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Research Article

Some Common Fixed Point Results for Modified Subcompatible Maps and Related Invariant Approximation Results

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We improve the class of subcompatible self-maps used by (Akbar and Khan, 2009) by introducing a new class of noncommuting self-maps called modified subcompatible self-maps. For this new class, we establish some common fixed point results and obtain several invariant approximation results as applications. In support of the proved results, we also furnish some illustrative examples.

1. Introduction and Preliminaries

From the last five decades, fixed point theorems have been used in many instances in invariant approximation theory. The idea of applying fixed point theorems to approximation theory was initiated by Meinardus [1] where he employs a fixed point theorem of Schauder to establish the existence of an invariant approximation. Later on, Brosowski [2] used fixed point theory to establish some interesting results on invariant approximation in the setting of normed spaces and generalized Meinardus's results. Singh [3], Habiniak [4], Sahab et al. [5], and Jungck and Sessa [6] proved some similar results in the best approximation theory. Further, Al-Thagafi [7] extended these works and proved some invariant approximation results for commuting self-maps. Al-Thagafi results have been further extended by Hussain and Jungck [8], Shahzad [9–14] and O'Regan and Shahzad [15] to various class of noncommuting self-maps, in particular to R-subweakly commuting and R-subcommuting self-maps. Recently, Akbar and Khan [16] extended the work of [7–15] to more general noncommuting class, namely, the class of subcompatible self-maps.

In this paper, we improve the class of subcompatible self-maps used by Akbar and Khan [16] by introducing a new class of noncommuting self-maps called modified subcompatible self-maps which contain commuting, R-subcommuting, R-subweakly, commuting, and subcompatible maps as a proper subclass. For this new class, we establish some common fixed point results for some families of self-maps and obtain several invariant approximation results as applications. The proved results improve and extend the corresponding results of [3–8, 10–15].

Before going to the main work, we need some preliminaries which are as follows.

Definition 1. Let (X, d) be a metric space, M be a subset of X, and S and T be self-maps of M. Then the family $\{A_i : i \in \mathbb{N} \cup \{0\}\}$ of self-maps of M is called (S, T):

(i) *contraction* if there exists k, $0 \le k < 1$ such that for all $x, y \in M$,

$$d(A_0 x, A_i y) \le k d(Sx, Ty), \quad \text{for each } i \in \mathbb{N}, \tag{1}$$

(ii) *nonexpansive* if for all $x, y \in M$,

$$d(A_0 x, A_i y) \le d(Sx, Ty), \quad \text{for each } i \in \mathbb{N}. \tag{2}$$

In Definition 1, if we take $T = S$, then this family $\{A_i : i \in \mathbb{N} \cup \{0\}\}$ is called S-contraction (resp., S-nonexpansive).

Definition 2. Let M be a subset of a metric space (X, d) and S, T be self-maps of M. A point $x \in M$ is a coincidence point (common fixed point) of S and T if $Sx = Tx$ ($Sx = Tx = x$).

The set of coincidence points of S and T is denoted by $C(S,T)$. The pair $\{S,T\}$ is called

(1) commuting if $STx = TSx$ for all $x \in M$;

(2) R-weakly commuting [17], provided there exists some positive real number R such that $d(STx, TSx) \leq Rd(Sx, Tx)$ for each $x \in M$;

(3) compatible [18] if $\lim_{n\to\infty} d(STx_n, TSx_n) = 0$ whenever $\{x_n\}$ is a sequence in M such that $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$ for some $t \in M$;

(4) weakly compatible [19] if $STx = TSx$ for all $x \in C(S,T)$.

For a useful discussion on these classes, that is, the class of commuting, R-weakly commuting, compatible, and weakly compatible maps, see also [20].

Definition 3. Let X be a linear space and let M be a subset of X. The set M is said to be star-shaped if there exists at least one point $q \in M$ such that the line segment $[x,q]$ joining x to q is contained in M for all $x \in M$; that is, $kx + (1-k)q \in M$ for all $x \in M$, where $0 \leq k \leq 1$.

Definition 4. Let X be a linear space and let M be a subset of X. A self-map $A : M \to M$ is said to be

(i) *affine* [21] if M is convex and

$$A(kx + (1-k)y) = kA(x) + (1-k)A(y) \quad \forall x, y \in M, \quad k \in (0,1), \tag{3}$$

(ii) *q-affine* [21] if M is q-star-shaped and

$$A(kx + (1-k)q) = kA(x) + (1-k)q \quad \forall x \in M, \quad k \in (0,1). \tag{4}$$

Here we observe that if A is q-affine then $Aq = q$.

Remark 5. Every affine map A is q-affine if $Aq = q$ but its converse need not be true even if $Aq = q$, as shown by the following examples.

Example 6. Let $X = R$ and $M = [0,1]$. Let $A : M \to M$ be defined as

$$A(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{1}{2} \\ 1-x & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases} \tag{5}$$

Then A is q-affine for $q = 1/2$, while A is not affine because for $x = 3/5$, $y = 0$, and $k = 1/3$

$$A(kx + (1-k)y) = kA(x) + (1-k)A(y) \tag{6}$$

does not hold.

