Research Article

On Certain Subclass of Harmonic Starlike Functions

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Coefficient conditions, distortion bounds, extreme points, convolution, convex combinations, and neighborhoods for a new class of harmonic univalent functions in the open unit disc are investigated. Further, a class preserving integral operator and connections with various previously known results are briefly discussed.

1. Introduction

A continuous complex-valued function $f = u + iv$ is said to be harmonic in a simply connected domain $D \subset \mathbb{C}$ if both u and v are real harmonic in D . There is a close interrelation between analytic functions and harmonic functions. For example, for real harmonic functions u and v , there exist analytic functions U and V so that $u = \operatorname{Re}(U)$ and $v = \operatorname{Im}(V)$. Then $f(z) = h(z) + \overline{g(z)}$, where h and g are, respectively, the analytic functions $(U + V)/2$ and $(U - V)/2$. In this case, the Jacobian of $f = h + \overline{g}$ is given by $J_f = |h'(z)|^2 - |g'(z)|^2$. The mapping $z \rightarrow f(z)$ is orientation preserving and locally one-to-one in D if and only if $J_f > 0$ in D . The function $f = h + \overline{g}$ is said to be harmonic univalent in D if the mapping $z \rightarrow f(z)$ is orientation preserving, harmonic, and one-to-one in D . We call h the analytic part and g the coanalytic part of f (see Clunie and Sheil-Small [1]).

Denote by $\mathcal{H}$ the class of functions $f = h + \overline{g}$ that are harmonic univalent and orientation preserving in the open unit disk $\mathbb{U} = \{z : |z| < 1\}$ for which $f(0) = f_z(0) - 1 = 0$. Then for $f = h + \overline{g} \in \mathcal{H}$, we may express the analytic functions f and g as

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (1)$$

$$g(z) = \sum_{n=1}^{\infty} b_n z^n, \quad |b_1| < 1.$$

Note that $\mathcal{H}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the coanalytic part of its members is zero. For this class the function $f(z)$ may be expressed as

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \quad (2)$$

A function $f = h + \overline{g}$ with h and g given by (1) is said to be harmonic starlike of order β ($0 \leq \beta < 1$) for $|z| = r < 1$, if

$$\frac{\partial}{\partial \theta} (\arg f(re^{i\theta})) = \operatorname{Re} \left(\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right) \geq \beta. \quad (3)$$

The class of all harmonic starlike functions of order β is denoted by $\mathcal{S}_H^*(\beta)$ and extensively studied by Jahangiri [2]. The cases $\beta = 0$ and $b_1 = 0$ were studied by Silverman and Silvia [3] and Silverman [4]. Other related works of the class $\mathcal{H}$ also appeared in [5–16].

Definition 1. Let $f = h + \overline{g}$ where h and g are given by (1). Let $0 \leq \beta < 1$ and $\alpha \geq 0$. Then $f \in \mathcal{S}_H^*(\alpha, \beta)$ if and only if

$$\operatorname{Re} \left(\frac{\alpha z^2 h''(z) + zh'(z) + \overline{\alpha z^2 g''(z) + (2\alpha - 1)zg'(z)}}{h(z) + \overline{g(z)}} \right) \geq \beta. \quad (4)$$

We note that for $\alpha = 0$, the class $\mathcal{S}_H^*(\alpha, \beta)$ reduces to the class $\mathcal{S}_H^*(\beta)$. Further, if the coanalytic part $g(z)$ is zero, the

class $\mathcal{S}_H^*(\alpha, \beta)$ reduces to the class $\mathcal{P}(\alpha, \beta)$ of functions $f \in \mathcal{S}$ which satisfy the condition

$$\operatorname{Re} \left(\frac{\alpha z^2 f''(z) + z f'(z)}{f(z)} \right) \geq \beta \quad (5)$$

for some β ($0 \leq \beta < 1$), $\alpha \geq 0$, $f(z)/z \neq 0$ and $z \in \mathbb{U}$. Observe that the classes $\mathcal{P}(\alpha, \beta)$ and $\mathcal{P}(\alpha, 0)$ were introduced and studied by many authors and these include, for example, by Obradovic and Joshi [17], Padmanabhan [18], Li and Owa [19], Xu and Yang [20], Singh and Gupta [21], and Lashin [22]. We also note that for $\alpha = 0$, the class $\mathcal{P}(0, \beta)$ was studied by Silverman [23].

