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Research Article

On Skew Circulant Type Matrices Involving Any Continuous Fibonacci Numbers

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Circulant and skew circulant matrices have become an important tool in networks engineering. In this paper, we consider skew circulant type matrices with any continuous Fibonacci numbers. We discuss the invertibility of the skew circulant type matrices and present explicit determinants and inverse matrices of them by constructing the transformation matrices. Furthermore, the maximum column sum matrix norm, the spectral norm, the Euclidean (or Frobenius) norm, and the maximum row sum matrix norm and bounds for the spread of these matrices are given, respectively.

1. Introduction

Skew circulant and circulant matrices have important applications in various networks engineering. Joy and Tavsanoglu [1] showed that feedback matrices of ring cellular neural networks, which can be described by the ODE, are block circulants. A special class of the feedback delay network using circulant matrices was proposed [2]. Jing and Jafarkhani [3] proposed distributed differential space-time codes that work for networks with any number of relays using circulant matrices. Exploiting the circulant structure of the channel matrices, Eghbali et al. [4] analysed the realistic near fast fading scenarios with circulant frequency selective channels. Rocchesso [5] presented particular choices of the feedback coefficients, namely, Galois sequences, arranged in a circulant matrix, to produce a maximum echo density in the time response. Sardellitti et al. [6] used an analytical expression for the eigenvalues of a block circulant matrix as a function of the coverage radius. Li et al. [7] gave a low-complexity binary frame-wise network coding encoder design based on circulant matrix. Hirt and Massey [8] introduced discrete time Fourier transform precoding to the proposed multihop relay system involving circulant matrix. When considering a single-input single-output transmission with CFO and omitting the relay index subscript, Wang et al. [9] proved that the intercarrier interference matrix is a circulant matrix. The system model of the OFDM based AF relay networks as well as the strategy of the superimposed training involves circulant matrix [10]. Two-way transmission model considered in [11] ensured the circular convolution between two frequency selective channels.

The skew circulant matrices as preconditioners for linear multistep formulae- (LMF-) based ordinary differential equations (ODEs) codes, Hermitian, and skew-Hermitian Toeplitz systems were considered in [12–15]. Lyness and Sørevik employed a skew circulant matrix to construct s-dimensional lattice rules in [16]. Compared with cyclic convolution algorithm, the skew cyclic convolution algorithm [17] was able to perform filtering procedure in approximately half of computational cost for real signals. In [18] two new normal-form realizations were presented which utilize circulant and skew circulant matrices as their state transition matrices. The well-known second-order coupled form is a special case of the skew circulant form. Li et al. [19] gave the style spectral decomposition of skew circulant matrix firstly and then dealt with the optimal backward perturbation analysis for the linear system with skew circulant coefficient matrix. In [20], a new fast algorithm for optimal design of block digital filters (BDFs) was proposed based on skew circulant matrix.

Besides, some scholars have given various algorithms for the determinants and inverses of nonsingular circulant matrices. Unfortunately, the computational complexity of

these algorithms is very amazing huge with the order of matrix increasing. However, some authors gave the explicit determinants and inverses of circulant and skew circulant matrices involving some famous numbers. For example, Yao and Jiang [21] considered the determinants, inverses, norm, and spread of skew circulant type matrices involving any continuous Lucas numbers. Shen et al. considered circulant matrices with Fibonacci and Lucas numbers and presented their explicit determinants and inverses by constructing the transformation matrices [22]. Gao et al. [23] gave explicit determinants and inverses of skew circulant and skew left circulant matrices with Fibonacci and Lucas numbers. Jiang et al. [24, 25] considered the skew circulant and skew left circulant matrices with the k-Fibonacci numbers and the k-Lucas numbers and discussed the invertibility of the these matrices and presented their determinant and the inverse matrix by constructing the transformation matrices, respectively. Jaiswal evaluated some determinants of circulant whose elements are the generalized Fibonacci numbers [26]. Lind presented the determinants of circulant and skew circulant involving Fibonacci numbers [27]. Dazheng [28] gave the determinant of the Fibonacci-Lucas quasi-cyclic matrices.

Recently, there are several papers on the norms of some special matrices. Solak [29] established the lower and upper bounds for the spectral norms of circulant matrices with classical Fibonacci and Lucas numbers entries. İpek [30] investigated an improved estimation for spectral norms of these matrices. Shen and Cen [31] gave upper and lower bounds for the spectral norms of r-circulant matrices in the form of $A = C_r(F_0, F_1, \ldots, F_{n-1})$, $B = C_r(L_0, L_1, \ldots, L_{n-1})$, and they also obtained some bounds for the spectral norms of Kronecker and Hadamard products of matrix A and matrix B. Akbulak and Bozkurt [32] found upper and lower bounds for the spectral norms of Toeplitz matrices such that $a_{ij} \equiv F_{i-j}$ and $b_{ij} \equiv L_{i-j}$. The convergence in probability and in distribution of the spectral norm of scaled Toeplitz, circulant, reverse circulant, symmetric circulant, and a class of k-circulant matrices were discussed in [33].

Beginning with Mirsky [34], several authors [35–37] have obtained bounds for the spread of a matrix.

The Fibonacci sequences are defined by the following recurrence relations [22, 23, 26–32]:

$$F_{n+1} = F_n + F_{n-1} \quad \text{where } F_0 = 0,\ F_1 = 1. \tag{1}$$

The $\{F_n\}$ is given by the formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \tag{2}$$

where α and β are the roots of the characteristic equation $x^2 - x - 1 = 0$.