Example 7. Let $X = R^2$ and $\lambda \in R^+ = [0,\infty)$. Let $M = M_1 \cup M_2$, where

$$M_1 = \{(x,y) \in R^2 : (x,y) = (\lambda, 3\lambda)\}, \quad M_2 = \{(x,y) \in R^2 : (x,y) = (\lambda,\lambda)\}. \tag{7}$$

Then M is q-star-shaped for $q = (0,0)$. Define $A : M \to M$ as

$$A(x,y) = \begin{cases} (0,0) & \text{if } (x,y) \in M_1 \\ (x,y) & \text{if } (x,y) \in M_2. \end{cases} \tag{8}$$

Then A is q-affine for $q = (0,0)$ but A is not affine, because for $x = (1,3) \in M$, $y = (1,1) \in M$, and $k = 1/2$, $kx + (1-k)y \notin M$, though $kA(x) + (1-k)A(y) = (1/2, 1/2) \in M$.

Definition 8. Let M be a subset of a normed linear space $(X, \|\cdot\|)$. The set $B_M(p) = \{x \in M : \|x - p\| = \operatorname{dist}(p, M)\}$ is called the set of best approximants to $p \in X$ out of M, where $\operatorname{dist}(p, M) = \inf\{\|y - p\| : y \in M\}$.

Definition 9 (see [11]). Let M be a subset of a normed linear space X and let S and T be self-maps of M. Then the pair (S,T) is called R-subweakly commuting on M with respect to q if M is q-star-shaped with $q \in F(S)$ (where $F(S)$ denote the set of fixed point of S) and $\|STx - TSx\| \leq R \operatorname{dist}(Sx, [q, Tx])$ for all $x \in M$ and some $R > 0$.

Definition 10. Let X be a Banach space. A map $S : M \subseteq X \to X$ is said to be demiclosed at 0 whenever $\{x_n\}$ is a sequence in M such that x_n converges weakly to $x \in M$ and Sx_n converges strongly to $0 \in M$; then $0 = Sx$.

Definition 11. A Banach space X is said to satisfy Opial's condition whenever $\{x_n\}$ is a sequence in X such that x_n converges weakly to $x \in X$; then

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \quad \text{holds } \forall y \neq x. \tag{9}$$

Note that Hilbert and l^p $(1 < p < \infty)$ spaces satisfy Opial's condition.

2. Common Fixed Point for Modified Subcompatible Self-Maps

First we introduce the notion of modified subcompatible maps.

Definition 12. Let M be a q-star-shaped subset of a normed linear space X and let S and T be self-maps of M with $q \in F(S)$. Define $\Lambda_q(S,T) = \bigcup_{k\in(0,1)} \Lambda(S, T_k)$, where $T_k(x) = (1-k)q + kTx$ and $\Lambda(S, T_k) = \{\{x_n\} \subset M : \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} T_k x_n = t \in M\}$. Then S and T are called modified subcompatible if $\lim_{n\to\infty} \|STx_n - TSx_n\| = 0$ for all sequences $\{x_n\} \in \Lambda_q(S,T)$.

In the definition of subcompatible maps (see [16]), $\Lambda_q(S,T) = \bigcup_{k\in[0,1]} \Lambda(S, T_k)$, but here $k \in (0,1)$. The following examples reveal the impact of this and show that R-subweakly commuting maps and also subcompatible maps of [16] form a proper subclass of modified subcompatible maps.

Example 13. Let $X = R$ with the usual norm and $M = [0, \infty)$. Define $S, T : M \to M$ by

$$S(x) = \begin{cases} \frac{x}{2}, & 0 \le x < 1 \\ 2x^2 - 1, & x \ge 1, \end{cases}$$
$$T(x) = \begin{cases} \frac{1}{2}, & 0 \le x < 1 \\ 4x - 3, & x \ge 1. \end{cases} \tag{10}$$

Then M is 1-star-shaped with $q = 1 \in F(S)$ and $\Lambda_q(S, T) = \{\{x_n\} : 1 \le x_n < \infty, \lim_{n\to\infty} x_n = 1\}$. Moreover, S and T are modified subcompatible but not subcompatible because for the sequence $\{1 - 1/n\}_{n\ge1}$, we have $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T_1(x_n) = 1/2$ and $\lim_{n\to\infty} \|ST(x_n) - TS(x_n)\| \ne 0$. Note that S and T are neither R-subweakly commuting nor R-subcommuting.

Example 14. Let $X = R$ with the usual norm and $M = [0, \infty)$. Define $S, T : M \to M$ by

$$S(x) = \begin{cases} \frac{1}{2}, & 0 \le x < 1 \\ x^2, & x \ge 1, \end{cases}$$
$$T(x) = \begin{cases} \frac{3}{2}, & 0 \le x < 1 \\ x, & x \ge 1. \end{cases} \tag{11}$$

Then M is 1/2-star-shaped with $q = 1/2 \in F(S)$ and $\Lambda_q(S, T) = \phi$. Clearly S and T are modified subcompatible but not subcompatible because for any sequence $\{x_n\}_{0\le x_n<1}$, we have $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T_0(x_n) = 1/2$ and $\lim_{n\to\infty} \|ST(x_n) - TS(x_n)\| \ne 0$. Also, S and T are not R-subweakly commuting.

The following two examples show that the modified subcompatible self-maps and compatible self-maps are of different classes.