2. Main Results

The first theorem of this section determines the sufficient coefficient condition for functions $f = h + \bar{g}$ to belong to the class $\mathcal{S}_H^*(\alpha, \beta)$. The following lemma obtained by Jahangiri is needed.

Lemma 2 (see [2, Theorem 1]). *Let $f = h + \bar{g}$ with h and g of the form (1) and let*

$$\sum_{n=2}^{\infty} \frac{n-\beta}{1-\beta} |a_n| + \sum_{n=1}^{\infty} \frac{n+\beta}{1-\beta} |b_n| \leq 1, \quad (6)$$

where $0 \leq \beta < 1$. Then f is harmonic, orientation preserving, and univalent in $\mathbb{U}$, and $f \in \mathcal{S}_H^*(\beta)$.

Theorem 3. *Let $f = h + \bar{g}$ where h and g are of the form (1). If*

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1 - \beta} |a_n| \\ & + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1 - \beta} |b_n| \leq 1 \end{aligned} \quad (7)$$

for some β , ($0 \leq \beta < 1$) and $\alpha \geq 0$, then f is harmonic, orientation preserving, and univalent in $\mathbb{U}$ and $f \in \mathcal{S}_H^*(\alpha, \beta)$.

Proof. Since $(n - \beta) \leq \alpha n(n - 1) + n - \beta$ and $n + \beta \leq \alpha n(n + 1) + n + \beta$, ($n \geq 1$), it follows from Lemma 2 that $f \in \mathcal{S}_H^*(\beta)$ and hence f is harmonic, orientation preserving, and univalent in $\mathbb{U}$. Now, we only need to show that if (7) holds then

$$\begin{aligned} & \operatorname{Re} \left(\frac{\alpha z^2 h''(z) + z h'(z) + \overline{\alpha z^2 g''(z) + (2\alpha - 1) z g'(z)}}{h(z) + \overline{g(z)}} \right) \\ & = \operatorname{Re} \frac{A(z)}{B(z)} \geq \beta. \end{aligned} \quad (8)$$

Using the fact that $\operatorname{Re}(w) \geq \beta$ if and only if $|1 - \beta + w| \geq |1 + \beta - w|$, it suffices to show that

$$|A(z) + (1 - \beta) B(z)| - |A(z) - (1 + \beta) B(z)| \geq 0, \quad (9)$$

where

$$\begin{aligned} A(z) &= \alpha z^2 h''(z) + z h'(z) + \overline{\alpha z^2 g''(z) + (2\alpha - 1) z g'(z)}, \\ B(z) &= h(z) + \overline{g(z)}. \end{aligned} \quad (10)$$

Substituting for $A(z)$ and $B(z)$ in (9), we obtain

$$\begin{aligned} & |A(z) + (1 - \beta) B(z)| - |A(z) - (1 + \beta) B(z)| \\ &= \left| (2 - \beta) z + \sum_{n=2}^{\infty} ((\alpha n + 1)(n - 1) + 2 - \beta) a_n z^n \right. \\ & \quad \left. + \sum_{n=1}^{\infty} (\alpha n(n + 1) - (n - 1 + \beta)) \bar{b}_n \bar{z}^n \right| \\ & \quad - \left| -\beta z + \sum_{n=2}^{\infty} ((\alpha n + 1)(n - 1) - \beta) a_n z^n \right. \\ & \quad \left. + \sum_{n=1}^{\infty} (\alpha n(n + 1) - (n + 1 + \beta)) \bar{b}_n \bar{z}^n \right| \\ & \geq (2 - \beta) |z| \\ & \quad - \sum_{n=2}^{\infty} ((\alpha n + 1)(n - 1) + 2 - \beta) |a_n| |z|^n \\ & \quad - \sum_{n=1}^{\infty} |\alpha n(n + 1) - (n - 1 + \beta)| |b_n| |z|^n - \beta |z| \\ & \quad - \sum_{n=2}^{\infty} ((\alpha n + 1)(n - 1) - \beta) |a_n| |z|^n \\ & \quad - \sum_{n=1}^{\infty} |\alpha n(n + 1) - (n + 1 + \beta)| |b_n| |z|^n \\ & \geq 2(1 - \beta) |z| \\ & \quad - 2 \sum_{n=2}^{\infty} (\alpha n(n - 1) + n - \beta) |a_n| |z|^n \\ & \quad - 2 \sum_{n=1}^{\infty} (\alpha n(n + 1) + (n + \beta)) |b_n| |z|^n \\ & \geq 2(1 - \beta) |z| \\ & \quad \times \left(1 - \sum_{n=2}^{\infty} \frac{\alpha n(n - 1) + n - \beta}{1 - \beta} |a_n| \right. \\ & \quad \left. - \sum_{n=1}^{\infty} \frac{\alpha n(n + 1) + (n + \beta)}{1 - \beta} |b_n| \right) \geq 0, \end{aligned} \quad (11)$$

