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Research Article

Analysis of the Symmetries and Conservation Laws of the Nonlinear Jaulent-Miodek Equation

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Lie symmetry method is performed for the nonlinear Jaulent-Miodek equation. We will find the symmetry group and optimal systems of Lie subalgebras. The Lie invariants associated with the symmetry generators as well as the corresponding similarity reduced equations are also pointed out. And conservation laws of the J-M equation are presented with two steps: firstly, finding multipliers for computation of conservation laws and, secondly, symbolic computation of conservation laws will be applied.

1. Introduction

The Jaulent-Miodek equation (J-M) is given by

$$u_t + u_{xxx} + \frac{3}{2}vv_{xxx} + \frac{9}{2}v_x v_{xx} - 6uu_x - 6uvv_x - \frac{3}{2}v^2 u_x = 0,$$
$$v_t + v_{xxx} - 6vu_x - 6uv_x - \frac{15}{2}v^2 v_x = 0. \tag{1}$$

The coupled system of (1) is associated with the J-M spectral problem [1]. The relation between this system and Euler-Darboux equation was found by Matsuno [2]. In recent years, much work associated with the J-M equation has been done [3–5]. The symmetry group method plays a fundamental role in the analysis of differential equations. The theory of Lie symmetry groups of differential equations called classical Lie method was first developed by Lie [6] at the end of the nineteenth century. Nowadays, the application of Lie transformations group theory for constructing the solutions of nonlinear partial differential equations (PDEs) is regarded as one of the most active fields of research in the theory of nonlinear PDEs and applications.

Many PDEs in the applied sciences and engineering are continuity equations which express conservation of mass, momentum, energy, or electric charge. Such equations occur in, for example, fluid mechanics, particle and quantum physics, plasma physics, elasticity, gas dynamics, electromagnetism, magnetohydrodynamics, nonlinear optics, and so forth. In the study of PDEs, conservation laws are important for investigating integrability and linearization mappings and for establishing existence and uniqueness of solutions. They are also used in the analysis of stability and global behavior of solutions [7–10].

The present paper is organized as follows. In Section 1, we obtain the symmetry of (1) and Lie symmetry groups of J-M equation are found. In Section 2, we construct the optimal system of one-dimensional subalgebras of (1). Lie invariants and similarity reduced equations corresponding to the infinitesimal symmetries of (1) are obtained in Section 3. In Section 4, the conservation laws of (1) are obtained with finding multipliers, and finally some new conservation laws of (1) are obtained with symbolic computation of conservation laws.

2. Lie Symmetries of the J-M Equation

In this section, we draw your attention to the general procedure for determining symmetries for J-M equation; see [11–13]. We consider the one parameter Lie group of

Table 1: The commutator table.

$[\ ,\ ]$	X_1	X_2	X_3
X_1	0	0	X_1
X_2	0	0	$\frac{1}{3}X_2$
X_3	$-X_1$	$-\frac{1}{3}X_2$	0

Table 2: Adjoint representation of the infinitesimal generators.

Ad	X_1	X_2	X_3
X_1	X_1	X_2	$X_3 - \varepsilon X_1$
X_2	X_1	X_2	$X_3 - \frac{1}{3}\varepsilon X_2$
X_3	$e^{\varepsilon}X_1$	$e^{(1/3)\varepsilon}X_2$	X_3

infinitesimal transformations on ($x^1 = x, x^2 = t, u^1 = u, u^2 = v$),

$$\begin{aligned} \tilde{x} &= x + s\xi^1(x,t,u,v) + O\left(s^2\right), \\ \tilde{t} &= x + s\xi^2(x,t,u,v) + O\left(s^2\right), \\ \tilde{u} &= x + s\varphi^1(x,t,u,v) + O\left(s^2\right), \\ \tilde{v} &= x + s\varphi^2(x,t,u,v) + O\left(s^2\right), \end{aligned} \tag{2}$$

where s is the group parameter and ξ^1, ξ^2, φ^1, and φ^2 are the infinitesimals of the transformations for the independent and dependent variables, respectively. The associated vector field is in the following form:

$$\begin{aligned} V = {} & \xi^1(t,x,u,v)\partial_t + \xi^2(t,x,u,v)\partial_x \\ & + \varphi^1(t,x,u,v)\partial_u + \varphi^2(t,x,u,v)\partial_v. \end{aligned} \tag{3}$$

The Lie algebra $\mathfrak{g}$ of infinitesimal symmetry of (1) is spanned by three vector fields:

$$\begin{gathered} X_1 = \partial_t, \qquad X_2 = \partial_x, \\ X_3 = t\partial_t + \frac{1}{3}x\partial_x - \frac{2}{3}u\partial_u - \frac{1}{3}v\partial_v. \end{gathered} \tag{4}$$

The commutation relations of the 3-dimensional Lie algebra $\mathfrak{g}$ spanned by the vector fields X_1, X_2, X_3 are shown in Table 1.

