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Research Article

Exponential Synchronization for Stochastic Neural Networks with Mixed Time Delays and Markovian Jump Parameters via Sampled Data

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The exponential synchronization issue for stochastic neural networks (SNNs) with mixed time delays and Markovian jump parameters using sampled-data controller is investigated. Based on a novel Lyapunov-Krasovskii functional, stochastic analysis theory, and linear matrix inequality (LMI) approach, we derived some novel sufficient conditions that guarantee that the master systems exponentially synchronize with the slave systems. The design method of the desired sampled-data controller is also proposed. To reflect the most dynamical behaviors of the system, both Markovian jump parameters and stochastic disturbance are considered, where stochastic disturbances are given in the form of a Brownian motion. The results obtained in this paper are a little conservative comparing the previous results in the literature. Finally, two numerical examples are given to illustrate the effectiveness of the proposed methods.

1. Introduction

Neural networks, such as Hopfield neural networks, cellular neural networks, the Cohen-Grossberg neural networks, and bidirectional associative neural networks, are very important nonlinear circuit networks and, in the past few decades, have been extensively studied due to their potential applications in classification, signal and image processing, parallel computing, associate memories, optimization, cryptography, and so forth; see [1–7]. Many results, which deal with the dynamics of various neural networks such as stability, periodic oscillation, bifurcation, and chaos, have been obtained by applying the Lyapunov stability theory; see, for example, [8–10] and the references therein. As a special case, synchronization issues of the neural network systems have been extensively investigated too, and a lot of criteria have been developed to guarantee the global synchronization of the network systems in [11–17].

It has been widely reported that a neural network sometimes has finite modes that switch from one mode to another at different times; such a switching (jumping) signal between different neural network models can be governed by a Markovian chain; see [18–25] and the references therein. This class of systems has the advantage of modeling the dynamic systems subject to abrupt variation in their structures and has many applications such as target tracking problems, manufactory processes, and fault-tolerant systems. In [24], delay-interval dependent stability criteria are obtained for neural networks with Markovian jump parameters and time-varying delays, which are based on free-weighing matrix method and LMIs technique. In [25], by introducing some free-weighting matrices, delay-dependent stochastic exponential synchronization conditions are derived for chaotic neural networks with Markovian jump parameters and mixed time delays in terms of the Jensen inequality and linear matrix inequalities.

It is well known that noise disturbance widely exists in biological networks due to environmental uncertainties, which is a major source of instability and can lead to poor performances in neural networks. Such systems are described by stochastic differential systems which have been used

efficiently in modeling many practical problems that arise in the fields of engineering, physics, and science as well. Therefore, the theory of stochastic differential equation is also attracting much attention in recent years and many results have been reported in the literature [26–30]. In addition to the noise disturbance, time delay is also a major source for causing instability and poor performances in neural networks; see, for example, [31–35]. It is known that time delays are often encountered in real neural networks, and the existence of time delays may cause oscillation or instability in neural networks, which are harmful to the applications of neural networks. Therefore, the stability analysis for neural networks with time delays has been widely studied in the literature.

On the other hand, as the rapid development of computer hardware, the sampled-data control technology has shown superiority over other control approaches because it is difficult to guarantee that the state variables transmitted to controllers are continuous in many real-world applications. In [36], Wu et al. investigated the synchronization problem of neural networks with time-varying delay under sampled-data control in the presence of a constant input delay. In [37], by using sampled-data controller, the global synchronization of the chaotic Lur'e systems is discussed and sufficient conditions are obtained in terms of effective synchronization linear matrix inequality by constructing the new discontinuous Lyapunov functionals. Wu et al. studied the sampled-data synchronization for Markovian jump neural networks with time-varying delay; some new and useful synchronization conditions in the framework of the input delay approach and the linear matrix inequality technique are derived in [38].

Motivated by the above discussion, in this paper we study the delay-dependent exponential synchronization of neural networks with stochastic perturbation, discrete and distributed time-varying delays, and Markovian jump parameters. Here, it should be mentioned that our results are delay dependent, which depend on not only the upper bounds of time delays but also their lower bounds. Moreover, the derivatives of time delays are not necessarily zero or smaller than one since several free matrices are introduced in our results. By constructing an appropriate Lyapunov-Krasovskii functional based on delay partitioning, several improved delay-dependent criteria are developed to achieve the exponential synchronization in mean square in terms of linear matrix inequalities. Two numerical examples are also provided to demonstrate the advantage of the theoretical results.