The Fibonacci sequences were introduced for the first time by the famous Italian mathematician Leonardo of Pisa (nicknamed Fibonacci). It is well known that the ratio of two consecutive classical Fibonacci numbers converges to the golden mean, or the golden section, $(1+\sqrt{5})/2$, which appears in modern research in many fields from architecture [38, 39] to physics of high energy particles [40]. As is shown in [41, 42], the hyperbolic Fibonacci functions can lead to creation of the Lobachevsky Fibonacci and Minkovsky Fibonacci geometry which are of great importance for theoretical physics. In the 19th century the French mathematician Francois Edouard Anatole Lucas (1842–1891) introduced the so-called Lucas numbers given by the recursive relation $L_n = L_{n-1} + L_{n-2}, n \geq 2$, with the seeds $L_0 = 2$ and $L_1 = 1$. The determinants, inverses, norm, and spread of skew circulant type matrices involving any continuous Lucas numbers are considered in [21].

The purpose of this paper is to obtain the explicit determinants, explicit inverses, norm, and spread of skew circulant type matrices involving any continuous Fibonacci numbers. And we generalize the result [23]. In passing, the norm and spread of skew circulant type matrices have not been research. It is hoped that this paper will help in changing this.

In the following, let r be a nonnegative integer. We adopt the following two conventions $0^0 = 1$ and, for any sequence $\{a_n\}$, $\sum_{k=i}^{n} a_k = 0$ in case $i > n$.

Definition 1 (see [21]). A skew circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$ is meant to be a square matrix of the form

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_{n-1} & a_n \\ -a_n & a_1 & a_2 & \cdots & a_{n-1} \\ \vdots & -a_n & a_1 & \ddots & \vdots \\ -a_3 & \vdots & \ddots & \ddots & a_2 \\ -a_2 & -a_3 & \cdots & -a_n & a_1 \end{pmatrix}_{n\times n}, \tag{3}$$

denoted by $\mathrm{SCirc}(a_1, a_2, \ldots, a_n)$.

Definition 2 (see [21]). A skew left circulant matrix with the first row $(a_1, a_2, \ldots, a_n)$ is meant to be a square matrix of the form

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ a_2 & a_3 & \cdots & a_n & -a_1 \\ a_3 & \iddots & \iddots & \iddots & \vdots \\ \vdots & a_n & -a_1 & \cdots & -a_{n-2} \\ a_n & -a_1 & \cdots & -a_{n-2} & -a_{n-1} \end{pmatrix}_{n\times n}, \tag{4}$$

denoted by $\mathrm{SLCirc}(a_1, a_2, \ldots, a_n)$.

Lemma 3 (see [30, 31]). *Let $\{F_n\}$ be Fibonacci numbers; then,*

$$(i)\ \sum_{i=0}^{n-1} F_i = F_{n+1} - 1, \tag{5}$$

$$(ii)\ \sum_{i=0}^{n-1} F_i^2 = F_n F_{n-1}, \tag{6}$$

$$(iii)\ \sum_{i=0}^{n-1} iF_i = (n-1)F_{n+1} - F_{n+2} + 2. \tag{7}$$

2. Determinant and Inverse of Skew Circulant Matrix with the Fibonacci Numbers

In this section, let $B_{r,n} = \mathrm{SCirc}(F_{r+1}, \ldots, F_{r+n})$ be a skew circulant matrix. Firstly, we give a determinant explicit formula for the matrix $B_{r,n}$. Afterwards, we prove that $B_{r,n}$ is an invertible matrix for $n \geq 2$, and then we find the inverse of the matrix $B_{r,n}$. In the following, let

$$\begin{aligned} x &= -\frac{F_r + F_{r+n}}{F_{r+1} + F_{r+n+1}}, \qquad s = \frac{F_{r+2}}{F_{r+1}}, \\ b &= F_{r+1} + F_{r+n+1}, \qquad a = F_r + F_{r+n}, \\ f_n &= F_{r+1} + sF_{r+n} \\ &\quad + \sum_{k=1}^{n-2} \left(sF_{r+k+1} - F_{r+k+2}\right) \cdot x^{n-(k+1)}, \\ f_{n'} &= \sum_{k=1}^{n-1} F_{r+k+1} \cdot x^{n-(k+1)}. \end{aligned} \tag{8}$$

Theorem 4. *Let $B_{r,n} = SCirc(F_{r+1}, \ldots, F_{r+n})$ be a skew circulant matrix; then,*

$$\det B_{r,n} = F_{r+1} \cdot f_n \cdot b^{n-2}, \tag{9}$$

where F_{r+n} is the $(r+n)$th Fibonacci number. In particular, when $r = 0$, we get the result of [23].

Proof. Obviously, $\det B_{r,2} = F_{r+1}^2 + F_{r+2}^2$ satisfies the equation. In case $n > 2$, let

$$\Gamma = \begin{pmatrix} 1 & & & & & & \\ s & & & & & & 1 \\ 1 & & & & & 1 & -1 \\ 0 & & 0 & & 1 & -1 & -1 \\ \vdots & & & \iddots & \iddots & \iddots & \\ 0 & & 1 & \iddots & \iddots & & \\ 0 & 1 & -1 & \iddots & & 0 & \\ 0 & 1 & -1 & -1 & & & \end{pmatrix}, \tag{10}$$

$$\Pi_1 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & x^{n-2} & 0 & \cdots & 0 & 1 \\ 0 & x^{n-3} & 0 & \cdots & 1 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & x & 1 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

be two $n \times n$ matrices; then, we have

$$\Gamma B_{r,n} \Pi_1 = \begin{pmatrix} F_{r+1} & f_n' & b_{13} & \cdots & b_{1(n-1)} & b_{1n} \\ 0 & f_n & b_{23} & \cdots & b_{2(n-1)} & b_{2n} \\ 0 & 0 & b & & & \\ 0 & 0 & a & \ddots & & \\ \vdots & \vdots & & \ddots & b & \\ 0 & 0 & & & a & b \end{pmatrix}, \tag{11}$$

where

$$\begin{aligned} b_{1j} &= F_{r+n+2-j}, \\ b_{2j} &= sF_{r+n+2-j} - F_{r+n+3-j}, \quad (j = 3, 4, \ldots, n). \end{aligned} \tag{12}$$