by the given condition (7).

The harmonic function

$$f(z) = z + \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n-1) + n - \beta} x_n z^n + \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n+1) + (n+\beta)} \bar{y}_n \bar{z}^n, \quad (12)$$

where $\sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1$ shows that the coefficient bound given in (7) is sharp. The functions of the form (12) are in $\mathcal{S}_H^*(\alpha, \beta)$ since

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1-\beta} |a_n| + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1-\beta} |b_n| \\ &= \sum_{n=2}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n| = 1. \end{aligned} \quad (13)$$

□

Remark 4. Setting $g(z) = 0$ in Theorem 3 yields the result obtained by Lashin [22, Theorem 2.1].

We denote by $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ the class of functions $f \in \mathcal{S}_H^*(\alpha, \beta)$ whose coefficients satisfy the condition (7).

Theorem 5. Let $0 \leq \alpha_1 < \alpha_2$ and $0 \leq \beta < 1$. Then $\overline{\mathcal{S}}_H^*(\alpha_2, \beta) \subset \overline{\mathcal{S}}_H^*(\alpha_1, \beta)$.

Proof. For $f \in \overline{\mathcal{S}}_H^*(\alpha_2, \beta)$, it follows from (7) that

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{\alpha_1 n(n-1) + n - \beta}{1-\beta} |a_n| + \sum_{n=1}^{\infty} \frac{\alpha_1 n(n+1) + n + \beta}{1-\beta} |b_n| \\ & < \sum_{n=2}^{\infty} \frac{\alpha_2 n(n-1) + n - \beta}{1-\beta} |a_n| \\ & \quad + \sum_{n=1}^{\infty} \frac{\alpha_2 n(n+1) + n + \beta}{1-\beta} |b_n| \leq 1. \end{aligned} \quad (14)$$

Hence $f \in \overline{\mathcal{S}}_H^*(\alpha_1, \beta)$. □

As a consequence of Theorem 5, the functions in $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ are starlike harmonic in $\mathbb{U}$.

Corollary 6. For $\alpha \geq 0$ and $0 \leq \beta < 1$, $\overline{\mathcal{S}}_H^*(\alpha, \beta) \subset \mathcal{S}_H^*(\beta)$.

3. Distortion Bounds and Extreme Points

In this section, we obtain the distortion bounds and extreme points for functions in the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$.

Theorem 7. Let $f = h + \bar{g}$ where h and g are of the form (1) and $f \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. Then for $|z| = r < 1$, we have

$$|f(z)| \leq (1 + |b_1|)r + \left(\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta} |b_1| \right) r^2, \quad (15)$$

$$|f(z)| \geq (1 - |b_1|)r - \left(\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta} |b_1| \right) r^2, \quad (16)$$

where

$$|b_1| \leq \frac{1-\beta}{2\alpha+1+\beta}. \quad (17)$$

The result is sharp.