Example 15. Let $X = R$ with usual norm and $M = [1, \infty)$. Let $S, T : M \to M$ be defined by

$$S(x) = 6x - 5, \qquad T(x) = 3x^2 - 2, \tag{12}$$

for all $x \in M$. Then

$$\begin{aligned} \|T(x_n) - S(x_n)\| &= 3\left\|(x_n - 1)^2\right\| \longrightarrow 0 \\ &\qquad \text{iff } x_n \longrightarrow 1, \\ \|ST(x_n) - TS(x_n)\| &= 90\left\|(x_n - 1)^2\right\| \longrightarrow 0 \\ &\qquad \text{if } x_n \longrightarrow 1. \end{aligned} \tag{13}$$

Thus S and T are compatible. Obviously M is q-star-shaped with $q = 1$ and $Sq = q$. Note that for any sequence $\{x_n\}$ in M with $x_n \to 2$, we have

$$\|T_{2/3}(x_n) - S(x_n)\| = 2\|(x_n - 1)(x_n - 2)\| \longrightarrow 0. \tag{14}$$

However, $\lim_{n\to\infty} \|ST(x_n) - TS(x_n)\| \ne 0$. Thus S and T are not modified subcompatible maps. Hence, they are not R-subweakly commuting.

Example 16. Let $X = R$ with norm $\|x\| = |x|$ and $M = [0, \infty)$. Let $S, T : M \to M$ be defined by

$$S(x) = \begin{cases} x, & 0 \le x < 1 \\ 3, & x \ge 1, \end{cases}$$
$$T(x) = \begin{cases} 3 - 2x, & 0 \le x < 1 \\ 3, & x \ge 1, \end{cases} \tag{15}$$
$$\forall x \in M.$$

Then M is 3-star-shaped with $S(3) = 3$ and $\Lambda_q(S, T) = \{\{x_n\} : 1 \le x_n < \infty\}$. Clearly S and T are modified subcompatible. Moreover, for any sequence $\{x_n\}$ in $[0, 1)$ with $\lim_{n\to\infty} x_n = 1$, we have $\lim_{n\to\infty} \|T(x_n) - S(x_n)\| = 0$. However, $\lim_{n\to\infty} \|ST(x_n) - TS(x_n)\| \ne 0$. Thus S and T are not compatible.

The following general common fixed point result is a consequence of Theorem 5.1 of Jachymski [22], which will be needed in the sequel.

Theorem 17. *Let S and T be self-maps of a complete metric space (X, d) and either S or T is continuous. Suppose $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ is a sequence of self-maps of X satisfying the following.*

(1) *$A_0(X) \subseteq T(X)$ and $A_i(X) \subseteq S(X)$ for each $i \in \mathbb{N}$.*

(2) *The pairs (A_0, S) and (A_i, T) are compatible for each $i \in \mathbb{N}$.*

(3) *For each $i \in \mathbb{N}$ and, for any $x, y \in M$,*

$$d(A_0x, A_iy) \le h \max M(x, y) \quad \text{for some } h \in (0, 1), \tag{16}$$

where

$$\begin{aligned} M(x, y) = \Big\{ & d(Sx, Ty), d(A_0x, Sx), \\ & d(A_iy, Ty), \\ & \frac{1}{2}\left[d(A_0x, Ty) + d(A_iy, Sx)\right] \Big\}; \end{aligned} \tag{17}$$

then there exists a unique point z in X such that $z = Sz = Tz = A_iz$, for each $i \in \mathbb{N} \cup \{0\}$.

The following result extends and improves [7, Theorem 2.2], [8, Theorem 2.2], [6, Theorem 6], and [13, Theorem 2.2].

Theorem 18. *Let M be a nonempty q-star-shaped subset of a normed space X and let S and T be continuous and q-affine*

self-maps of M. Let $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ be a family of self-maps of M satisfying the following.

(1) *$A_0(M) \subseteq T(M)$ and $A_i(M) \subseteq S(M)$ for each $i \in \mathbb{N}$.*

(2) *(A_0, S) and (A_i, T) are modified subcompatible for each $i \in \mathbb{N}$.*

(3) *For each $i \in \mathbb{N}$ and, for any $x, y \in M$*

$$\|A_0x - A_iy\| \leq \max M(x, y), \tag{18}$$

where

$$\begin{aligned} M(x, y) = \Big\{ &\|Sx - Ty\|, \operatorname{dist}(Sx, [A_0x, q]), \\ &\operatorname{dist}(Ty, [A_iy, q]), \\ &\frac{1}{2}[\operatorname{dist}(Sx, [A_iy, q]) \\ &\quad + \operatorname{dist}(Ty, [A_0x, q])] \Big\}; \end{aligned} \tag{19}$$

then all the A_i ($i \in \mathbb{N} \cup \{0\}$), S and T have a common fixed point provided one of the following conditions hold.

(a) *M is sequentially compact and A_i is continuous for each $i \in \mathbb{N} \cup \{0\}$.*

(b) *M is weakly compact, $(S - A_i)$ is demiclosed at 0 for each $i \in \mathbb{N} \cup \{0\}$, and X is complete.*

Proof. For each $i \in \mathbb{N} \cup \{0\}$, define $A_i^n : M \to M$ by

$$A_i^n x = (1 - k_n) q + k_n A_i x \tag{20}$$

for all $x \in M$ and a fixed sequence of real numbers k_n ($0 < k_n < 1$) converging to 1. Then, A_i^n is a self-map of M for each $i \in \mathbb{N} \cup \{0\}$ and for each $n \geq 1$.

Firstly, we prove $A_0^n(M) \subseteq T(M)$; for this let $y \in A_0^n(M)$, which implies $y = A_0^n x$ for some $x \in M$.