So it holds that

$$\begin{aligned} &\det \Gamma \det B_{r,n} \det \Pi_1 \\ &\quad = F_{r+1} \cdot f_n \cdot b^{n-2} \\ &\quad = F_{r+1} \Bigg[F_{r+1} + sF_{r+n} \\ &\qquad\qquad + \sum_{k=1}^{n-2} \left(sF_{r+1+k} - F_{r+2+k}\right) \cdot x^{n-(k+1)} \Bigg] \\ &\qquad \times \left(F_{r+1} + F_{r+n+1}\right)^{n-2}, \end{aligned} \tag{13}$$

while taking $\det \Gamma = \det \Pi_1 = (-1)^{(n-1)(n-2)/2}$, we can get

$$\begin{aligned} \det B_{r,n} &= F_{r+1} \Bigg[\frac{F_{r+2}}{F_{r+1}} \cdot F_{r+n} + F_{r+1} \\ &\qquad + \sum_{k=1}^{n-2} \left(\frac{F_{r+2}}{F_{r+1}} \cdot F_{r+1+k} - F_{r+2+k} \right) \cdot x^{n-(k+1)} \Bigg] \\ &\qquad \times \left(F_{r+1} + F_{r+n+1}\right)^{n-2}. \end{aligned} \tag{14}$$

This completes the proof. □

Theorem 5. *Let $B_{r,n} = SCirc(F_{r+1}, \ldots, F_{r+n})$ be a skew circulant matrix; then, $B_{r,n}$ is an invertible matrix. Specially, when $r = 0$, we get the result of [23].*

Proof. Taking $n = 2$ in Theorem 4, we have $\det B_{r,2} = F_{r+1}^2 + F_{r+2}^2 \neq 0$. Hence $B_{r,2}$ is invertible. In case $n > 2$, since $F_{r+n} = (\alpha^{r+n} - \beta^{r+n})/(\alpha - \beta)$, where $\alpha + \beta = 1, \alpha\beta = -1$, we obtain

$$\begin{aligned} &f\left(\omega^k \eta\right) \\ &\quad = \sum_{j=1}^{n} F_{r+j} \left(\omega^k \eta\right)^{j-1} \\ &\quad = \frac{1}{\alpha - \beta} \sum_{j=1}^{n} \left(\alpha^{r+j} - \beta^{r+j}\right) \left(\omega^k \eta\right)^{j-1} \\ &\quad = \frac{1}{\alpha - \beta} \left[\frac{\alpha^{r+1}\left(1 + \alpha^n\right)}{1 - \alpha\omega^k\eta} - \frac{\beta^{r+1}\left(1 + \beta^n\right)}{1 - \beta\omega^k\eta} \right] \end{aligned}$$

$$= \frac{1}{\alpha-\beta}\left[\frac{\left(\alpha^{r+1}-\beta^{r+1}\right)+\left(\alpha^{r+n+1}-\beta^{r+n+1}\right)}{1-(\alpha+\beta)\omega^k\eta+\alpha\beta\omega^{2k}\eta^2}\right.$$
$$\left.-\frac{\alpha\beta\left(\alpha^r-\beta^r+\alpha^{r+n}-\beta^{r+n}\right)\omega^k\eta}{1-(\alpha+\beta)\omega^k\eta+\alpha\beta\omega^{2k}\eta^2}\right]$$
$$= \frac{F_{r+1}+F_{r+n+1}+\left(F_r+F_{r+n}\right)\omega^k\eta}{1-\omega^k\eta-\omega^{2k}\eta^2}$$
$$(k=1,2,\ldots,n-1), \tag{15}$$

where $\omega = \exp(2\pi i/n), \eta = \exp(\pi i/n)$. If there exists $\omega^l\eta(l = 1,2,\ldots,n-1)$ such that $f(\omega^l\eta) = 0$, we obtain $F_{r+1}+F_{r+n+1}+(F_r+F_{r+n})\omega^k\eta = 0$ for $1-\omega^l\eta-\omega^{2l}\eta^2 \neq 0$, and hence it follows that $\omega^l\eta = -(F_{r+1}+F_{r+n+1})/(F_r+F_{r+n})$ is a real number. Since

$$\omega^l\eta = \exp\left(\frac{(2l+1)\pi i}{n}\right) = \cos\frac{(2l+1)\pi}{n} + i\sin\frac{(2l+1)\pi}{n}, \tag{16}$$

it yields that $\sin((2l+1)\pi/n) = 0$, so we have $\omega^l\eta = -1$ for $0 < (2l+1)\pi/n < 2\pi$. Since $x = -1$ is not the root of the equation,

$$F_{r+1}+F_{r+n+1}+\left(F_r+F_{r+n}\right)x = 0, \quad (n>2). \tag{17}$$

We obtain $f(\omega^k\eta) \neq 0$ for any $\omega^k\eta(k = 1,2,\ldots,n-1)$, while

$$f(\eta) = \sum_{j=1}^{n} F_{r+j}\eta^{j-1} = \frac{F_{r+1}+F_{r+n+1}+\left(F_r+F_{r+n}\right)\eta}{1-\eta-\eta^2} \neq 0. \tag{18}$$

It follows from Lemma 3 in [21] that the conclusion holds. □

Lemma 6. *Let the matrix $\mathcal{G} = [g_{i,j}]_{i,j=1}^{n-2}$ be of the form*

$$g_{ij} = \begin{cases} F_{r+1}+F_{r+n+1}, & i=j, \\ F_r+F_{r+n}, & i=j+1, \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

Then the inverse $\mathcal{G}^{-1} = [g'_{i,j}]_{i,j=1}^{n-2}$ of $\mathcal{G}$ is equal to

$$g'_{i,j} = \begin{cases} \dfrac{\left[-(F_r+F_{r+n})\right]^{i-j}}{\left(F_{r+1}+F_{r+n+1}\right)^{i-j+1}}, & i \geq j, \\ 0, & i < j. \end{cases} \tag{20}$$

In particular, when $r = 0$, we get the result of [23].