Theorem 1. *If $u = f(t,x)$ and $v = g(t,x)$ are a solution of* (1), *then so are the functions*

$$\begin{aligned} G_1(s)\cdot f(t,x) &= f(t-s,x), \\ G_1(s)\cdot g(t,x) &= g(t-s,x), \\ G_2(s)\cdot f(t,x) &= f(t,x-s), \\ G_2(s)\cdot g(t,x) &= g(t,x-s), \\ G_3(s)\cdot f(t,x) &= e^{-(2/3)s} f\left(e^{-s}t, e^{-(1/3)s}x\right), \\ G_3(s)\cdot g(t,x) &= e^{-(1/3)s} g\left(e^{-s}t, e^{-(1/3)s}x\right). \end{aligned} \tag{5}$$

3. Optimal System of the Jaulent-Miodek Equation

In this section, we obtain the optimal system and reduced forms of (1) by using symmetry group properties obtained in previous section. Since the original partial differential equation has two independent variables, this partial differential equation transforms into the ordinary differential equation after reduction.

A well-known standard procedure [11] allows us to classify all the one-dimensional subalgebras into subsets of conjugate subalgebras. This involves constructing the adjoint representation group, which introduces a conjugate relation in the set of all one-dimensional subalgebras. In fact, for one-dimensional subalgebras, the classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation. Since each one-dimensional subalgebra is determined by nonzero vector in $\mathfrak{g}$, this problem is attacked by the naive approach of taking a general element V in $\mathfrak{g}$ and subjecting it to various adjoint transformations so as to "simplify" it as much as possible. Thus, we will deal with the construction of the optimal system of subalgebras of $\mathfrak{g}$. To compute the adjoint representation, we use the Lie series

$$\begin{aligned} \mathrm{Ad}\left(\exp\left(\varepsilon\left(X_i\right)\right)X_j\right) = {} & X_j - \varepsilon\left[X_i, X_j\right] \\ & + \frac{\varepsilon^2}{2}\left[X_i, \left[X_i, X_j\right]\right] - \cdots, \end{aligned} \tag{6}$$

where $[X_i, X_j]$ is the commutator for the Lie algebra, ε is a parameter, and $i, j = 1, 2, 3$. Then we have Table 2.

Theorem 2. *An optimal system of one-dimensional Lie algebras of the J-M equation is provided by (1) X_3, (2) $\alpha X_1 + X_2$, and (3) X_1, where $\alpha \in \mathbb{R}$ and $\alpha \neq 0$.*

Proof. Let $\mathfrak{g}$ be the symmetry group of (1), with adjoint representation determined in Table 2, and

$$X = a_1X_1 + a_2X_2 + a_3X_3 \tag{7}$$

is a nonzero vector field of $\mathfrak{g}$. We will simplify as many of the coefficients of a_i, $i = 1, 2, 3$, as possible through judicious applications of adjoint maps to X.

Case 1. Suppose first that $a_3 \neq 0$. Scaling X if necessary, we can assume that $a_3 = 1$. Referring to Table 2, if we act on such a X by $\mathrm{Ad}(\exp(a_1X_1))$ and $\mathrm{Ad}(\exp(3a_2X_2))$, respectively, we can make the coefficients of v_1 and X_2 vanish. Thus, every one-dimensional subalgebra generated by a X with $a_3 \neq 0$ is equivalent to the subalgebra spanned by X_3.

Case 2. The remaining one-dimensional subalgebras are spanned by vectors of the above form with $a_3 = 0$. If $a_2 \neq 0$, we can scale to make $a_2 = 1$. Referring to Table 2, we cannot do anything in this case. Thus, every one-dimensional subalgebra generated by a v with $a_3 = 0$ and $a_2 \neq 0$ is

equivalent to the subalgebra spanned by $\alpha X_1 + X_2$, where α is arbitrary constant.

Case 3. Consider $a_3 = 0$, $a_2 = 0$, and $a_1 \neq 0$. Thus, every one-dimensional subalgebra generated by X is equivalent to the subalgebra spanned by X_1. □

4. Symmetry Reduction of the J-M Equation

We can now compute the invariants associated with the symmetry operators by integrating the characteristic equations. For example, for the operator characteristic equation $X_3 = t\partial_t + (1/3)x\partial_x - (2/3)u\partial_u - (1/3)v\partial_v$,

$$\frac{dt}{t} = \frac{3dx}{x} = -\frac{3du}{2u} = -\frac{3v}{dv}. \tag{8}$$

The corresponding invariants are $\lambda = xt^{-1/3}$, $U = ut^{2/3}$, and $V = vt^{1/3}$. Therefore, solution of our equation in this case is $u = Ut^{-2/3}$, $v = Vt^{-1/3}$. Substituting derivatives of u and v in terms of λ, U, and V into (1), the coupled system of ordinary differential equation is obtained as follows:

$$\begin{gathered} 4U + 2\lambda U_\lambda - 6U_{\lambda\lambda\lambda} - 9VV_{\lambda\lambda\lambda} - 27V_\lambda V_{\lambda\lambda} \\ + 36UU_\lambda + 36UVV_\lambda + 9V^2U_\lambda = 0 \\ 2V + 2\lambda V_\lambda - 6V_{\lambda\lambda\lambda} + 36VU_\lambda + 36UV_\lambda + 45V^2V_\lambda = 0. \end{gathered} \tag{9}$$