The rest of this paper is organized as follows. In Section 2, the model of stochastic neural network with both mixed time delays and Markovian jump parameters under sampled-data control is introduced, together with some definitions and lemmas. Exponential synchronization is proposed for neural networks with both Markovian jump parameters and mixed time delays via sampled data in Section 3. In Section 4, exponential synchronization is proved for stochastic neural networks with both Markovian jump parameters and mixed time delays under sampled-data control. In Section 5, two illustrative examples are given to demonstrate the validity of the proposed results. Finally, some conclusions are drawn in Section 6.

Notations. Throughout this paper, R denotes the set of real numbers, R^n denotes the n-dimensional Euclidean space, and $R^{m\times n}$ denotes the set of all $m\times n$ real matrices. For any matrix A, A^T denotes the transpose of A. If A is a real symmetric matrix, $A > 0$ ($A < 0$) means that A is positive definite (negative definite). $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ represent minimum and maximum eigenvalues of a real symmetric matrix, respectively. $(\Omega\ \mathscr{F}\ \mathscr{P})$ is a complete probability space, where Ω is the sample space, $\mathscr{F}$ is the σ-algebra of subsets of the sample space, and $\mathscr{P}$ is the probability measure on $\mathscr{F}$. $\operatorname{diag}\{\cdots\}$ denotes a block-diagonal matrix and $\operatorname{col}\{\cdots\}$ stands for a matrix column with blocks given by the matrices in $\{\cdots\}$. $\mathbb{E}\{\cdot\}$ denotes the expectation operator with respect to some probability measure $\mathscr{P}$. Given the column vectors $x = (x_1,\ldots,x_n)^T$, $y = (y_1,\ldots,y_n)^T \in R^n$, $x^Ty = \sum_{i=1}^n x_iy_i$, $|x| = (|x_1|,\ldots,|x_n|)^T$, and $\|x\| = (\sum_{i=1}^n x_i^2)^{1/2}$. $\dot{x}(t)$ denotes the derivative of $x(t)$ and $*$ represents the symmetric form of matrix. Matrices, if their dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Model Description and Preliminaries

Let $\{r(t), t \geq 0\}$ be a right-continuous Markovian chain on the probability space $(\Omega\ \mathscr{F}\ \mathscr{P})$ taking values in a finite state space $\mathscr{S} = \{1,2,\ldots,s\}$ with generator $\Upsilon = (\pi_{ij})_{s\times s}$ given by

$$\begin{aligned} &P\{r(t+\Delta t) = j \mid r(t) = i\} \\ &\quad = P_{ij}(\Delta t) = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & \text{if } i \neq j, \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & \text{if } i = j, \end{cases} \end{aligned} \tag{1}$$

where $\Delta t > 0$ and $\lim_{\Delta t\to 0}(o(\Delta t)/\Delta t) = 0$. Here, $\pi_{ij} \geq 0$ $(j \neq i)$ is the transition rate from i to j if $i \neq j$ at time $t + \Delta t$, and π_{ii} is the transition rate from i to i at time $t + \Delta t$.

Remark 1. The probability defined in (1) is called time-homogeneous transition probability, which is only relevant to the time internal Δt; that is, $P_{ij}(\Delta t)$ is not relevant to the starting point t. Moreover, for the time-homogeneous transition probability defined in (1), the following two properties should be satisfied:

$$P_{ij}(\Delta t) \geq 0, \quad \sum_{j=1}^{s} P_{ij}(\Delta t) = 1. \tag{2}$$

Accordingly, for any Δt satisfying the conditions in (2), the matrix $P = (P_{ij}(\Delta t))_{s\times s}$ is called the probability transition matrix for the right-continuous Markovian chain $\{r(t), t \geq 0\}$.