Now, by using (20)

$$\begin{aligned} y = A_0^n x &= (1 - k_n) q + k_n A_0 x \\ &= (1 - k_n) q + k_n T z, \quad \text{for some } z \in M \\ &\Longrightarrow y \in T(M), \quad \text{as } T \text{ is } q\text{-affine}, M \text{ is } q\text{-star-shaped}. \end{aligned} \tag{21}$$

Hence $A_0^n(M) \subseteq T(M)$ for each $n \geq 1$.

Similarly, it can be shown that for each $i \in \mathbb{N}$ and each $n \geq 1$, $A_i^n(M) \subseteq S(M)$, as S is q-affine and M is q-star-shaped.

Now, we prove that for each $n \geq 1$, the pair (A_0^n, S) is compatible; for this let $\{x_m\} \subseteq M$ with $\lim_{m\to\infty} Sx_m = \lim_{m\to\infty} A_0^n x_m = t \in M$. Since the pair (A_0, S) is modified subcompatible, therefore, by the assumption of A_{0_k}, we have

$$\lim_{m\to\infty} A_{0_{k_n}} x_m = \lim_{m\to\infty} A_0^n x_m = t. \tag{22}$$

As the pair (A_0, S) is modified subcompatible and S is q-affine, therefore

$$\begin{aligned} &\lim_{m\to\infty} \|A_0^n S x_m - S A_0^n x_m\| \\ &\quad = k_n \lim_{m\to\infty} \|A_0 S x_m - S A_0 x_m\| = 0. \end{aligned} \tag{23}$$

Hence, the pair (A_0^n, S) is compatible for each n.

Similarly, we can prove that the pair (A_i^n, T) is compatible for each $i \in \mathbb{N}$ and each $n \geq 1$.

Also, using (18) and (20) we have

$$\begin{aligned} \|A_0^n x - A_i^n y\| &= k_n \|A_0 x - A_i y\| \\ &\leq k_n \max \Big\{ \|Sx - Ty\|, \operatorname{dist}(Sx, [A_0x, q]), \\ &\qquad \operatorname{dist}(Ty, [A_iy, q]), \\ &\qquad \frac{1}{2}[\operatorname{dist}(Sx, [A_iy, q]) \\ &\qquad\quad + \operatorname{dist}(Ty, [A_0x, q])] \Big\} \\ &\leq k_n \max \Big\{ \|Sx - Ty\|, \|Sx - A_0^n x\|, \\ &\qquad \|Ty - A_i^n y\|, \\ &\qquad \frac{1}{2}[\|Sx - A_i^n y\| \\ &\qquad\quad + \|Ty - A_0^n x\|] \Big\} \end{aligned} \tag{24}$$

for each $x, y \in M$ and $0 < k_n < 1$. By Theorem 17, for each $n \geq 1$, there exists $x_n \in M$ such that $x_n = Sx_n = Tx_n = A_i^n x_n$, for each $i \in \mathbb{N} \cup \{0\}$.

(a) As M is sequentially compact and $\{x_n\}$ is a sequence in M, so $\{x_n\}$ has a convergent subsequence $\{x_m\}$ such that $x_m \to z \in M$. Thus, by the continuity of S, T and all A_i ($i \in \mathbb{N}\cup\{0\}$), we can say that z is a common fixed point of S, T and all A_i ($i \in \mathbb{N}\cup\{0\}$). Thus $F(T)\cap F(S)\cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$.

(b) Since M is weakly compact, there is a subsequence $\{x_m\}$ of $\{x_n\}$ converging weakly to some $u \in M$. But, S and T being q-affine and continuous are weakly continuous, and the weak topology is Hausdorff, so u is a common fixed point of S and T. Again the set M is bounded, so $(S - A_i)(x_m) = x_m - x_m k_m^{-1} - q(1 - k_m^{-1}) \to 0$ as $m \to \infty$. Now demiclosedness of $(S - A_i)$ at 0 gives that $(S - A_i)(u) = 0$ for each $i \in \mathbb{N} \cup \{0\}$, and hence $F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$. □

Theorem 19. *Let M be a nonempty q-star-shaped subset of a normed space X, and let S and T be continuous and q-affine self-maps of M. Let $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ be a family of self-maps with $A_0(M) \subseteq T(M)$ and $A_i(M) \subseteq S(M)$ for each $i \in \mathbb{N}$. If the pairs (A_0, S) and (A_i, T) are modified subcompatible for each $i \in \mathbb{N}$*

and also the family $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ of maps is (S,T)-nonexpansive, then $F(T)\cap F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}}F(A_i))\neq\phi$, provided one of the following conditions hold.

(a) *M is sequentially compact.*

(b) *M is weakly compact, $(S-A_i)$ is demiclosed at 0 for each $i\in\mathbb{N}\cup\{0\}$, and X is complete.*

(c) *M is weakly compact and X is a complete space satisfying Opial's condition.*

Proof. (a) The proof follows from Theorem 18(a).

(b) The proof follows from Theorem 18(b).