And for the operator $\alpha X_1 + X_2$, we have

$$\begin{gathered} \alpha U_\lambda + U_{\lambda\lambda\lambda} + \frac{3}{2}VV_{\lambda\lambda\lambda} + \frac{9}{2}V_\lambda V_{\lambda\lambda} \\ - 6UU_\lambda - 6UVV_\lambda - \frac{3}{2}V^2U_\lambda = 0 \\ \alpha V_\lambda + V_{\lambda\lambda\lambda} - 6VU_\lambda - 6UV_\lambda - \frac{15}{2}V^2V_\lambda = 0, \end{gathered} \tag{10}$$

and the corresponding invariants associated with the above operator are $\lambda = x - \alpha t$, $U = u$, and $V = v$.

5. Conservation Laws for the J-M Equation

To deal with the conservation laws, many methods, such as the method based on the Noethers theorem and the multiplier method, are derived by the relationship between the conserved vector of the PDE and the Lie-Bäcklund symmetry generators of the PDE, the direct method, and so forth [7, 8, 11].

Definition 3. A local conservation law of the PDE system

$$\Delta_\nu\left(x, u^{(n)}\right) = 0, \quad \nu = 1, \ldots, l, \tag{11}$$

involving $\tilde{x} = x(x^1, \ldots, x^p)$, $\tilde{u} = (u^1, \ldots, u^q)$, and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n, is a divergence expression

$$D_i\Phi^i[u] = D_1\Phi^1[u] + \cdots + D_n\Phi^n[u] = 0 \tag{12}$$

holding for all solutions of the system (11). $\Phi^i[u] = \Phi^i(x, u, \partial u, \ldots, \partial_u^r)$, $i = 1, \ldots, n$, are called fluxes of the conservation law, and the highest-order derivative (r) present in the fluxes $\Phi^i[u]$ is called the order of a conservation law [8].

Remark 4. If one of the independent variables of (11) is time t, the conservation law (12) takes the form

$$D_t\Psi[u] + \operatorname{div}\Phi[u] = 0, \tag{13}$$

where $\operatorname{div}\Phi[u] = D_i\Phi^i[u] = D_1\Phi^1[u] + \cdots + D_n\Phi^n[u]$ is a spatial divergence and $x = (x^1, \ldots, x^{n-1})$ are $n-1$ spatial variables. Here $\Psi[u]$ is referred to as a density, and $\Phi^i[u]$ as spatial fluxes of the conservation law (13).

5.1. Computation of Conservation Laws with Finding Multiplier. In this study, we derive the conservation law from the multiplier method. In particular, a set of multipliers $\{\Lambda_\nu\}_{\nu=1}^l = \{\Lambda_\nu(x, U, \partial U, \ldots, \partial_U^r)\}_{\nu=1}^l$ yields a divergence expression for the system (11) if the identity

$$\Lambda_\nu[U]\,\Delta_\nu[U] \equiv D_i\Phi^i[U] \tag{14}$$

holds identically for arbitrary functions $U(x)$. Then, on the solutions $U(x) = u(x)$ of the system (11), if $\Delta_\nu[U]$ is nonsingular, one has local conservation law $\Lambda_\nu[u]\Delta_\nu[u] \equiv D_i\Phi^i[u] = 0$.

Definition 5. The Euler operator with respect to U^j is the operator defined by

$$E_{U^j} = \frac{\partial}{\partial U^j} - D_i\frac{\partial}{\partial U_i^j} + \cdots + (-1)^s D_{i_1}\cdots D_{i_s}\frac{\partial}{\partial U_{i_1\cdots i_s}^j} + \cdots \tag{15}$$

for $j = 1, \ldots, q$ [8].

Theorem 6. *A set of nonsingular local multipliers $\{\Lambda_\nu(x, U, \partial U, \ldots, \partial_U^r)\}_{\nu=0}^l$ yields a local conservation law for the system $\Delta_\nu(x, u^{(n)})$ if and only if the set of identities*

$$E_{U^j}\left(\Lambda_\nu\left(x, U, \partial U, \ldots, \partial^r U\right)\Delta_\nu\left(x, u^{(n)}\right)\right) = 0, \quad j = 1, \ldots, q, \tag{16}$$

holds for arbitrary functions $U(x)$ (Theorem 1.3.3, [8]).