Fix a probability space $(\Omega\ \mathscr{F}\ \mathscr{P})$ and consider the neural networks with mixed time delays and Markovian jump described by the following differential equation system:

$$\begin{aligned} \dot{x}(t) = &-C(r(t))\,x(t) + W_0(r(t))\,g(x(t)) \\ &+ W_1(r(t))\,g(x(t-d(t))) \\ &+ W_2(r(t))\int_{t-\tau(t)}^{t} g(x(s))\,ds + I(t), \end{aligned} \tag{3}$$

where $x(t) = [x_1(t)\ x_2(t) \cdots x_n(t)]^T \in R^n$ is the neuron state vector and $x_i(t)$ is the state of the ith neuron at time t; $g(x(t)) = [g_1(x_1(t))\ g_2(x_2(t)) \cdots g_n(x_n(t))]^T$ denotes the neuron activation function; $C(r(t)) = \mathrm{diag}\{c_1, c_2, \ldots, c_n\}$ is a diagonal matrix with positive entries; $W_0(r(t)) = (w_{0ij})_{n\times n}$, $W_1(r(t)) = (w_{1ij})_{n\times n}$, and $W_2(r(t)) = (w_{2ij})_{n\times n}$ are, respectively, the connection weight matrix, the discretely delayed connection weight matrix, and the distributively delayed connection weight matrix; $I(t) = [I_1(t)\ I_2(t) \cdots I_n(t)]^T$ is an external input vector; $d(t)$ and $\tau(t)$ denote the discrete delay and the distributed delay.

Throughout this paper, we make the following assumptions.

($\mathscr{H}_1$) There exist positive constants d, μ, and τ such that

$$d_1 \le d(t) \le d_2, \quad \dot{d}(t) \le \mu, \quad 0 \le \tau(t) \le \tau. \tag{4}$$

($\mathscr{H}_2$) Each activation function g_i in (3) is continuous and bounded, and there exist constants F_i^- and F_i^+ such that

$$F_i^- \le \frac{g_i(\alpha_1) - g_i(\alpha_2)}{\alpha_1 - \alpha_2} \le F_i^+, \quad i = 1, 2, \ldots, n, \tag{5}$$

where $\alpha_1, \alpha_2 \in R$ and $\alpha_1 \ne \alpha_2$.

Remark 2. In the earlier literature, the activation functions g_i $(i = 1, 2, \ldots, n)$ are supposed to be continuous, differentiable, monotonically increasing, and bounded. Moreover, the constants $F_i^- \equiv 0$ for $i = 1, 2, \ldots, n$ or the constants $F_i^- = -F_i^+$ for $i = 1, 2, \ldots, n$. However, in this paper, the resulting activation functions may be not monotonically increasing and more general than the usual Lipschitz-type conditions. Moreover, the constants F_i^- and F_i^+ $(i = 1, 2, \ldots, n)$ are allowed to be positive, negative, or zero. Hence, Assumption 2 of this paper is weaker than those given in the earlier literature (see, e.g., [39, 40]).

In this paper, we consider system (3) as the master system and a slave system for (3) can be described by the following equation:

$$\begin{aligned}\dot{y}(t) = {} & -C(r(t))\, y(t) + W_0(r(t))\, g(y(t)) \\ & + W_1(r(t))\, g(y(t-d(t))) \\ & + W_2(r(t)) \int_{t-\tau(t)}^{t} g(y(s))\, ds + I(t) + u(t),\end{aligned} \tag{6}$$

where $C(r(t))$ and $W_i(r(t))$ for $i = 0, 1, 2, \ldots$, are matrices given in (3) and $u(t) \in R^n$ is the appropriate control input.

In order to investigate the problem of exponential synchronization between systems (3) and (6), we define the error signal $e(t) = y(t) - x(t)$. Therefore, the error dynamical system between (3) and (6) is given as follows:

$$\begin{aligned}\dot{e}(t) = {} & -C(r(t))\, e(t) + W_0(r(t))\, f(e(t)) \\ & + W_1(r(t))\, f(e(t-d(t))) \\ & + W_2(r(t)) \int_{t-\tau(t)}^{t} f(e(s))\, ds + u(t),\end{aligned} \tag{7}$$

where $f(e(t)) = g(y(t)) - g(x(t))$. It can be found that the functions $f_i(\cdot)$ satisfy the following condition:

$$F_i^- \le \frac{f_i(\alpha)}{\alpha} \le F_i^+, \quad i = 1, 2, \ldots, n, \tag{8}$$

where $\alpha \in \mathbb{R}$ and $\alpha \ne 0$.

