Remark 4. From (11) and (22), we derive that

\[
\int_0^1 K_X(t, t) \, dt = \int_0^1 \mathbb{E}(\hat{W}_2(t)^2) \, dt = \mathbb{E} \int_0^1 \hat{W}_2(t)^2 \, dt
\]

\[
= \sum_{k \geq 1} \frac{1}{4z^{3/2}_k} + \sum_{k \geq 1} \frac{1}{4z^{5/2}_k}
\]

\[
= \frac{1}{40} + \frac{1}{56} = \frac{3}{140}
\]

(41)

by using the Rayleigh’s formula, for \(v = 3/2\) and \(v = 5/2\) (see, e.g., [3, (1.91), page 77] and [6, page 502]).

To check (41), from (11), we infer that

\[
\int_0^1 K_X(t, t) \, dt
\]

\[
= \int_0^1 \left[ t - \sum_{p=q=1}^3 a_{pq} \left( \frac{t^p}{p} - \frac{t^{p+q}}{p(p+1)} \right) \right.
\]

\[
- \sum_{i,j=1} a_{ij} \left( \frac{t^i}{i} - \frac{t^{i+j}}{i(i+1)} \right)
\]

\[
+ \sum_{p,q=1}^3 a_{pq} \sum_{i,j=1} t^{i+j} \left( \frac{t^p}{p} - \frac{t^{p+q}}{p(p+1)(i+1)} \right)
\]

\[
\times \frac{p+i+2}{p+1(p+i+1)(i+1)} \left( t^{i+q} \right) \right] \, dt
\]

\[
= \frac{3}{140}
\]

(42)

which is in agreement with (41).

3. Applications

In this section, the relevant applications of Karhunen-Loève expansion are given.

Proposition 5. For each \(\theta \in \mathbb{R}\), one has

\[
\mathbb{E} \left( -\frac{\theta^2}{2} \int_0^1 \hat{W}_2(t)^2 \, dt \right)
\]

\[
= \left\{ -720 \left( \left( 24\theta^7 - \theta^5 \right) \sin \theta + \left( -24\theta^8 + 8\theta^6 \right) \cos \theta + 24\theta^8 + 4\theta^6 \right) \right\}^{1/2}.
\]

(43)

Proof.

\[
\mathbb{E} \left( -\frac{\theta^2}{2} \int_0^1 \hat{W}_2(t)^2 \, dt \right)
\]

\[
= \left\{ -720 \left( \left( 24\theta^7 - \theta^5 \right) \sin \theta + \left( -24\theta^8 + 8\theta^6 \right) \cos \theta + 24\theta^8 + 4\theta^6 \right) \right\}^{1/2},
\]

(44)

where \(\lambda_1 > \lambda_2 > \cdots > 0\) and \(\sum_{k=1}^{\infty} \lambda_k < \infty\).

Proposition 6. If \(x > 0\), then

\[
P \left( \int_0^1 \hat{W}_2(t)^2 \, dt > x \right)
\]

\[
= \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^k \frac{(-1)^{k+1}}{\gamma_k} \times \left( u \left( \left| -720 \left( (24u^{7/2} - u^{-5/2}) \sin u^{1/2} \right. \right. \right. \right. \right.
\]

\[
\left. \left. + (24u^4 - 8u^3) \cos u^{1/2} \right) \right) \left( 24u^4 - 8u^3 \right)^{1/2} \right] \, du.
\]

(45)

where \(\gamma_k = \lambda_k^{-1}, k = 1, 2, \ldots\).

Proof. It can be proved by the Smirnov formula [8, 9], formula (23), and the definition of the Fredholm determinants. Similar proof method can be found from Proposition 3.3 in [10].

Next, we give the large deviation and small deviation probabilities of the second order detrended Brownian motion with respect to the norm in the Hilbert Space \(L^2[0, 1]\).
Proposition 7. Consider \( x \to \infty \),
\[
P \left( \int_0^1 \hat{W}_2(t)^2\,dt > x \right) = (1 + o(1)) \left( \frac{2}{\pi} \right)^{1/2} \left( 2z_{3/2,1} \right)^{-2} x^{-1/2} \exp \left( -2z_{3/2,1}^2 x \right) \]
\[\cdot \left\{ 720 \left( \left( \frac{3}{2}z_{3/2,1} - 9 \right) \sin 2z_{3/2,1} + \left( \frac{3}{2}z_{3/2,1} - 10 \right) - \frac{1}{2}z_{3/2,1}^6 \right) \right. \]
\[\left. \times \cos 2z_{3/2,1} + \frac{3}{2}z_{3/2,1} \right\}^{-1/2} \]
\[\cdot \left( \sum_{k \in N} \frac{a_k}{b_k} \right) \]
\[\text{as } \varepsilon \to 0.
\]
\[\text{we take } D(\lambda) \text{ and } \gamma_1 = (2z_{3/2,1})^2 \text{ into (47), and then the proof is completed.} \]