The set of (16) yields the set of linear determining equations to find all sets of local conservation law multipliers of the system (11). Now, we consider all local conservation law multipliers of the forms $\Lambda_1 = \alpha(t, x, u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx})$ and

$\Lambda_2 = \beta(t, x, u, v, u_t, v_t, u_x, v_x, u_{tt}, v_{tt}, u_{xx}, v_{xx})$ of (1). The determining equation (16) for J-M equation is

$$\begin{aligned} E_u \Big[\Lambda_1 \Big(& u_t + u_{xxx} + \frac{3}{2} v v_{xxx} + \frac{9}{2} v_x v_{xx} \\ & -6 u u_x - 6 u v v_x - \frac{3}{2} v^2 u_x \Big) \\ & + \Lambda_2 \Big(v_t + v_{xxx} - 6 v u_x - 6 u v_x - \frac{15}{2} v^2 v_x \Big) \Big] \equiv 0, \\ E_v \Big[\Lambda_1 \Big(& u_t + u_{xxx} + \frac{3}{2} v v_{xxx} + \frac{9}{2} v_x v_{xx} \\ & - 6 u u_x - 6 u v v_x - \frac{3}{2} v^2 u_x \Big) \\ & + \Lambda_2 \Big(v_t + v_{xxx} - 6 v u_x - 6 u v_x - \frac{15}{2} v^2 v_x \Big) \Big] \equiv 0, \end{aligned} \tag{17}$$

where $u(x,t)$ and $v(x,t)$ are arbitrary functions. Equation (17) splits with respect to third order derivatives of u to yield the determining PDE system whose solutions are the sets of local multipliers of all nontrivial local conservation laws of the J-M equation.

The solution of the determining system (17) for J-M equation is given by

$$\begin{aligned} \alpha &= c_1 x + 6 c_1 t u + \frac{9}{2} c_1 t v^2 + c_2 u v + c_3 u_{1/6} c_2 v_{xx} \\ &\quad + \frac{5}{12} c_2 v^3 + \frac{3}{4} c_3 v^2 + c_4 v + c_5, \\ \beta &= -\frac{1}{6} c_2 u_{xx} + \frac{1}{2} c_1 x v - \frac{5}{24} c_2 v_x^2 - \frac{5}{12} v_{xx} - c_2 V - \frac{3}{2} c_1 t v_{xx} \\ &\quad - \frac{1}{4} c_3 v_{xx} + \frac{15}{4} c_1 t v^3 + 9 c_1 t u v + \frac{1}{2} c_2 u^2 + \frac{5}{4} c_2 u v^2 \\ &\quad + \frac{3}{2} c_3 u v + c_4 u + \frac{35}{96} c_2 v^4 + \frac{5}{8} c_3 v^3 + \frac{3}{4} c_4 v^2 + \frac{1}{2} c_5 v + c_6, \end{aligned} \tag{18}$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are arbitrary constants. So local multipliers are given by

(1) $\alpha = 0, \quad \beta = 1,$

(2) $\alpha = 1, \quad \beta = \frac{1}{2} v,$

(3) $\alpha = v, \quad \beta = u + \frac{3}{4} v^2,$

(4) $\alpha = u + \frac{3}{4} v^2, \quad \beta = -\frac{1}{4} v_{xx} + \frac{3}{2} u v + \frac{5}{8} v^3,$

(5) $\alpha = x + 6 t u + \frac{9}{2} t v^2,$

$$\beta = \frac{1}{2} x v - \frac{3}{2} t v_{xx} + \frac{15}{4} t v^3 + 9 t u v,$$

(6) $\alpha = u v - \frac{1}{6} v_{xx} + \frac{5}{12} v^3,$

$$\beta = -\frac{1}{6} u_{xx} - \frac{5}{24} v_x^2 - \frac{5}{12} v v_{xx} + \frac{1}{2} u^2 + \frac{5}{4} u v^2 + \frac{35}{96} v^4. \tag{19}$$

Multipliers α and β determine a nontrivial local conservation law $D_t \Psi + D_x \Phi = 0$ with the characteristic form

$$\begin{aligned} D_t \Psi + D_x \Phi \equiv \alpha \Big(& u_t + u_{xxx} + \frac{3}{2} v v_{xxx} + \frac{9}{2} v_x v_{xx} \\ & - 6 u u_x - 6 u v v_x - \frac{3}{2} v^2 u_x \Big) \\ & + \beta \Big(v_t + v_{xxx} - 6 v u_x - 6 u v_x - \frac{15}{2} v^2 v_x \Big). \end{aligned} \tag{20}$$

The total divergence operator must be inverted to calculate the conserved quantities Φ and Ψ. To do this, we need to integrate (by parts) one of the expressions in multidimensions involving arbitrary functions and its derivatives, which is a difficult task. The homotopy operator [14] is a powerful and useful algorithmic tool (explicit formula) that originates from homological algebra and variational bicomplexes.