The control signal is assumed to be generated by using a zero-order-hold function with a sequence of hold times $0 = t_0 < t_1 < \cdots < t_k < \cdots$. Therefore, the mode-independent state feedback controller takes the following form:

$$u(t) = Ke(t_k), \quad t_k \le t < t_{k+1}, \tag{9}$$

where K is a sampled-data feedback controller gain matrix to be determined, $e(t_k)$ is a discrete measurement of $e(t)$ at the sampling instant t_k, and $\lim_{k\to\infty} t_k = +\infty$. It is assumed that $t_{k+1} - t_k = h_k \le h$ for any integer $k \ge 0$, where h is a positive scalar and represents the largest sampling interval.

By substituting (9) into (7), we obtain

$$\begin{aligned}\dot{e}(t) = {} & -C(r(t))\, e(t) + W_0(r(t))\, f(e(t)) \\ & + W_1(r(t))\, f(e(t-d(t))) \\ & + W_2(r(t)) \int_{t-\tau(t)}^{t} f(e(s))\, ds + Ke(t_k).\end{aligned} \tag{10}$$

For convenience, in the following, each possible value of $e(t)$ is denoted by i, $i \in \mathscr{S}$. Then we have $C_i = C(r(t))$, $W_{0i} = W_0(r(t))$, $W_{1i} = W_1(r(t))$, and $W_{2i} = W_2(r(t))$, where C_i, W_{0i}, and $W_{1i}W_{2i}$, for any $i \in \mathscr{S}$, are known constant matrices of appropriate dimensions. The system (10) can be written as

$$\begin{aligned}\dot{e}(t) = {} & -C_i e(t) + W_{0i} f(e(t)) \\ & + W_{1i} f(e(t-d(t))) \\ & + W_{2i} \int_{t-\tau(t)}^{t} f(e(s))\, ds + Ke(t_k).\end{aligned} \tag{11}$$

The first purpose of this paper is to design a controller with the form (9) to achieve the exponential synchronization of the master system (3) and slave system (6). In other words, we are interested in finding a feedback gain matric K such that the error system (11) is exponentially stable.

As mentioned earlier, it is often the case in practice that the neural network is disturbed by environmental noises that affect the stability of the equilibrium. Motivated by this we express a stochastic system whose consequent parts are a set of stochastic uncertain recurrent neural networks with mixed time delays:

$$\begin{aligned}dx(t) & \\ = {} & \Big\{ -C(r(t))\, x(t) + W_0(r(t))\, g(x(t)) \\ & + W_1(r(t))\, g(x(t-d(t))) \\ & + W_2(r(t)) \int_{t-\tau(t)}^{t} g(x(s))\, ds + I(t) \Big\}\, dt \\ & + \rho(t, x(t), x(t-d(t)), x(t-\tau(t)), r(t))\, d\omega(t),\end{aligned} \tag{12}$$

where $\omega(t) = (\omega_1(t), \omega_2(t), \ldots, \omega_n(t))^T$ is an n-dimensional Brownian motion defined on a complete probability space $(\Omega\ \mathcal{F}\ \mathcal{P})$ satisfying $\mathbb{E}\{d\omega(t)\} = 0$ and $\mathbb{E}\{d\omega^2(t)\} = dt$ and $\rho(t, x(t), x(t-d(t)), x(t-\tau(t)), r(t)) : R^+ \times R^n \times R^n \times R^n \times \mathcal{S} \to R^{n\times m}$ is the noise intensity function matrix.

And a slave system for (12) can be described by the following equation:

$$\begin{aligned} dy(t) &= \Big\{ -C(r(t))\, y(t) + W_0(r(t))\, g(y(t)) \\ &\quad + W_1(r(t))\, g(y(t-d(t))) \\ &\quad + W_2(r(t)) \int_{t-\tau(t)}^{t} g(y(s))\, ds + I(t) + u(t) \Big\}\, dt \\ &\quad + \rho(t, y(t), y(t-d(t)), y(t-\tau(t)), r(t))\, d\omega(t). \end{aligned} \tag{13}$$