(c) Following the proof of Theorem 18(b), we have $Su=u=Tu$ and for each $i\in\mathbb{N}\cup\{0\}$, $\|Sx_m-A_ix_m\|\to 0$ as $m\to\infty$. Since the family $\{A_i\}_{i=0}^{\infty}$ is (S,T)-nonexpansive, therefore, for each $i\in\mathbb{N}$, we have $A_0u=A_iu$. Now we have to show that $Su=A_0u$. If not, then by Opial's condition of X and (S,T)-nonexpansiveness of the family $\{A_i\}_0^{\infty}$, we get

$$\begin{aligned}\liminf_{m\to\infty}\|Sx_m-Tu\| &= \liminf_{m\to\infty}\|Sx_m-Su\|\\ &< \liminf_{m\to\infty}\|Sx_m-A_0u\|\\ &\le \liminf_{m\to\infty}\|Sx_m-A_ix_m\|\\ &\quad+\liminf_{m\to\infty}\|A_ix_m-A_0u\|,\\ &\qquad\text{where } i\in\mathbb{N}\\ &= \liminf_{m\to\infty}\|A_0u-A_ix_m\|\\ &\le \liminf_{m\to\infty}\|Su-Tx_m\|\\ &= \liminf_{m\to\infty}\|Tu-Sx_m\|,\end{aligned}\tag{25}$$

which is a contradiction. Therefore, $Su=A_0u$ and, hence, $F(T)\cap F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}}F(A_i))\neq\phi$. □

In Theorems 18 and 19, if we take $A_i=A$ for each $i\in\mathbb{N}\cup\{0\}$, we obtain the following corollary which generalizes Theorems 2.2 and 2.3 of Hussain and Jungck [8], respectively.

Corollary 20. *Let M be a nonempty q-star-shaped subset of a normed space X, and let S and T be continuous and q-affine self-maps of M. Let A be a self-map of M satisfying the following.*

(1) *$A(M)\subseteq S(M)\cap T(M)$.*

(2) *The pairs (A,S) and (A,T) are modified subcompatible.*

(3) *For all $x,y\in M$,*

$$\|Ax-Ay\|\le\max\ M(x,y),\tag{26}$$

where

$$\begin{aligned}M(x,y)=\Big\{&\|Sx-Ty\|,\operatorname{dist}(Sx,[Ax,q]),\\ &\operatorname{dist}(Ty,[Ay,q]),\\ &\frac{1}{2}[\operatorname{dist}(Sx,[Ay,q])\\ &\quad+\operatorname{dist}(Ty,[Ax,q])]\Big\}.\end{aligned}\tag{27}$$

Then S, T, and A have a common fixed point provided one of the following conditions hold.

(a) *M is sequentially compact and A is continuous.*

(b) *M is weakly compact, $(S-A)$ is demiclosed at 0, and X is complete.*

(c) *M is complete, $\mathrm{cl}(A(M))$ is compact, and A is continuous.*

Proof. (a) and (b) follow from Theorem 18 by taking $A_i=A$ for each $i\in\mathbb{N}\cup\{0\}$.

(c) Define $A^n\colon M\to M$ by

$$A^nx=(1-k_n)q+k_nAx.\tag{28}$$

As we have done in Theorem 18, for each $n\ge 1$, there exists $x_n\in M$ such that $x_n=Sx_n=Tx_n=A^nx_n$. Then, compactness of $\mathrm{cl}(A(M))$ implies that there exists a subsequence $\{Ax_m\}$ of $\{Ax_n\}$ such that $Ax_m\to z$ as $m\to\infty$. Then the definition of A^mx_m implies $x_m\to z$; thus, by continuity of A, S, and T, we can say that z is a common fixed point of A, S, and T. □

Corollary 21. *Let M be a nonempty q-star-shaped subset of a normed space X, and let S and T be continuous and q-affine self-maps of M. Let A be a self-map of M with $A(M)\subseteq S(M)\cap T(M)$. If the pairs (A,S) and (A,T) are modified subcompatible and also the map A is (S,T)-nonexpansive, then $F(T)\cap F(S)\cap F(A)\neq\phi$, provided one of the following conditions hold.*

(a) *M is sequentially compact.*

(b) *M is weakly compact, $(S-A)$ is demiclosed at 0, and X is complete.*

(c) *M is weakly compact and X is complete space satisfying Opial's condition.*

(d) *M is complete and $\mathrm{cl}(A(M))$ is compact.*

In Corollary 20(b), if we take $T=S$, then we obtain the following corollary as a generalization of Theorem 4 proved by Shahzad [12].

Corollary 22. *Let M be a nonempty weakly compact q-star-shaped subset of a Banach space X, and let A and S be self-maps of M. Suppose that S is q-affine and continuous, and $A(M)\subseteq S(M)$. If $(S-A)$ is demiclosed at 0, the pair (A,S) is modified subcompatible and satisfies*

$$\|Ax-Ay\|\le\max\ M(x,y),\tag{29}$$

where

$$M(x,y) = \Big\{\|Sx - Sy\|, \operatorname{dist}(Sx,[Ax,q]), \operatorname{dist}(Sy,[Ay,q]), \frac{1}{2}[\operatorname{dist}(Sx,[Ay,q]) + \operatorname{dist}(Sy,[Ax,q])]\Big\} \tag{30}$$

for all $x, y \in M$; then $F(S) \cap F(A) \neq \phi$.

In Theorems 18 and 19, if we take $T = S$, then we obtain the following corollary.

Corollary 23. *Let M be a nonempty q-star-shaped subset of a normed space X. Suppose that S is continuous and is a q-affine self-map of M. Let $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ be a family of self-maps of M satisfying the following.*

(1) *$\bigcup_{i=0}^{\infty} A_i(M) \subseteq S(M)$ and for each $i \in \mathbb{N}\cup\{0\}$, the pair (A_i, S) is modified subcompatible.*

(2) *For each $i \in \mathbb{N}$ and, for any $x, y \in M$*

$$\|A_0x - A_iy\| \leq \max M(x,y), \tag{31}$$

where

$$M(x,y) = \Big\{\|Sx - Sy\|, \operatorname{dist}(Sx,[A_0x,q]), \operatorname{dist}(Sy,[A_iy,q]), \frac{1}{2}[\operatorname{dist}(Sx,[A_iy,q]) + \operatorname{dist}(Sy,[A_0x,q])]\Big\}; \tag{32}$$

then S and all the A_i $(i \in \mathbb{N}\cup\{0\})$ have a common fixed point provided one of the following conditions hold.