Definition 7. The 2-dimensional homotopy operator is a vector operator with two components, $(H^{(x)}_{\mathbf{u}(x,t)} f, H^{(t)}_{\mathbf{u}(x,t)} f)$, where

$$\begin{aligned} H^{(x)}_{\mathbf{u}(x,t)} f &= \int_0^1 \left(\sum_{j=0}^{q} I^{(x)}_{u^j} f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}, \\ H^{(t)}_{\mathbf{u}(x,t)} f &= \int_0^1 \left(\sum_{j=0}^{q} I^{(t)}_{u^j} f \right) [\lambda \mathbf{u}] \frac{d\lambda}{\lambda}. \end{aligned} \tag{21}$$

The x-integrand, $I^{(x)}_{u(x,t)^j} f$, is given by

$$\begin{aligned} I^{(x)}_{u^j} f = \sum_{k_1=1}^{M_1^j} \sum_{k_2=0}^{M_2^j} \Bigg(& \sum_{i_1=0}^{k_1-1} \sum_{i_2=0}^{k_2} B^{(x)} u^j_{x^{i_1} x^{i_2}} (-D_x)^{k_1-i_1-1} \\ & \times (-D_t)^{k_2-i_2} \Bigg) \frac{\partial f}{\partial u^j_{x^{k_1} t^{k_2}}}, \end{aligned} \tag{22}$$

where M_1^j, M_2^j are the order of f in u to x and t, respectively, with combinatorial coefficient $B^{(x)} = B(i_1, i_2, k_1, k_2)$, where

$$B(i_1, i_2, k_1, k_2) = \frac{\binom{i_1+i_2}{i_1} \binom{k_1+k_2-i_1-i_2-1}{k_1-i_1-1}}{\binom{k_1+ki_2}{k_1}}. \tag{23}$$

Similarly, t-integrand, $I^{(t)}_{u(x,t)^j} f$, defined as

$$\begin{aligned} I^{(t)}_{u^j} f = \sum_{k_1=0}^{M_1^j} \sum_{k_2=1}^{M_2^j} \Bigg(& \sum_{i_1=0}^{k_1} \sum_{i_2=0}^{k_2-1} B^{(t)} u^j_{x^{i_1} x^{i_2}} (-D_x)^{k_1-i_1} \\ & \times (-D_t)^{k_2-i_2-1} \Bigg) \frac{\partial f}{\partial u^j_{x^{k_1} t^{k_2}}}, \end{aligned} \tag{24}$$

where $B^{(t)} = B(i_2, i_1, k_2, k_1)$.

We apply homotopy operator to find conserved quantities Ψ and Φ which yield multipliers $\alpha = 0$ and $\beta = 1$. We have

$$\begin{aligned} f &= \alpha\left(u_t + u_{xxx} + \frac{3}{2}vv_{xxx} + \frac{9}{2}v_x v_{xx} \right. \\ &\quad \left. - 6uu_x - 6uvv_x - \frac{3}{2}v^2 u_x\right) \\ &\quad + \beta\left(v_t + v_{xxx} - 6vu_x - 6uv_x - \frac{15}{2}v^2 v_x\right) \\ &= v_t + v_{xxx} - 6vu_x - 6uv_x - \frac{15}{2}v^2 v_x. \end{aligned} \tag{25}$$

The integrands (22) and (24) are

$$\begin{aligned} &I_u^{(t)} f = 0, \qquad I_u^{(x)} f = -6uv, \qquad I_v^{(t)} f = v, \\ &I_v^{(x)} f = -6uv - \frac{15}{2}v^3 + v_{xx}. \end{aligned} \tag{26}$$

Apply (21) to the integrands (26); therefore

$$\begin{aligned} \Phi &= H_{\mathbf{u}(x,t)}^{(x)} f = 6uv - v_{xx} + \frac{5}{2}v^3, \\ \Psi &= H_{\mathbf{u}(x,t)}^{(t)} f = -v. \end{aligned} \tag{27}$$

So, we have the conservation law of the J-M equation with respect to multipliers $\alpha = 0$ and $\beta = 1$:

$$D_t(-v) + D_x\left(6uv - v_{xx} + \frac{5}{2}v^3\right) = 0. \tag{28}$$

And similarly, conservation laws with respect to other multipliers are given as follows:

(1) $\alpha = 1$ and $\beta = (1/2)v$:

$$\begin{aligned} \Phi &= -\frac{1}{4}v^2 - u, \\ \Psi &= \frac{15}{16}v^4 + \frac{9}{2}uv^2 - 2vv_{xx} - \frac{5}{4}v_x^2 + 3u^2 - u_{xx}; \end{aligned} \tag{29}$$

(2) $\alpha = v$ and $\beta = u + (3/4)v^2$:

$$\begin{aligned} \Phi &= -\frac{1}{4}v^3 - uv, \\ \Psi &= \frac{9}{8}v^5 + 6uv^3 - \frac{9}{4}v^2 v_{xx} + 6u^2 v - uv_{xx} - vu_{xx} + u_x v_x; \end{aligned} \tag{30}$$