Proof. Let $c_{ij} = \sum_{k=1}^{n-2} g_{ik}g'_{kj}$.

Then $c_{i,j} = 0$, for $i < j$, $c_{ii} = g_{ii}g'_{ii} = (F_{r+1}+F_{r+n+1}) \cdot 1/(F_{r+1}+F_{r+n+1}) = 1$, for $i = j$, and

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n-2} g_{ik}g'_{kj} = g_{i,i-1}g'_{i-1,j} + g_{i,i}g'_{i,j} \\ &= \left(F_r+F_{r+n}\right)\cdot\frac{\left[-(F_r+F_{r+n})\right]^{i-j-1}}{\left(F_{r+1}+F_{r+n+1}\right)^{i-j}} \\ &\quad + \left(F_{r+1}+F_{r+n+1}\right)\cdot\frac{\left[-(F_r+F_{r+n})\right]^{i-j}}{\left(F_{r+1}+F_{r+n+1}\right)^{i-j+1}} \\ &= 0, \end{aligned} \tag{21}$$

for $i \geq j+1$.

Hence, we get $\mathcal{G}\mathcal{G}^{-1} = I_{n-2}$, where I_{n-2} is an $(n-2)\times(n-2)$ identity matrix. Similarly, we can verify $\mathcal{G}^{-1}\mathcal{G} = I_{n-2}$. Thus, the proof is completed. □

Theorem 7. *Let $B_{r,n} = SCirc(F_{r+1},\ldots,F_{r+n})$ be a skew circulant matrix; then,*

$$B_{r,n}^{-1} = \frac{1}{f_n}\cdot SCirc\left(x'_1,x'_2,\ldots,x'_n\right), \tag{22}$$

where

$$\begin{aligned} x'_1 &= 1-\left(F_{r+3}-sF_{r+2}\right)\cdot\frac{(-a)^{n-3}}{b^{n-1}} \\ &\quad -\sum_{i=1}^{n-3}\left(F_{r+n+2-i}-sF_{r+n+1-i}\right)\cdot\frac{(-a)^{i-1}}{b^i}, \\ x'_2 &= -s-\sum_{i=1}^{n-2}\left(F_{r+n+1-i}-sF_{r+n-i}\right)\cdot\frac{(-a)^{i-1}}{b^i}, \\ x'_3 &= -\left(F_{r+3}-sF_{r+2}\right)\cdot\frac{1}{b}, \\ x'_4 &= -\sum_{i=1}^{2}\left(F_{r+1+i}-sF_{r+i}\right)\cdot\frac{(-a)^{i-1}}{b^i}, \\ x'_k &= -\frac{1}{f_n}\sum_{i=1}^{2}\left(F_{r+1+i}-sF_{r+i}\right)\cdot\frac{(-a)^{k-5+i}}{b^{k-4+i}}, \\ &\qquad (k=5,6,\ldots,n). \end{aligned} \tag{23}$$

In particular, when $r = 0$, we get the result of [23].

Proof. Let

$$\Pi_2 = \begin{pmatrix} 1 & -\dfrac{f'_n}{F_{r+1}} & \pi_{13} & \pi_{14} & \cdots & \pi_{1n} \\ 0 & 1 & \pi_{23} & \pi_{24} & \cdots & \pi_{2n} \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \tag{24}$$

where

$$\pi_{1j} = \frac{1}{F_{r+1}}\left[\frac{f_n'}{f_n}\left(sF_{r+n+2-j} - F_{r+n+3-j}\right) - F_{r+n+2-j}\right], \quad (j = 3, 4, \ldots, n), \tag{25}$$

$$\pi_{2j} = -\frac{sF_{r+n+2-j} - F_{r+n+3-j}}{f_n}, \quad (j = 3, 4, \ldots, n).$$

Then we have

$$\Gamma B_{r,n}\Pi_1\Pi_2 = \begin{pmatrix} F_{r+1} & 0 & 0 & 0 & \cdots & 0 \\ 0 & f_n & 0 & 0 & \cdots & 0 \\ 0 & 0 & b & 0 & \cdots & 0 \\ 0 & 0 & a & b & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & b \end{pmatrix}. \tag{26}$$

$\Gamma B_{r,n}\Pi_1\Pi_2 = D_1 \oplus \mathscr{G}$, $D_1 = \operatorname{diag}(F_{r+1}, f_n)$ is a diagonal matrix, and $\mathscr{D}_1 \oplus \mathscr{G}$ is the direct sum of $\mathscr{D}_1$ and $\mathscr{G}$. If we denote $\Pi = \Pi_1\Pi_2$, then we obtain $B_{r,n}^{-1} = \Pi(\mathscr{D}_1^{-1} \oplus \mathscr{G}^{-1})\Gamma$.