(a) *M is sequentially compact and A_i is continuous for each $i \in \mathbb{N}\cup\{0\}$.*

(b) *M is weakly compact, $(S - A_i)$ is demiclosed at 0 for each $i \in \mathbb{N}\cup\{0\}$, and X is complete.*

Corollary 24. *Let M be a nonempty q-star-shaped subset of a normed space X. Suppose that S is continuous and is a q-affine self-map of M. Let $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ be a family of self-maps with $\bigcup_{i=0}^{\infty} A_i(M) \subseteq S(M)$ and the pairs (A_i, S) are modified subcompatible for each $i \in \mathbb{N}\cup\{0\}$. If this family $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ of maps is S-nonexpansive then $F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$, provided one of the following conditions hold.*

(1) *M is sequentially compact.*

(2) *M is weakly compact, $(S - A_i)$ is demiclosed at 0 for each $i \in \mathbb{N}\cup\{0\}$, and X is complete.*

(3) *M is weakly compact and X is a complete space satisfying Opial's condition.*

3. Applications to Best Approximation

The following theorem extends and generalizes [5, Theorem 2], [8, Theorem 2.8], and main result of [3].

Theorem 25. *Let M be a subset of a normed space X and let $S, T, A_i : X \to X$ be mappings for each $i \in \mathbb{N}\cup\{0\}$ such that $u \in F(T)\cap F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}} F(A_i))$ for some $u \in X$ and for each $i \in \mathbb{N}\cup\{0\}$, $A_i(\partial M \cap M) \subseteq M$. Suppose that S and T are q-affine and continuous on $P_M(u)$ and also $P_M(u)$ is q-star-shaped and $S(P_M(u)) = P_M(u) = T(P_M(u))$.*

Moreover, if

(1) *the pairs (A_0, S) and (A_i, T) are modified subcompatible for each $i \in \mathbb{N}$.*

(2) *for each $i \in \mathbb{N}$, and for all $x \in P_M(u)\bigcup\{u\}$,*

$$\|A_0x - A_iy\| \leq \begin{cases} \|Sx - Tu\|, & \text{if } y = u \\ \max\Big\{\|Sx - Ty\|, \operatorname{dist}(Sx,[q,A_0x]), \operatorname{dist}(Ty,[q,A_iy]), \frac{1}{2}[\operatorname{dist}(Sx,[q,A_iy]) + \operatorname{dist}(Ty,[q,A_0x])]\Big\} & \text{if } y \in P_M(u), \end{cases} \tag{33}$$

$$\|A_ix - A_0u\| \leq \|A_0x - A_iu\|. \tag{34}$$

Then $P_M(u)\cap F(T)\cap F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$, provided one of the following conditions hold.

(a) *$P_M(u)$ is sequentially compact and A_i is continuous for each $i \in \mathbb{N}\cup\{0\}$.*

(b) *$P_M(u)$ is weakly compact, X is complete, and $(S - A_i)$ is demiclosed at 0 for each $i \in \mathbb{N}\cup\{0\}$.*

Proof. Let $x \in P_M(u)$. Then $\|x - u\| = d(u, M)$. Note that for any $k \in (0,1)$,

$$\|ku + (1-k)x - u\| = (1-k)\|x - u\| < d(u,M). \tag{35}$$

It follows that the line segment $\{ku + (1-k)x : 0 < k < 1\}$ and the set M are disjoint. Thus, x is not interior of M and so $x \in \partial M \cap M$. As $A_i(\partial M \cap M) \subseteq M$ for each $i \in \mathbb{N}\cup\{0\}$, therefore, for each $i \in \mathbb{N}\cup\{0\}$, $A_ix \in M$. Now we have to show that $A_0x \in P_M(u)$ and for each $i \in \mathbb{N}$, $A_ix \in P_M(u)$. Since $Sx \in P_M(u)$, $u \in F(T)\cap F(S)\cap F(A_0)\cap(\bigcap_{i\in\mathbb{N}} F(A_i))$ and S, T, and A_i's satisfy (33); therefore, we have

$$\begin{aligned} \|A_0x - u\| &= \|A_0x - A_iu\| \\ &\leq \|Sx - Tu\| = \|Sx - u\| \\ &= d(u,M), \quad \text{where } i \in \mathbb{N}. \end{aligned} \tag{36}$$

Then the definition of $P_M(u)$ implies

$$A_0x \in P_M(u). \tag{36a}$$

Again using (33) and (34), for each $i \in \mathbb{N}$, we have

$$\begin{aligned} \|A_i x - u\| &= \|A_i x - A_0 u\| \\ &\le \|A_0 x - A_i u\| \le \|Sx - Tu\| \\ &= \|Sx - u\| = d(u, M). \end{aligned} \tag{37}$$

This yields that

$$A_i x \in P_M(u), \quad \text{for each } i \in \mathbb{N}. \tag{37a}$$

Then combining (36a) and (37a), we get $A_i x \in P_M(u)$ for each $i \in \mathbb{N} \cup \{0\}$. Consequently, $A_i(P_M(u)) \subseteq P_M(u)$, for each $i \in \mathbb{N} \cup \{0\}$. Since $S(P_M(u)) = P_M(u) = T(P_M(u))$, therefore we have

$$\begin{aligned} A_0(P_M(u)) &\subseteq S(P_M(u)), \\ A_i(P_M(u)) &\subseteq T(P_M(u)), \\ &\text{for each } i \in \mathbb{N}. \end{aligned} \tag{38}$$

Hence, by Theorem 18 $P_M(u) \cap F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$. □

The following corollary improves and extends [4, Theorem 8], [8, Corollary 2.9], and [10, Theorem 4].