Proof. We shall prove the first inequality. Let $f \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. Then we have

$$\begin{aligned} |f(z)| & \leq (1 + |b_1|)r + \sum_{n=2}^{\infty} (|a_n| + |b_n|)r^n \\ & \leq (1 + |b_1|)r + r^2 \sum_{n=2}^{\infty} (|a_n| + |b_n|) \\ & = (1 + |b_1|)r + \frac{1-\beta}{2\alpha+2-\beta} \\ & \quad \times \sum_{n=2}^{\infty} \frac{2\alpha+2-\beta}{1-\beta} (|a_n| + |b_n|)r^2 \end{aligned} \quad (18)$$

and so

$$\begin{aligned} |f(z)| & \leq (1 + |b_1|)r + \frac{1-\beta}{2\alpha+2-\beta} \\ & \quad \times \sum_{n=2}^{\infty} \left(\frac{\alpha n(n-1) + n - \beta}{1-\beta} |a_n| + \frac{\alpha n(n+1) + n + \beta}{1-\beta} |b_n| \right) r^2 \\ & \leq (1 + |b_1|)r + \frac{1-\beta}{2\alpha+2-\beta} \left(1 - \frac{2\alpha+1+\beta}{1-\beta} |b_1| \right) r^2 \\ & = (1 + |b_1|)r + \left(\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta} |b_1| \right) r^2. \end{aligned} \quad (19)$$

The proof of the inequality (16) is similar to the proof of the inequality (15), thus we omit it.

The upper bound given for $f \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$ is sharp and the equality occurs for the function

$$f(z) = z + |b_1| \bar{z} + \left(\frac{1-\beta}{2\alpha+2-\beta} - \frac{2\alpha+1+\beta}{2\alpha+2-\beta} |b_1| \right) \bar{z}^2 \quad (z=r), \quad (20)$$

where $|b_1| \leq (1-\beta)/(2\alpha+1+\beta)$. This completes the proof of Theorem 7. $\square$

Now, we determine the extreme points of the closed convex hull of the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ denoted by $\text{clco } \overline{\mathcal{S}}_H^*(\alpha, \beta)$.

Theorem 8. Let $f = h + \bar{g}$ where h and g are given by (1). Then $f \in \text{clco } \overline{\mathcal{S}}_H^*(\alpha, \beta)$ if and only if

$$f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n), \quad (21)$$

where

$$\begin{aligned} h_1(z) &= z, \\ h_n(z) &= z + \frac{1-\beta}{\alpha n(n-1) + n - \beta} z^n \quad (n=2, 3, \dots); \\ g_n(z) &= z + \frac{1-\beta}{\alpha n(n+1) + n + \beta} \bar{z}^n \quad (n=1, 2, 3, \dots), \\ \sum_{n=1}^{\infty} (X_n + Y_n) &= 1, \quad X_n \geq 0, \quad Y_n \geq 0. \end{aligned} \quad (22)$$

In particular, the extreme points of the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ are $\{h_n\}$ and $\{g_n\}$, respectively.

Proof. For a function f of the form (21), we have

$$\begin{aligned} f(z) &= \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n) \\ &= \sum_{n=1}^{\infty} (X_n + Y_n) z \\ &\quad + \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n-1) + n - \beta} X_n z^n \\ &\quad + \sum_{n=1}^{\infty} \frac{1-\beta}{\alpha n(n+1) + n + \beta} Y_n \bar{z}^n \\ &= z + \sum_{n=2}^{\infty} \frac{1-\beta}{\alpha n(n-1) + n - \beta} X_n z^n \\ &\quad + \sum_{n=1}^{\infty} \frac{1-\beta}{\alpha n(n+1) + n + \beta} Y_n \bar{z}^n. \end{aligned} \quad (23)$$

But

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1-\beta} \left(\frac{1-\beta}{\alpha n(n-1) + n - \beta} X_n \right) \\ &\quad + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1-\beta} \left(\frac{1-\beta}{\alpha n(n+1) + n + \beta} Y_n \right) \\ &= \sum_{n=2}^{\infty} X_n + \sum_{n=1}^{\infty} Y_n = 1 - X_1 \leq 1. \end{aligned} \quad (24)$$

Thus $f \in \text{clco } \overline{\mathcal{S}}_H^*(\alpha, \beta)$.