Proof. By Deheuvels [2] and Martynov [8], we have for all \( x > 0 \)
\[
P \left( \int_0^1 \hat{W}_2(t)^2\,dt > x \right) = (1 + o(1)) \left( \frac{2}{\pi} \right)^{1/2} \gamma_1^{-1} \left( -D'(\gamma_1) \right)^{-1/2} \]
\[\times x^{-1/2} \exp \left( -\frac{\gamma_1 x}{2} \right); \]

we have \( D(\lambda) \) and \( \gamma_1 = (2z_{3/2,1})^2 \) into (47), and then the proof is completed.

Proposition 8. There exists a constant \( c > 0 \) such that
\[
P \left( \int_0^1 \hat{W}_2(t)^2\,dt \leq \varepsilon \right) = (c + o(1)) \varepsilon^{-2} \exp \left( -\frac{1}{8\varepsilon} \right), \quad \text{as } \varepsilon \to 0.
\]

Proof. We start with proving (48) by recalling Li, 1992 [11, 12].
Given two sequences \( a_k > 0 \) and \( b_k > 0 \) with
\[
\sum_{k \geq 1} a_k < \infty, \quad \sum_{k \geq 1} b_k < \infty, \quad \sum_{k \geq 1} \left| \frac{a_k}{b_k} \right| < \infty, \quad (49)
\]
we have, as \( \varepsilon \to 0 \),
\[
P \left( \sum_{k \geq 1} a_k b_k \varepsilon_k \leq \varepsilon \right) = (1 + o(1)) \left( \prod_{k \geq 1} \frac{b_k}{a_k} \right)^{1/2} P \left( \sum_{k \geq 1} b_k \varepsilon_k^2 \leq \varepsilon \right) \]
\[\text{By the asymptotic formula for zeros of Bessel function}
\]
\[z_{3/2,2k} = \left( k + \frac{1}{2} \right) \pi + O(k^{-1}), \quad k \to \infty,
\]
\[z_{3/2,2k} = (k + 1) \pi + O(k^{-1}), \quad k \to \infty,
\]
then \( a_k = \lambda_k, b_{2k-1} = ((2k + 1)\pi)^2, \) and \( b_{2k} = ((2k + 2)\pi)^2, \) \( k \in N, \) which satisfy (49) and by the distribution identity
\[
\int_0^1 \hat{W}_2(t)^2\,dt = \sum_{k \geq 1} \lambda_2 \eta_k^2 + \sum_{k \geq 1} \lambda_3 \eta_{2k}^2 \text{ and (50), there exists a constant } c_1, \text{ such that}
\]
\[
P \left( \int_0^1 \hat{W}_2(t)^2\,dt \leq \varepsilon \right) = (1 + o(1)) \left( \prod_{k \geq 1} \frac{b_k}{a_k} \right)^{1/2} P \left( \sum_{k \geq 1} b_k \varepsilon_k^2 \leq \varepsilon \right) \]
\[= (1 + o(1)) \left( \prod_{k \geq 1} \frac{b_k}{a_k} \right)^{1/2} P \left( \sum_{k \geq 1} b_k \varepsilon_k^2 \leq \varepsilon \right) \]
\[= (1 + o(1)) c_1 P \left( \sum_{k \geq 1} \left( (k + 2) \pi \right)^2 \leq \varepsilon \right), \quad \text{as } \varepsilon \to 0.
\]

Also, for all \( d > -1 \), there exists a constant \( c_2 > 0 \), such that, as \( \varepsilon \to 0 \),
\[
P \left( \sum_{k \geq 1} (k + d)^{-2} \varepsilon_k^d \leq \varepsilon \right) = (1 + o(1)) c_2 \varepsilon^{-d} \exp \left( -\frac{1}{8\varepsilon} \right).
\]

Connecting (52) with (53), we can obtain the proposition.

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

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