Since the last row elements of the matrix Π are $(0, 1, \pi_{23}, \pi_{24}, \ldots, \pi_{2(n-1)}, \pi_{2n})$, then the last row elements of the matrix $\Pi(\mathscr{D}_1^{-1} \oplus \mathscr{G}^{-1})$ are $(0, 1/f_n, U_{23}, U_{24}, \ldots, U_{2n})$, where

$$U_{23} = \sum_{i=1}^{n-2} \pi_{2(2+i)} \cdot \frac{(-a)^{i-1}}{b^i},$$
$$U_{2k} = \sum_{i=1}^{n+1-k} \pi_{2(k-1+i)} \cdot \frac{(-a)^{i-1}}{b^i}, \quad (k = 3, 4, \ldots, n). \tag{27}$$

Hence it follows from Lemma 6 that letting $B_{r,n}^{-1} = \operatorname{SCirc}(x_1, x_2, \ldots, x_n)$ then its last row elements are $(-x_2, -x_3, \ldots, -x_n, x_1)$ which are given by the following equations:

$$\begin{aligned}
-x_2 &= \frac{s}{f_n} + U_{23} = \frac{s}{f_n} + \sum_{i=1}^{n-2} \pi_{2(2+i)} \cdot \frac{(-a)^{i-1}}{b^i} \\
&= \frac{s}{f_n} + \frac{1}{f_n} \cdot \sum_{i=1}^{n-2} \left(F_{r+n+1-i} - sF_{r+n-i}\right) \cdot \frac{(-a)^{i-1}}{b^i}, \\
-x_3 &= U_{2n} = \frac{1}{f_n} \cdot \frac{F_{r+3} - sF_{r+2}}{b}, \\
-x_4 &= U_{2(n-1)} - U_{2n} = \frac{1}{f_n}\left[\left(F_{r+2} - sF_{r+1}\right) \cdot \frac{1}{b} + \left(F_{r+3} - sF_{r+2}\right) \cdot \frac{-a}{b^2}\right] \\
&= \frac{1}{f_n} \sum_{i=1}^{2} \left(F_{r+1+i} - sF_{r+i}\right) \cdot \frac{(-a)^{i-1}}{b^i}, \\
-x_k &= U_{2(k-1)} - U_{2k} - U_{2(k+1)} \\
&= \frac{1}{f_n}\left[\left(F_{r+2} - sF_{r+1}\right) \cdot \frac{(-a)^{k-4}}{b^{k-3}} + \left(F_{r+3} - sF_{r+2}\right) \cdot \frac{(-a)^{k-3}}{b^{k-2}}\right] \\
&= \frac{1}{f_n} \sum_{i=1}^{2} \left(F_{r+1+i} - sF_{r+i}\right) \cdot \frac{(-a)^{k-5+i}}{b^{k-4+i}}, \\
&\vdots \\
-x_n &= U_{23} - U_{24} - U_{25} \\
&= \frac{1}{f_n} \sum_{i=1}^{n-2} \left(F_{r+n+1-i} - sF_{r+n-i}\right) \cdot \frac{(-a)^{i-1}}{b^i} \\
&\quad - \frac{1}{f_n} \sum_{i=1}^{n-3} \left(F_{r+n-i} - sF_{r+n-1-i}\right) \cdot \frac{(-a)^{i-1}}{b^i} \\
&\quad - \frac{1}{f_n} \sum_{i=1}^{n-1} \left(F_{r+n-1-i} - sF_{r+n-2-i}\right) \cdot \frac{(-a)^{i-1}}{b^i} \\
&= \frac{1}{f_n}\left[\left(F_{r+2} - sF_{r+1}\right) \cdot \frac{(-a)^{n-4}}{b^{n-3}} + \left(F_{r+3} - sF_{r+2}\right) \cdot \frac{(-a)^{n-3}}{b^{n-2}}\right] \\
&= \frac{1}{f_n} \sum_{i=1}^{2} \left(F_{r+1+i} - sF_{r+i}\right) \cdot \frac{(-a)^{n-5+i}}{b^{n-4+i}}, \\
x_1 &= \frac{1}{f_n} - U_{23} - U_{24} \\
&= \frac{1}{f_n} - \frac{1}{f_n} \sum_{i=1}^{n-2} \left(F_{r+n+1-i} - sF_{r+n-i}\right) \cdot \frac{(-a)^{i-1}}{b^i} \\
&\quad - \frac{1}{f_n} \sum_{i=1}^{n-3} \left(F_{r+n+2-i} - sF_{r+n+1-i}\right) \cdot \frac{(-a)^{i-1}}{b^i} \\
&= \frac{1}{f_n}\left[1 - \left(F_{r+3} - sF_{r+2}\right) \cdot \frac{(-a)^{n-3}}{b^{n-2}} - \sum_{i=1}^{n-3} \left(F_{r+n+2-i} - sF_{r+n+1-i}\right) \cdot \frac{(-a)^{i-1}}{b^i}\right].
\end{aligned} \tag{28}$$

Hence, we obtain

$$\begin{aligned}
x_1 &= \frac{1}{f_n}\left[1-\left(F_{r+3}-sF_{r+2}\right)\cdot\frac{(-a)^{n-3}}{b^{n-1}}\right.\\
&\qquad\left.-\sum_{i=1}^{n-3}\left(F_{r+n+2-i}-sF_{r+n+1-i}\right)\cdot\frac{(-a)^{i-1}}{b^i}\right],\\
x_2 &= -\frac{s}{f_n}-\frac{1}{f_n}\sum_{i=1}^{n-2}\left(F_{r+n+1-i}-sF_{r+n-i}\right)\cdot\frac{(-a)^{i-1}}{b^i},\\
x_3 &= -\frac{1}{f_n}\cdot\frac{F_{r+3}-sF_{r+2}}{b},\\
x_4 &= -\frac{1}{f_n}\sum_{i=1}^{2}\left(F_{r+1+i}-sF_{r+i}\right)\cdot\frac{(-a)^{i-1}}{b^i},\\
x_k &= -\frac{1}{f_n}\sum_{i=1}^{2}\left(F_{r+1+i}-sF_{r+i}\right)\cdot\frac{(-a)^{k-5+i}}{b^{k-4+i}},\\
&\vdots\\
x_n &= -\frac{1}{f_n}\sum_{i=1}^{2}\left(F_{r+1+i}-sF_{r+i}\right)\cdot\frac{(-a)^{n-5+i}}{b^{n-4+i}},\\
B_{r,n}^{-1} &= \frac{1}{f_n}\cdot\mathrm{SCirc}\left(x_1',x_2',\ldots,x_n'\right),
\end{aligned}\tag{29}$$