The mode-independent state feedback controller is made as the form of (9) and each possible value of $e(t)$ is denoted by i, $i \in \mathcal{S}$; then we have the final stochastic error system:

$$\begin{aligned} de(t) &= \Big\{ -C_i e(t) + W_{0i} f(e(t)) + W_{1i} f(e(t-d(t))) \\ &\quad + W_{2i} \int_{t-\tau(t)}^{t} f(e(s))\, ds + K e(t_k) \Big\}\, dt \\ &\quad + \rho(t, e(t), e(t-d(t)), e(t-\tau(t)), e(t_k), i)\, d\omega(t). \end{aligned} \tag{14}$$

We impose the following assumption:

$(\mathcal{H}_3)$ $\rho : R^+ \times R^n \times R^n \times R^n \times R^n \times \mathcal{S} \to R^{n\times m}$ is locally Lipschitz continuous and satisfies

$$\begin{aligned} &\operatorname{trace}\left[\rho^T(t, e_1, e_2, e_3, e_4, i)\, \rho(t, e_1, e_2, e_3, e_4, i)\right] \\ &\quad \le e_1^T Y_{1i} e_1 + e_2^T Y_{2i} e_2 + e_3^T Y_{3i} e_3 + e_4^T Y_{4i} e_4. \end{aligned} \tag{15}$$

Our second purpose of this paper is to find a feedback gain matric K in the controller with the form (9) to ensure that the error system (14) is exponentially stable, so that the master system (12) and slave system (13) are exponentially synchronous.

To state our main results, the following definition and lemmas are first introduced, which are essential for the proof in the sequel.

Definition 3. Master system and slave system are said to be exponentially synchronous if error system is exponentially stable; that is, for any initial condition $e(s) = \varphi(s)$ defined on the interval $[-\varpi\ \ 0]$, $\varpi = \max\{d_2, \tau, h\}$, the following condition is satisfied:

$$\mathbb{E}\left\{\|e(t)\|^2\right\} \le \zeta e^{-\epsilon t} \mathbb{E}\left\{\sup_{-\varpi \le s \le 0} \|\varphi(s)\|^2\right\}. \tag{16}$$

Lemma 4 (the Jensen inequality, see [41]). *For any constant matrix $P \in R^{m\times m}$, $P > 0$, scalar $0 < z(t) < z$, vector function $V : [t-z, t] \to R^m$, $t \ge 0$, such that the integrations concerned are well defined; then*

$$\begin{aligned} &\left(\int_0^{z(t)} V(s)\, ds\right)^T P \left(\int_0^{z(t)} V(s)\, ds\right) \\ &\quad \le z(t) \left(\int_0^{z(t)} V^T(s) P V(s)\, ds\right). \end{aligned} \tag{17}$$

Lemma 5 (the Schur complement). *Given one positive definite matrix $G_2 > 0$ and constant matrices G_1 and G_3, where $G_1 = G_1^T$, $G_1 + G_3^T G_2^{-1} G_3 < 0$ if and only if*

$$\begin{pmatrix} G_1 & G_3^T \\ G_3 & -G_2 \end{pmatrix} < 0 \quad or \quad \begin{pmatrix} -G_2 & G_3^T \\ G_3 & G_1 \end{pmatrix} < 0. \tag{18}$$

Lemma 6 (see [42]). *For any constant matric $M \in R^{k\times n}$, symmetric positive definite matrix $R \in R^{n\times n}$, two functions $\nu_1(t)$ and $\nu_2(t)$ satisfying $0 < \nu_m \le \nu_1(t) < \nu_2(t) \le \nu_M$ $(t \ge 0)$, and vector function $V : [\nu_m, \nu_M] \to R^n$ such that the integrations concerned are well defined, let*

$$\int_{\nu_1(t)}^{\nu_2(t)} V(s)\, ds = \widetilde{\xi}^T \phi(t), \tag{19}$$

where $\widetilde{\xi} \in R^{k\times n}$ and $\phi(t) \in R^k$. Then the following inequality holds:

$$\begin{aligned} &\phi(t)\left[M\widetilde{\xi}^T + \widetilde{\xi} M^T - (\nu_2(t) - \nu_1(t)) M R^{-1} M\right] \phi(t) \\ &\quad \le \int_{\nu_1(t)}^{\nu_2(t)} V^T(s) R V(s)\, ds. \end{aligned} \tag{20}$$