Conversely, suppose that $f \in \text{clco } \overline{\mathcal{S}}_H^*(\alpha, \beta)$. Set

$$\begin{aligned} X_n &= \frac{\alpha n(n-1) + n - \beta}{1-\beta} |a_n| \quad (n=2, 3, \dots), \\ Y_n &= \frac{\alpha n(n+1) + n + \beta}{1-\beta} |b_n| \quad (n=1, 2, 3, \dots). \end{aligned} \quad (25)$$

Then by the inequality (7), we have $0 \leq X_n \leq 1$ ($n=2, 3, \dots$) and $0 \leq Y_n \leq 1$ ($n=1, 2, \dots$). Define $X_1 = 1 - \sum_{n=2}^{\infty} X_n - \sum_{n=1}^{\infty} Y_n$ and note that $X_1 \geq 0$. Thus we obtain $f(z) = \sum_{n=1}^{\infty} (X_n h_n + Y_n g_n)$. This completes the proof of the theorem. $\square$

4. Convolution and Convex Combinations

For two harmonic functions

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{z}^n, \quad (26)$$

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \bar{B}_n \bar{z}^n$$

we define their convolution

$$(f * F)(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} \bar{b}_n \bar{B}_n \bar{z}^n. \quad (27)$$

Using this definition, we show that the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ is closed under convolution.

Theorem 9. For $0 \leq \beta < 1$ and $\alpha \geq 0$, let $f, F \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. Then $f * F \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$.

Proof. We note that $|A_n| \leq 1$ and $|B_n| \leq 1$. For the convolution $(f * F)$, we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1 - \beta} |A_n a_n| \\ & + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1 - \beta} |B_n b_n| \\ & \leq \sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1 - \beta} |a_n| \\ & + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1 - \beta} |b_n| \leq 1. \end{aligned} \quad (28)$$

Therefore $f * F \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. $\square$

We show that the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ is closed under convex combination of its members.

Theorem 10. *The class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ is closed under convex combination.*

Proof. For $i = 1, 2, 3, \dots$, let $f_i \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$ where $f_i(z)$ is given by

$$f_i(z) = z + \sum_{n=2}^{\infty} a_{ni} z^n + \sum_{n=1}^{\infty} \bar{b}_{ni} \bar{z}^n. \quad (29)$$

Then by (7), we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1 - \beta} |a_{ni}| \\ & + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1 - \beta} |b_{ni}| \leq 1. \end{aligned} \quad (30)$$

For $\sum_{i=1}^{\infty} t_i = 1$, $0 \leq t_i \leq 1$, the convex combination of f_i may be written as

$$\sum_{i=1}^{\infty} t_i f_i = z + \sum_{n=2}^{\infty} \left(\sum_{i=1}^{\infty} t_i a_{ni} \right) z^n + \sum_{n=1}^{\infty} \left(\sum_{i=1}^{\infty} t_i \bar{b}_{ni} \right) \bar{z}^n. \quad (31)$$

Then by (7), we have

$$\begin{aligned} & \sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1 - \beta} \left| \sum_{i=1}^{\infty} t_i a_{ni} \right| \\ & + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1 - \beta} \left| \sum_{i=1}^{\infty} t_i \bar{b}_{ni} \right| \\ & \leq \sum_{i=1}^{\infty} t_i \left(\sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1 - \beta} |a_{ni}| \right. \\ & \quad \left. + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1 - \beta} |b_{ni}| \right) \\ & \leq \sum_{i=1}^{\infty} t_i = 1. \end{aligned} \quad (32)$$

Therefore $\sum_{i=1}^{\infty} t_i f_i \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. $\square$

5. Neighborhood Results

Following the earlier investigations by Goodman [24], Ruscheweyh [25], Altintas et al. [26], and Porwal and Aouf [27], we define the δ -neighborhood of function $f(z) \in \mathcal{H}$ by

$$\begin{aligned} N_{\delta}(f) = \left\{ F \in \mathcal{H} : F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \bar{B}_n \bar{z}^n, \right. \\ \left. \sum_{n=2}^{\infty} n |a_n - A_n| + \sum_{n=1}^{\infty} n |b_n - B_n| \leq \delta \right\}. \end{aligned} \quad (33)$$

In particular, for the identity function $e(z) = z$, we immediately have

$$\begin{aligned} N_{\delta}(e) = \left\{ F \in \mathcal{H} : F(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} \bar{B}_n \bar{z}^n, \right. \\ \left. \sum_{n=2}^{\infty} n |A_n| + \sum_{n=1}^{\infty} n |B_n| \leq \delta \right\}. \end{aligned} \quad (34)$$