(3) $\alpha = u + (3/4)v^2$ and $\beta = -(1/4)v_{xx} + (3/2)uv + (5/8)v^3$:

$$\begin{aligned} \Phi &= -\frac{5}{32}v^4 - \frac{3}{4}uv^2 + \frac{1}{8}vv_{xx} - \frac{1}{2}u^2, \\ \Psi &= \frac{25}{32}v^6 + \frac{39}{8}uv^4 - \frac{7}{8}v^3 v_{xx} + \frac{15}{2}v^2 u^2 - \frac{3}{2}uv_x^2 \\ &\quad - 3uvv_{xx} - \frac{3}{4}v^2 u_{xx} + \frac{3}{2}vu_x v_x + 2u^3 + \frac{1}{2}u_x^2 + \frac{1}{8}v_{xx}^2 \\ &\quad - uu_{xx} - \frac{1}{8}vv_{tx} + \frac{1}{8}v_t v_x; \end{aligned} \tag{31}$$

(4) $\alpha = uv - (1/6)v_{xx} + (5/12)v^3$ and $\beta = -(1/6)u_{xx} - (5/24)v_x^2 - (5/12)vv_{xx} + (1/2)u^2 + (5/4)uv^2 + (35/96)v^4$:

$$\begin{aligned} \Phi &= -\frac{7}{96}v^5 - \frac{5}{12}uv^3 + \frac{5}{72}vv_x^2 - \frac{1}{2}vu^2 \\ &\quad + \frac{5}{36}v^2 v_{xx} = \frac{1}{12}vu_{xx} + \frac{1}{12}uv_{xx}, \\ \Psi &= \frac{25}{64}v^7 + \frac{45}{16}uv^5 - \frac{25}{48}v^3 v_x^2 + \frac{23}{4}v^3 u^2 - \frac{95}{96}v^4 v_{xx} \\ &\quad - \frac{11}{4}uv^2 v_{xx} - \frac{5}{12}v^3 u_{xx} + 3vu^3 - \frac{5}{4}uvv_x^2 - \frac{1}{2}u^2 v_{xx} \\ &\quad - \frac{5}{36}v^2 v_{tx} + \frac{5}{36}vv_t v_x + \frac{1}{3}vv_{xx}^2 + \frac{5}{24}v_x^2 v_{xx} - uvu_{xx} \\ &\quad + \frac{1}{12}u_t v_x - \frac{1}{12}vu_{tx} + \frac{1}{12}v_t u_x + \frac{1}{6}u_{xx} v_{xx} - \frac{1}{12}uv_{tx}; \end{aligned} \tag{32}$$

(5) $\alpha = x + 6tu + (9/2)tv^2$ and $\beta = (1/2)xv - (3/2)tv_{xx} + (15/4)tv^3 + 9tuv$:

$$\begin{aligned} \Phi &= -\frac{15}{16}tv^4 - \frac{9}{2}tuv^2 + \frac{3}{4}tvv_{xx} - \frac{1}{4}xv^2 - 3tu^2 - xu, \\ \Psi &= \frac{75}{16}tv^6 + \frac{117}{4}tuv^4 + \frac{15}{16}xv^4 + 45tu^2 v^2 - \frac{21}{2}tv^3 v_{xx} \\ &\quad + 9tvu_x v_x - 9tuv_x^2 + \frac{9}{2}xuv^2 - 18tuvv_{xx} - \frac{9}{2}tv^2 u_{xx} \\ &\quad + 12tu^3 + \frac{5}{4}vv_x + \frac{3}{4}tv_{xx}^2 - 2xvv_{xx} + \frac{3}{4}tv_t v_x - 6tuu_{xx} \\ &\quad + 3xu^2 - \frac{5}{4}xv_x^2 + 3tu_x^2 - \frac{3}{4}tvv_{tx} - xu_{xx} + u_x. \end{aligned} \tag{33}$$

5.2. Symbolic Computation for Finding Conservation Laws Equation. This subsection covers the application of the homotopy operator to the computation of conservation laws of J-M equation. Finding a conservation law needs computing the density Ψ first, followed by computing of the flux Φ. Computing flux Φ will require using homotopy operator. Following the approach by Hereman et al. [9, 14, 15], a candidate density is built as a linear combination (with undetermined coefficients of differential terms) which is invariant under the scaling symmetry of the given PDE. By determining Ψ we can compute $D_t\Psi$ and remove all time derivatives; $D_t\Psi$ must be a divergence. Thus, using Theorem 4.4 of [10], one requires that

$$E_{U^j}(D_t\Psi) = 0, \quad j = 1, \ldots, N. \tag{34}$$

This leads to a linear system for the undetermined coefficients. Substituting its solution into the candidate for Ψ gives the actual density. Finally, the $\Phi = \mathrm{div}^{-1}(D_t\Psi)$ is computed with the homotopy operator.