Corollary 26. *Let M be a subset of a normed space X and let $S, T, A_i: X \to X$ be mappings for each $i \in \mathbb{N} \cup \{0\}$ such that $u \in F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i))$ for some $u \in X$ and $A_i(\partial M \cap M) \subseteq M$ for each $i \in \mathbb{N} \cup \{0\}$. Suppose that S and T are q-affine and continuous on $P_M(u)$ and also $P_M(u)$ is q-star-shaped and $S(P_M(u)) = P_M(u) = T(P_M(u))$. If the pairs (A_0, S) and (A_i, T) are modified subcompatible for each $i \in \mathbb{N}$ and also the family $\{A_i\}_{i\in\mathbb{N}\cup\{0\}}$ of maps is (S, T)-nonexpansive, then $P_M(u) \cap F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$, provided one of the following conditions hold.*

(a) *$P_M(u)$ is sequentially compact.*

(b) *$P_M(u)$ is weakly compact, X is complete, and $(S - A_i)$ is demiclosed at 0 for each $i \in \mathbb{N} \cup \{0\}$.*

(c) *$P_M(u)$ is weakly compact and X is complete space satisfying Opial's condition.*

The following corollary generalizes [12, Theorem 5] and [8, Corollary 2.10].

Corollary 27. *Let M be a subset of a normed space X and let $S, A: X \to X$ be mappings such that $u \in F(A) \cap F(S)$ for some $u \in X$ and $A(\partial M \cap M) \subseteq M$. Suppose that S is q-affine and continuous on $P_M(u)$ and also $P_M(u)$ is q-star-shaped and $S(P_M(u)) = P_M(u)$. If the pair (A, S) is modified subcompatible and satisfies for all $x \in P_M(u) \cup \{u\}$*

$$\begin{aligned} &\|Ax - Ay\| \\ &\le \begin{cases} \|Sx - Su\|, & \text{if } y = u \\ \max\Big\{\|Sx - Sy\|, \operatorname{dist}(Sx, [q, Ax]), \\ \qquad \operatorname{dist}(Sy, [q, Ay]), \\ \qquad \frac{1}{2}[\operatorname{dist}(Sx, [q, Ay]) \\ \qquad\qquad + \operatorname{dist}(Sy, [q, Ax])]\Big\} & \text{if } y \in P_M(u), \end{cases} \end{aligned} \tag{39}$$

then $P_M(u) \cap F(S) \cap F(A) \neq \phi$, provided one of the following conditions hold.

(a) *$P_M(u)$ is sequentially compact.*

(b) *$P_M(u)$ is complete and $\operatorname{cl}(A(P_M(u)))$ is compact.*

(c) *$P_M(u)$ is weakly compact, X is complete, and $(S - A)$ is demiclosed at 0.*

4. Examples

Now, we present some examples which demonstrate the validity of the proved results.

Example 28. Let $X = R$ with usual norm $\|x\| = |x|$ and $M = [0, 1]$. Suppose $A_0, A_i : M \to M$ are defined as

$$\begin{aligned} A_0(x) &= 1, \quad \text{for } 0 \le x \le 1, \\ A_i(x) &= \frac{x+i}{i+1}, \quad \text{for each } i \in \mathbb{N}, 0 \le x \le 1 \end{aligned} \tag{40}$$

and also $S, T : M \to M$ are defined as

$$S(x) = \frac{x+1}{2}, \quad T(x) = x, \quad \text{for } 0 \le x \le 1. \tag{41}$$

Here $A_0(M) = \{1\}$, $T(M) = [0, 1]$, $S(M) = [1/2, 1]$, and $A_i(M) = [i/(i+1), 1]$ for each $i \in \mathbb{N}$, so that $A_0(M) \subseteq T(M)$ and $A_i(M) \subseteq S(M)$ for each $i \in \mathbb{N}$. Besides M is compact and the pairs of mappings $\{A_0, S\}$ and $\{A_i, T\}$ are modified subcompatible for each $i \in \mathbb{N}$ and also the maps S and T are q-affine for $q = 1$. Further the mappings S, T, and A_i for each $i \in \mathbb{N} \cup \{0\}$ satisfy the inequality (18). Hence all the conditions of Theorem 18(a) are satisfied. Therefore S, T, and all A_i ($i \in \mathbb{N} \cup \{0\}$) have a common fixed point and $x = 1$ is such a unique common fixed point.

Remark 29. (1) In Example 28, if we define $A_0(x) = A_i(x) = S(x) = T(x) = x$ for all $x \in X \sim M$, then S, T, and all A_i ($i \in \mathbb{N} \cup \{0\}$) are self-maps of X and $u = 2 \in F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i))$. Clearly, $P_M(u) = \{1\}$ is q-star-shaped and $S(P_M(u)) = P_M(u) = T(P_M(u))$. Therefore, all the conditions of Theorem 25 are satisfied and, hence, $P_M(u) \cap F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$. Here, $x = 1 \in P_M(u) \cap F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) \neq \phi$.