Theorem 11. $\overline{\mathcal{S}}_H^*(\alpha, \beta) \subseteq N_{\delta}(e)$, where

$$\delta = \frac{(1 - \beta)(2\alpha + 1)}{(\alpha + 1)(2\alpha + 1 - \beta)}. \quad (35)$$

Proof. Let $f \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. Then, in view of (7), since $\alpha n(n-1) + n - \beta$ and $\alpha n(n+1) + n + \beta$ are increasing functions of n ($n \geq 1$), we have

$$\begin{aligned} & (2\alpha + 1 - \beta) \left(\sum_{n=2}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \right) \\ & \leq (2\alpha + 2 - \beta) \sum_{n=2}^{\infty} |a_n| + (2\alpha + 1 + \beta) \sum_{n=1}^{\infty} |b_n| \\ & \leq \sum_{n=2}^{\infty} (\alpha n(n-1) + n - \beta) |a_n| \\ & \quad + \sum_{n=1}^{\infty} (\alpha n(n+1) + n + \beta) |b_n| \leq (1 - \beta), \end{aligned} \quad (36)$$

which yields

$$\sum_{n=2}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \leq \frac{1 - \beta}{(2\alpha + 1 - \beta)}. \quad (37)$$

On the other hand, we also find from (7)

$$\begin{aligned}
 & \left((\alpha + 1) \sum_{n=2}^{\infty} n |a_n| - \beta \sum_{n=2}^{\infty} |a_n| \right) \\
 & + \left((\alpha + 1) \sum_{n=1}^{\infty} n |b_n| - \beta \sum_{n=1}^{\infty} |b_n| \right) \\
 & \leq \sum_{n=2}^{\infty} ((\alpha(n-1) + 1)n - \beta) |a_n| \\
 & + \sum_{n=1}^{\infty} ((\alpha(n+1) + 1)n + \beta) |b_n| \\
 & \leq (1 - \beta).
 \end{aligned} \quad (38)$$

From (37) and (38), we obtain

$$\begin{aligned}
 & (\alpha + 1) \left(\sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \right) \\
 & \leq (1 - \beta) + \beta \left(\sum_{n=2}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \right) \\
 & = (1 - \beta) + \beta \frac{1 - \beta}{(2\alpha + 1 - \beta)},
 \end{aligned} \quad (39)$$

which is equivalent to

$$\sum_{n=2}^{\infty} n |a_n| + \sum_{n=1}^{\infty} n |b_n| \leq \frac{(1 - \beta)(2\alpha + 1)}{(\alpha + 1)(2\alpha + 1 - \beta)} = \delta. \quad (40)$$

□

6. A Family of Class Preserving Integral Operator

In this section, we consider the closure property of the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$ under the Bernardi integral operator $F(z)$, which is defined by

$$F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} h(t) dt + \overline{\frac{c+1}{z^c} \int_0^z t^{c-1} g(t) dt} \quad (41)$$

$(c > -1).$

Theorem 12. Let $f = h + \bar{g}$ be in the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$, where h and g are given by (1). Then $F(z)$ defined by (41) also belongs to the class $\overline{\mathcal{S}}_H^*(\alpha, \beta)$.

Proof. From the representation of F , it follows that

$$F(z) = z + \sum_{n=2}^{\infty} \frac{c+1}{c+n} a_n z^n + \sum_{n=1}^{\infty} \frac{c+1}{c+n} \overline{b_n} \bar{z}^n. \quad (42)$$

Now

$$\begin{aligned}
 & \sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1 - \beta} \left(\frac{c+1}{c+n} |a_n| \right) \\
 & + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1 - \beta} \left(\frac{c+1}{c+n} |b_n| \right) \\
 & \leq \sum_{n=2}^{\infty} \frac{\alpha n(n-1) + n - \beta}{1 - \beta} |a_n| \\
 & + \sum_{n=1}^{\infty} \frac{\alpha n(n+1) + n + \beta}{1 - \beta} |b_n| \leq 1
 \end{aligned} \quad (43)$$

by (7). Thus $F(z) \in \overline{\mathcal{S}}_H^*(\alpha, \beta)$. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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