Jaulent-Miodek equation is invariant under the scaling (dilation) symmetry (4):

$$(t, x, u, v) \longrightarrow \left(\lambda^3 t, \lambda x, \lambda^{-2} u, \lambda^{-1} v\right). \tag{35}$$

Conservation law (13) must hold on solutions of (1). Therefore, we search for polynomial conservation laws that obey the scaling symmetry of the PDE. Indeed, we have to find a polynomial conservation law that does not adhere to the scaling symmetry. We choose a scaling factor for one of the components of (13). The selected scaling factor will be called the rank (R) of that component. Then, we construct a candidate for that component as a linear combination of monomial terms (all of rank R) with undetermined coefficients. If we remove divergence and divergence-equivalent terms dynamically that candidate will be shorted and of low order.

For J-M equation we will compute the density Ψ of a fixed rank; for example, $R = -3$. We construct a list of differential terms which contains all powers of dependent variables and their derivatives and products of them of rank -3:

$$\begin{aligned} Q = \{ & u_x^3, v^3, tu^3, xu^2, uu_x, uv, u^2u_{xxx}, u_x^3u_{xx}, v^3u_{xx}, vu_x^2, \\ & v_xu_x^3, v^3v_x, tu^3u_{xx}, xu^2u_{xx}, uu_xu_{xx}, uvu_{xx}, \\ & u^2u_{xx}u_{xxx}, vu^2u_{xx}, v^2u_xu_{xx}, tu^3v_x, xu^2v_x, uv_xu_x, \\ & uvv_x, u^2v_xu_{xxx}, vv_xu_x^2, v^2u_xv_x \}. \end{aligned} \tag{36}$$

By removing all terms that are divergences or divergence-equivalent to other terms in Q, we have

$$\begin{aligned} Q = \{ & u_x^3, v^3, tu^3, xu^2, v^2u_x, v^3u_{xx}, vu_x^2, v_xu_x^3, \\ & tu^3u_{xx}, xu^2u_{xx}, uvu_{xx}, u^2u_{xx}u_{xxx}, vu_x^2u_{xx}, \\ & v^2u_xu_{xx}, tu^3v_x, xu^2v_x, uu_xv_x, u^2v_xu_{xxx}, \\ & vv_xu_x^2, v^2v_xu_x \}. \end{aligned} \tag{37}$$

Now, by forming a candidate density combining the terms in Q linearly with undetermined coefficients c_i,

$$\begin{aligned} \Psi = {} & c_1u_x^3 + c_2v^3 + c_3tu^3 + c_4xu^2 + c_5v^2u_x + c_6v^3u_{xx} \\ & + c_7vu_x^2 + c_8v_xu_x^3 + c_9tu^3u_{xx} + c_{10}xu^2u_{xx} \\ & + c_{11}uvu_{xx} + c_{12}u^2u_{xx}u_{xxx} + c_{13}vu_x^2u_{xx} \\ & + c_{14}v^2u_xu_{xx} + c_{15}tu^3v_x + c_{16}xu^2v_x + c_{17}uu_xv_x \\ & + c_{18}u^2v_xu_{xxx} + c_{19}vv_xu_x^2 + c_{20}v^2v_xu_x. \end{aligned} \tag{38}$$

Compute the total derivative with respect to t of (38), and set

$$F = -D_t\Psi. \tag{39}$$

After replacing u_t with $-u_{xxx} - (3/2)vv_{xxx} - (9/2)v_xv_{xx} + 6uu_x + 6uvv_x + (3/2)v^2u_x$ and v_t by $-v_{xxx} + 6vu_x + 6uv_x + (15/2)v^2v_x$, (39) must be a divergence, use (34), and require

$$E_{u(t,x)}F = 0, \qquad E_{v(x,t)}F = 0. \tag{40}$$

The solution of system (40) is

$$\begin{aligned} & c_1 = 0, \quad c_2 = 0, \quad c_3 = 0, \quad c_4 = 0, \\ & c_5 = 0, \quad c_6 = c_{20}, \quad c_7 = c_{11}, \quad c_8 = \frac{1}{3}c_{13}, \\ & c_9 = 0, \quad c_{10} = 0, \quad c_{12} = 0, \quad c_{14} = c_{19}, \\ & c_{15} = 0, \quad c_{16} = 0, \quad c_{17} = c_{11}, \quad c_{18} = 0, \end{aligned} \tag{41}$$

where c_{11}, c_{13}, c_{19}, and c_{20} is arbitrary.