(2) If inequality (18) in Theorem 18 is replaced with the weaker condition

$$\begin{aligned}&\|A_0x - A_iy\|\\&\quad\le \max\Big\{\|Sx - Ty\|, \|Sx - A_0x\|,\\&\qquad \|Ty - A_iy\|,\\&\qquad \frac{1}{2}\left[\|Sx - A_iy\| + \|Ty - A_0x\|\right]\Big\},\end{aligned} \tag{42}$$

for each $i \in \mathbb{N}$ and, for any $x, y \in M$. Then, Theorem 18 need not be true. This can be seen by the following example.

Example 30. Let $X = R$ with usual norm $\|x\| = |x|$ and $M = [0, 1]$. Suppose $A_0, A_i : M \to M$ are defined as

$$\begin{aligned}A_0(x) &= \frac{1}{2}, \quad \text{for } 0 \le x \le 1,\\ A_i(x) &= \frac{3}{4}, \quad \text{for each } i \in \mathbb{N}, 0 \le x \le 1\end{aligned} \tag{43}$$

and also $S, T : M \to M$ are defined as

$$S(x) = \begin{cases}\frac{1}{2} & \text{if } 0 \le x < \frac{1}{2}\\ \frac{1}{2}x + \frac{1}{4} & \text{if } \frac{1}{2} \le x \le 1,\end{cases} \tag{44}$$

$$T(x) = \begin{cases}x & \text{if } 0 \le x < \frac{1}{2}\\ 1 - x & \text{if } \frac{1}{2} \le x \le 1.\end{cases} \tag{45}$$

Here $A_0(M) = \{1/2\}$, $T(M) = [0, 1/2]$, $S(M) = [1/2, 3/4]$, and $A_i(M) = \{3/4\}$ for each $i \in \mathbb{N}$, so that $A_0(M) \subseteq T(M)$ and $A_i(M) \subseteq S(M)$ for each $i \in \mathbb{N}$. Besides M is compact and the pairs of mappings $\{A_0, S\}$ and $\{A_i, T\}$ are modified subcompatible for each $i \in \mathbb{N}$ and also the maps S and T are q-affine for $q = 1/2$. Further, the mappings S, T, and A_i for each $i \in \mathbb{N} \cup \{0\}$ are continuous and satisfy the inequality (42). Note that $F(T) \cap F(S) \cap F(A_0) \cap (\bigcap_{i\in\mathbb{N}} F(A_i)) = \phi$.

Remark 31. Clearly mappings S, T, and A_i for each $i \in \mathbb{N} \cup \{0\}$ defined in Example 30 satisfy all of the conditions of Theorem 18(a) except the inequality (18) at $x = 1/2$, $y = 1/2$. Note that there is no common fixed point of S, T, and A_i for each $i \in \mathbb{N} \cup \{0\}$.

Example 32. Let $X = R$ with usual norm $\|x\| = |x|$ and $M = [0, 1]$. Suppose $T, S, A : M \to M$ are defined as

$$\begin{aligned}T(x) &= \begin{cases}x & \text{if } 0 \le x < \frac{1}{2}\\ 1 - x & \text{if } \frac{1}{2} \le x \le 1,\end{cases}\\ S(x) &= \begin{cases}x & \text{if } 0 \le x < \frac{1}{2}\\ \frac{1}{2}x + \frac{1}{4} & \text{if } \frac{1}{2} \le x \le 1,\end{cases}\\ A(x) &= \frac{1}{2}, \quad \text{for } 0 \le x \le 1.\end{aligned} \tag{46}$$

Here we observe that $A(M) = \{1/2\}$, $S(M) = [0, 3/4]$, and $T(M) = [0, 1/2]$ so that $A(M) \subseteq S(M) \cap T(M)$. Also, M is q-star-shaped and the maps S and T are q-affine with $q = 1/2$. We also observe that the pairs (A, S) and (A, T) are modified subcompatible and M is sequentially compact. Further, the mappings A, S, and T satisfy (26). Hence, the mappings A,S, and T satisfy all the conditions of Corollary 20(a) and $x = 1/2$ is the unique common fixed point of mappings A, S, and T.

Remark 33. In Example 32, S and T are not affine because for $x = 3/5$, $y = 0$, and $k = 1/3$, $S(kx + (1 - k)y) = kS(x) + (1 - k)S(y)$ and $T(kx + (1 - k)y) = kT(x) + (1 - k)T(y)$ do not hold. Therefore, Theorem 2.2 of Hussain and Jungck [8] cannot apply to Example 32; hence Corollary 20 is more general than Theorem 2.2 of [8].

Example 34. Take X, M, and S as in Example 32 and define

$$\begin{aligned}A(x) &= \frac{1}{4}, \quad \text{for } 0 \le x \le 1,\\ T(x) &= \begin{cases}\frac{1}{2} & \text{if } 0 \le x < \frac{1}{2}\\ 1 - x & \text{if } \frac{1}{2} \le x \le 1.\end{cases}\end{aligned} \tag{47}$$

Then all of the conditions of Corollary 20(a) are satisfied except that the pair (A, T) is modified subcompatible. Note that $F(T) \cap F(S) \cap F(A) = \phi$.

Remark 35. All results of the paper can be proved for Hausdorff locally convex spaces defined and studied by various authors (see [16, 23–27]).

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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