where

$$\begin{aligned}
x_1' &= 1-\left(F_{r+3}-sF_{r+2}\right)\cdot\frac{(-a)^{n-3}}{b^{n-1}}\\
&\qquad-\sum_{i=1}^{n-3}\left(F_{r+n+2-i}-sF_{r+n+1-i}\right)\cdot\frac{(-a)^{i-1}}{b^i},\\
x_2' &= -s-\sum_{i=1}^{n-2}\left(F_{r+n+1-i}-sF_{r+n-i}\right)\cdot\frac{(-a)^{i-1}}{b^i},\\
x_3' &= -\left(F_{r+3}-sF_{r+2}\right)\cdot\frac{1}{b},\\
x_4' &= -\sum_{i=1}^{2}\left(F_{r+1+i}-sF_{r+i}\right)\cdot\frac{(-a)^{i-1}}{b^i},\\
x_k' &= -\sum_{i=1}^{2}\left(F_{r+1+i}-sF_{r+i}\right)\cdot\frac{(-a)^{k-5+i}}{b^{k-4+i}},\\
&\qquad(k=5,6,\ldots,n).
\end{aligned}\tag{30}$$

This completes the proof. □

3. Norm and Spread of Skew Circulant Matrix with the Fibonacci Numbers

Theorem 8. *Let $B_{r,n} = SCirc(F_{r+1},\ldots,F_{r+n})$ be a skew circulant matrix; then three kinds of norms of $B_{r,n}$ are given by*

$$\|B_{r,n}\|_1 = \|B_{r,n}\|_\infty = F_{r+n+2}-F_{r+2},\tag{31}$$

$$\|B_{r,n}\|_F = \sqrt{n\left(F_{r+n}F_{r+n+1}-F_rF_{r+1}\right)}.\tag{32}$$

Proof. By Definition 8 in [21] and (5), we have

$$\|B_{r,n}\|_1 = \|B_{r,n}\|_\infty = \sum_{i=1}^{n}F_{r+i} = F_{r+n+2}-F_{r+2}.\tag{33}$$

According to Definition 8 in [21] and (6), we know

$$\begin{aligned}
\left(\|B_{r,n}\|_F\right)^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n}\left|a_{ij}\right|^2\\
&= n\sum_{i=1}^{n}F_{r+i}^2\\
&= n\left(\sum_{i=0}^{r+n}F_i^2-\sum_{i=0}^{r}F_i^2\right)\\
&= n\left(F_{r+n}F_{r+n+1}-F_rF_{r+1}\right).
\end{aligned}\tag{34}$$

Thus

$$\|B_{r,n}\|_F = \sqrt{n\left(F_{r+n}F_{r+n+1}-F_rF_{r+1}\right)}.\tag{35}$$

□

Theorem 9. *Let*

$$B_{r,n}' = SCirc\left(F_{r+1},-F_{r+2},\ldots,-F_{r+n-1},F_{r+n}\right)\tag{36}$$

be an odd-order alternative skew circulant matrix and let n be odd. Then

$$\|B_{r,n}'\|_2 = \sum_{i=1}^{n}F_{r+i} = F_{r+n+2}-F_{r+2}.\tag{37}$$

Proof. By Lemma 3 in [21], we have

$$\lambda_j\left(B_{r,n}'\right) = \sum_{i=1}^{n}(-1)^{i-1}F_{r+i}\left(\omega^j\eta\right)^{i-1}.\tag{38}$$

So

$$\begin{aligned}
\left|\lambda_j\left(B_{r,n}'\right)\right| &\le \sum_{i=1}^{n}\left|(-1)^{i-1}F_{r+i}\right|\cdot\left|\left(\omega^j\eta\right)^{i-1}\right|\\
&= \sum_{i=1}^{n}F_{r+i},
\end{aligned}\tag{39}$$

for all $j=0,1,\ldots,n-1$.

Since n is odd, $\sum_{i=1}^{n} F_{r+i}$ is an eigenvalue of $B'_{r,n}$; that is

$$\begin{pmatrix} F_{r+1} & -F_{r+2} & \cdots & F_{r+n} \\ -F_{r+n} & F_{r+1} & \cdots & -F_{r+n-1} \\ \vdots & \vdots & \ddots & \vdots \\ F_{r+2} & -F_{r+3} & \cdots & F_{r+1} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix} = \sum_{i=1}^{n} F_{r+i} \cdot \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ \vdots \\ 1 \end{pmatrix}. \tag{40}$$

To sum up, we can get

$$\max_{0 \le j \le n-1} \left| \lambda_j \left(B'_{r,n} \right) \right| = \sum_{i=1}^{n} F_{r+i}. \tag{41}$$