Case 1. Substitute (41) and $c_{11} = 1$, $c_{13} = 0$, $c_{19} = 0$, and $c_{20} = 0$ into (38) and (39) given $\Psi = vu_x^2 + uvu_{xx} + uu_xv_x$ and

$$\begin{aligned} F = {} & u_xv_xu_{xxx} + 2vu_xu_{xxxx} + \frac{9}{2}u_xv_x^2v_{xx} + 9vu_xv_{xx}^2 \\ & + uu_xv_{xxxx} + 3v^2u_xv_{xxxx} + \frac{9}{2}uv_xv_{xx} + uv_xu_{xxxx} \\ & + 6uv_x^2v_{xxx} + \frac{3}{2}uv^2v_{xxxxx} + \frac{3}{2}v^2u_{xx}v_{xxx} + vu_{xx}u_{xxx} \\ & + vuu_{xxxxx} + uu_{xx}v_{xxx} - 12u^2v_xu_{xx} - \frac{9}{2}v^3u_xu_{xx} \\ & - \frac{3}{2}uv^3u_{xxx} - 6u^2v^2v_{xxx} - 6vu^2u_{xxx} - 6u^2u_xv_{xx} \\ & - 30uu_x^2v_x - 27v^2v_xu_x^2 + u_x^2v_{xxx} - 6u^2v_x^3 - 18vu_x^3 \\ & + \frac{27}{3}vu_xv_xv_{xxx} + \frac{9}{2}vv_xv_{xx}u_{xx} + 9uvv_xv_{xxxx} \\ & + 15uvv_{xx}v_{xxx} - \frac{69}{2}uv^2u_xv_{xx} - 27uv^2v_xu_{xx} \\ & - 48uvu_xu_{xx} - 57uvu_xv_x^2 - 24vu^2v_xv_{xx}. \end{aligned} \tag{42}$$

Since $F = \operatorname{div}\Phi$, the flux Φ can be computed with the 1D homotopy operator which inverts divergences. Applying 1D homotopy operator formulas in (21) and removing curl term of flux Φ yield

$$\begin{aligned} \Phi = {} & -\frac{3}{2}v^3u_x^2 - \frac{3}{2}uv^3u_{xx} - 6v^2u^2v_{xx} - \frac{45}{2}uv^2u_xv_x \\ & - 18uvu_x^2 - \frac{3}{2}v^2u_xv_{xxx} + \frac{9}{2}vv_xu_xu_{xx} + \frac{3}{2}uv^2v_{xxxx} \\ & - 6vu^2u_{xx} + 6uvv_xv_{xxx} + \frac{9}{2}uvv_{xx}^2 - 6u^2u_xv_x \\ & + vu_xu_{xxx} + uvu_{xxxx} + uu_xv_{xxx}. \end{aligned} \tag{43}$$

Case 2. Substitute $c_{11} = 0$, $c_{13} = 1$, $c_{19} = 0$, and $c_{20} = 0$ into (38), given

$$\Psi = \frac{1}{3}v_xu_x^3 + vu_x^2u_{xx},$$

$$\Phi = -\frac{1}{6}u_x^2\left(36uv^2v_{xx} + 9v^3u_{xx} + 69v^2u_xv_x\right.$$

$$+ 36uvv_x^2 - 36vv_xv_{xxx} + 12uu_xv_x - 27vv_{xx}^2 + 36uvu_{xx} + 48vu_x^2 - 9v^2v_{xxxx} - 6vu_{xxxx} - 2u_xv_{xxx}\big). \tag{44}$$

Case 3. Substitute $c_{11} = 0$, $c_{13} = 0$, $c_{19} = 1$, $c_{20} = 0$,

$$\begin{aligned}\Psi &= v^2u_xu_{xx} + vv_xu_x^2,\\ \Phi &= -\frac{1}{2}vu_x\big(12uv^2v_{xx} + 3v^3u_{xx} + 33v^2u_xv_x + 12uvv_x^2 - 12vv_xv_{xxx} + 12uvu_{xx} + 12uu_xv_x - 3v^2v_{xxxx} + 24vu_x^2 - 9vv_{xx}^2 - 2vu_{xxxx} - 2u_xv_{xxx}\big).\end{aligned} \tag{45}$$

And finally, $c_{11} = 0$, $c_{13} = 0$, $c_{19} = 0$, $c_{20} = 1$,

$$\begin{aligned}\Psi &= \frac{1}{3}v^3u_{xx} + v^2u_xv_x,\\ \Phi &= -\frac{1}{6}v^2\big(3v^3u_{xx} + 12uv^2v_{xx} + 63v^2v_xu_x + 12uvv_x^2 + 48vu_x^2 - 9vv_{xx}^2 - 12vv_xv_{xxx} + 36uu_xv_x - 3v^2v_{xxxx} + 12uvu_{xx} - 2vu_{xxxx} - 6u_xv_{xxx}\big).\end{aligned} \tag{46}$$

6. Conclusion

In this paper, we studied Jaulent-Miodek equation using the Lie symmetry group of infinitesimal transformations of the equation. We found that the underlying equation admits a three-dimensional Lie algebra. We obtained the optimal system of one-dimensional subalgebras of the Lie algebra of the equation. These subalgebras were then used to reduce the underlying equation to nonlinear third order ordinary system of differential equations. Further conservation laws are constructed for this equation in two methods. First, conservation laws of the equation are obtained by finding multipliers; then some other conservation laws of J-M equation are obtained with symbolic computation of conservation laws.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publishing of this paper.

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