Since all skew circulant matrices are normal, by Lemma 9 in [21], (5), and (41), we obtain

$$\left\| B'_{r,n} \right\|_2 = \sum_{i=1}^{n} F_{r+i} = F_{r+n+2} - F_{r+2}, \tag{42}$$

which completes the proof. □

Theorem 10. *Let $B_{r,n} = SCirc(F_{r+1}, \ldots, F_{r+n})$ be a skew circulant matrix; then, the bounds for the spread of $B_{r,n}$ are*

$$\begin{aligned} s\left(B_{r,n}\right) &\leqslant \sqrt{2n\left(F_{r+n}F_{r+n+1} - F_{r+1}F_{r+2}\right)}, \\ s\left(B_{r,n}\right) &\geq \frac{1}{n-1}\left|2F_{r+n+4} - nF_{r+n+2} - nF_{r+3} - 2F_{r+4}\right|. \end{aligned} \tag{43}$$

Proof. The trace of $B_{r,n}$ is denoted by $\operatorname{tr} B_{r,n} = nF_{r+1}$. By (18) in [21] and (32), we know

$$s\left(B_{r,n}\right) \leqslant \sqrt{2n\left(F_{r+n}F_{r+n+1} - F_{r+1}F_{r+2}\right)}. \tag{44}$$

Since

$$\begin{aligned} \sum_{i \neq j} a_{ij} &= \sum_{k=2}^{n} (n-(k-1)) F_{r+k} - \sum_{k=2}^{n} (k-1) F_{r+k} \\ &= (n+2) \sum_{k=2}^{n} F_{r+k} - 2\sum_{k=2}^{n} kF_{r+k} \\ &= (n+2)\left(F_{r+n+2} - F_{r+3}\right) \\ &\quad - 2\left[\sum_{k=2}^{n} (r+k) F_{r+k} - \sum_{k=2}^{n} rF_{r+k}\right], \end{aligned} \tag{45}$$

furthermore, by (5) and (7),

$$\begin{aligned} \sum_{i \neq j} a_{ij} &= (n+2)\left(F_{r+n+2} - F_{r+3}\right) \\ &\quad - 2\left[(r+n) F_{r+n+2} - F_{r+n+3} - (r+1) F_{r+3} \right. \\ &\quad \left. + F_{r+4} - rF_{r+n+2} + rF_{r+3}\right] \\ &= 2F_{r+n+4} - nF_{r+n+2} - nF_{r+3} - 2F_{r+4}. \end{aligned} \tag{46}$$

By (19) in [21], we have

$$s\left(B_{r,n}\right) \geq \frac{1}{n-1}\left|2F_{r+n+4} - nF_{r+n+2} - nF_{r+3} - 2F_{r+4}\right|. \tag{47}$$

□

4. Determinant and Inverse of Skew Left Circulant Matrix with the Fibonacci Numbers

In this section, let $B''_{r,n} = \mathrm{SLCirc}(F_{r+1}, \ldots, F_{r+n})$ be a skew left circulant matrix. By using the obtained conclusions in Section 2, we give a determinant explicit formula for the matrix $B''_{r,n}$. Afterwards, we prove that $B''_{r,n}$ is an invertible matrix for any positive interger n. The inverse of the matrix $B''_{r,n}$ is also presented.

According to Lemma 5 in [21], Lemma 6 in [21], and Theorems 4, 5, and 7, we can obtain the following theorems.

Theorem 11. *Let $B''_{r,n} = SLCirc(F_{r+1}, \ldots, F_{r+n})$ be a skew left circulant matrix; then,*

$$\begin{aligned} &\det B''_{r,n} \\ &= (-1)^{n(n-1)/2} F_{r+1} \left[\frac{F_{r+2}}{F_{r+1}} \cdot F_{r+n} + F_{r+1} \right. \\ &\qquad \left. + \sum_{k=1}^{n-2} \left(\frac{F_{r+2}}{F_{r+1}} \cdot F_{r+1+k} - F_{r+2+k}\right) x^{n-(k+1)}\right] \\ &\quad \times \left(F_{r+1} + F_{r+n+1}\right)^{n-2}, \end{aligned} \tag{48}$$

where F_{r+n} is the $(r+n)$th Fibonacci number.

Theorem 12. *Let $B''_{r,n} = SLCirc(F_{r+1}, \ldots, F_{r+n})$ be a skew left circulant matrix; then, $B''_{r,n}$ is an invertible matrix.*

Theorem 13. *Let $B''_{r,n} = SLCirc(F_{r+1}, \ldots, F_{r+n})$ be a skew left circulant matrix; then,*

$$B''^{\,-1}_{r,n} = \frac{1}{f_n} SLCirc\left(x''_1, x''_2, \ldots, x''_n\right), \tag{49}$$

where

$$\begin{aligned} x_1'' &= 1-\left(F_{r+3}-sF_{r+2}\right)\cdot\frac{(-a)^{n-3}}{b^{n-2}} \\ &\quad -\sum_{i=1}^{n-3}\left(F_{r+n+2-i}-sF_{r+n+1-i}\right)\cdot\frac{(-a)^{i-1}}{b^{i}}, \\ x_k'' &= -x_{n-k+2}' \\ &= \sum_{i=1}^{2}\left(f_{r+1+i}-tf_{r+i}\right)\cdot\frac{(-a)^{n-k-3+i}}{b^{n-k-2+i}}, \\ &\qquad (k=2,3,\ldots,n-2), \\ x_{n-1}'' &= -x_3' = \left(F_{r+3}-sF_{r+2}\right)\cdot\frac{1}{b}, \\ x_n'' &= -x_2' \\ &= s+\sum_{i=1}^{n-2}\left(F_{r+n+1-i}-sF_{r+n-i}\right)\cdot\frac{(-a)^{i-1}}{b^{i}}. \end{aligned} \tag{50}$$

5. Norm and Spread of Skew Left Circulant Matrix with the Fibonacci Numbers

Theorem 14. *Let $B_{r,n}'' = SLCirc(F_{r+1},\ldots,F_{r+n})$ be a skew left circulant matrix. Then three kinds of norms of $B_{r,n}''$ are given by*

$$\begin{aligned} \left\|B_{r,n}''\right\|_1 &= \left\|B_{r,n}''\right\|_\infty = F_{r+n+2}-F_{r+2}, \\ \left\|B_{r,n}''\right\|_F &= \sqrt{n\left(F_{r+n}F_{r+n+1}-F_rF_{r+1}\right)}. \end{aligned} \tag{51}$$

Proof. Using the method in Theorem 8 similarly, the conclusion is obtained. □

Theorem 15. *Let*

$$B_{r,n}''' = SLCirc\left(F_{r+1},-F_{r+2},\ldots,-F_{r+n-1},F_{r+n}\right) \tag{52}$$

be an odd-order alternative skew left circulant matrix; then,

$$\left\|B_{r,n}'''\right\|_2 = \sum_{i=1}^{n}F_{r+i} = F_{r+n+2}-F_{r+2}. \tag{53}$$

Proof. According to Lemma 4 in [21],

$$\lambda_j\left(B_{r,n}'''\right) = \pm\left|\sum_{i=1}^{n}(-1)^{i-1}F_{r+i}\omega^{(j-(1/2))(k-1)}\right|, \tag{54}$$

for $j=1,2,\ldots,(n-1)/2$, and

$$\lambda_{(n+1)/2}\left(B_{r,n}'''\right) = \sum_{i=1}^{n}F_{r+i}. \tag{55}$$

So

$$\begin{aligned} \left|\lambda_j\left(B_{r,n}'''\right)\right| &\le \sum_{i=1}^{n}\left|(-1)^{i-1}F_{r+i}(-1)^{i-1}\right| \\ &= \sum_{i=1}^{n}F_{r+i}, \quad \left(j=1,2,\cdots,\frac{n+1}{2}\right). \end{aligned} \tag{56}$$

By (55) and (56), we know

$$\max_{0\le i\le(n+1)/2}\left|\lambda_i\left(B_{r,n}'''\right)\right| = \sum_{i=1}^{n}F_{r+i}. \tag{57}$$

Since all skew left circulant matrices are symmetrical, by Lemma 9 in [21], (5), and (57), we obtain

$$\left\|B_{r,n}'''\right\|_2 = F_{r+n+2}-F_{r+2}. \tag{58}$$

□

Theorem 16. *Let $B_{r,n}'' = SLCirc(F_{r+1},\ldots,F_{r+n})$ be skew left circulant matrix, if n is odd, then*

$$\begin{aligned} 2F_{r+n} &\le s\left(B_{r,n}''\right) \\ &\le \sqrt{2n\left(F_{r+n}F_{r+n+1}-F_{r+1}F_r\right)-\frac{2}{n}\left(F_{r+n-1}+F_{r-1}\right)^2}; \end{aligned} \tag{59}$$

if n is even, then

$$2F_{r+n} \le s\left(B_{r,n}''\right) \le \sqrt{2n\left(F_{r+n}F_{r+n+1}-F_{r+1}F_r\right)}. \tag{60}$$

Proof. Since $B_{r,n}''$ is a symmetric matrix, by (20) in [21],

$$s\left(B_{r,n}''\right) \ge 2\max_{i\ne j}\left|a_{ij}\right| = 2F_{r+n}. \tag{61}$$

If n is odd, the trace of $B_{r,n}''$ is

$$\begin{aligned} &\operatorname{tr}\left(B_{r,n}''\right) \\ &\quad = F_{r+1}-F_{r+2}+F_{r+3}-\cdots+F_{r+n} \\ &\quad = F_{r+1}+F_{r+1}+F_{r+3}+\cdots+F_{r+n-2} \\ &\quad = 2F_{r+1}+F_{r+1}+F_{r+2}+\cdots+F_{r+n-3} \\ &\quad = 2F_{r+1}+\sum_{i=1}^{n-3}F_{r+i}; \end{aligned} \tag{62}$$

by (5), we know

$$\operatorname{tr}\left(B_{r,n}''\right) = F_{r+n-1}+F_{r-1}. \tag{63}$$

By (18) in [21], (51), and (63), we obtain

$$\begin{aligned} &s\left(B_{r,n}''\right) \\ &\quad \le \sqrt{2n\left(F_{r+n}F_{r+n+1}-F_rF_{r+1}\right)-\frac{2}{n}\left(F_{r+n-1}+F_{r-1}\right)^2}. \end{aligned} \tag{64}$$

If n is even, the trace of $B_{r,n}''$ is

$$\begin{aligned} \operatorname{tr}\left(B_{r,n}''\right) &= F_{r+1}-F_{r+1}+F_{r+3} \\ &\quad -F_{r+3}\cdots+F_{r+n-1}-F_{r+n-1} = 0. \end{aligned} \tag{65}$$

By (18) in [21], (51), and (65), we can get

$$s\left(B_{r,n}''\right) \le \sqrt{2n\left(F_{r+n}F_{r+n+1}-F_rF_{r+1}\right)}. \tag{66}$$

So the result follows. □

6. Conclusion

We discuss the invertibility of skew circulant type matrices with any continuous Fibonacci numbers and present the determinant and the inverse matrices by constructing the transformation matrices. The four kinds of norms and bounds for the spread of these matrices are given, respectively. In [20], a new fast algorithm for optimal design of block digital filters (BDFs) was proposed based on skew circulant matrix. The reason why we focus our attentions on skew circulant is to explore the application of skew circulant in the related field in real-time tracking and networks engineering. On the basis of method of [17] and ideas of [43], we will exploit real-time tracking with kernel matrix of skew circulant structure. On the basis of existing application situation [1–11], we will exploit application of network engineering based on skew circulant matrix.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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