3. Exponential Synchronization for Markovian Jump Neural Networks with Mixed Time Delays via Sampled Data

To present the main results of this section, we denote $F_1 = \operatorname{diag}\{F_1^- F_1^+, F_2^- F_2^+, \ldots, F_n^- F_n^+\}$, $F_2 = \operatorname{diag}\{(F_1^- + F_1^+)/2, (F_2^- + F_2^+)/2, \ldots, (F_n^- + F_n^+)/2\}$, $g_1 = [I\ \ 0]$, and $g_2 = [0\ \ I]$.

Here, some LMI conditions will be developed to ensure that master system (3) and slave system (6) are exponential synchronous by employing the Lyapunov functionals.

Theorem 7. *Under Assumptions $\mathcal{H}_1$ and $\mathcal{H}_2$, for a given scalar γ, if there exist matrices $P_i > 0$, $Q_1 > 0$, $Q_2 > 0$, $Q_3 > 0$, $Z_1 > 0$, $Z_2 > 0$, $Z_3 > 0$, $U > 0$, S, X, X_1, G, L, and $H_i = [H_{1i}\ \ H_{2i}\ \ H_{3i}]$ and diagonal matrices $V_{1i} > 0$, $V_{2i} > 0$,*

$V_{3i} > 0$, and $V_{4i} > 0$ such that, for any $i \in \mathcal{S}$, inequalities (21)–(24) *hold,*

$$\begin{pmatrix} Z_2 & S \\ * & Z_2 \end{pmatrix} > 0, \tag{21}$$

$$\Sigma_i(h) = \begin{pmatrix} P_i + h\dfrac{X+X^T}{2} & -hX + hX_1 \\ * & -hX_1 - hX_1^T + h\dfrac{X+X^T}{2} \end{pmatrix} > 0, \tag{22}$$

$$\Sigma_{1i}(h) = \begin{pmatrix} \Sigma_{11i} & \Sigma_{12} & \Sigma_{13i} & 0 & \Sigma_{15i} & \Sigma_{16i} & \Sigma_{17i} + \Lambda_1(h) \\ * & \Sigma_{22i} & \Sigma_{23} & \Sigma_{24} & 0 & 0 & 0 \\ * & * & \Sigma_{33i} & \Sigma_{34} & 0 & 0 & \Sigma_{37i} \\ * & * & * & \Sigma_{44} & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & \Sigma_{57i} \\ * & * & * & * & * & \Sigma_{66i} & \Sigma_{67i} + \Lambda_2(h) \\ * & * & * & * & * & * & \Sigma_{77} + hU \end{pmatrix} < 0, \tag{23}$$

$$\Sigma_{2i} = \begin{pmatrix} \Sigma_{11i} & \Sigma_{12} & \Sigma_{13i} & 0 & \Sigma_{15i} & \Sigma_{16i} & \Sigma_{17i} & hg_1^T H_{1i}^T \\ * & \Sigma_{22i} & \Sigma_{23} & \Sigma_{24} & 0 & 0 & 0 & 0 \\ * & * & \Sigma_{33i} & \Sigma_{34} & 0 & 0 & \Sigma_{37i} & 0 \\ * & * & * & \Sigma_{44} & 0 & 0 & 0 & 0 \\ * & * & * & * & \Sigma_{55} & 0 & \Sigma_{57i} & 0 \\ * & * & * & * & * & \Sigma_{66i} & \Sigma_{67i} & hH_{2i}^T \\ * & * & * & * & * & * & \Sigma_{77} & hH_{3i}^T \\ * & * & * & * & * & * & * & -hU \end{pmatrix} < 0 \tag{24}$$

are satisfied, where

$$\begin{aligned}
\Sigma_{11i} &= g_1^T\left(\sum_{j=1}^{s}\pi_{ij}P_j\right)g_1 + Q_1 + Q_2 + Q_3 \\
&\quad - g_1^T Z_1 g_1 + \tau^2 g_1^T Z_3 g_1 - g_1^T\frac{X+X^T}{2}g_1 \\
&\quad + g_1^T H_{1i} g_1 + g_1^T H_{1i}^T g_1 - g_1^T F_1 V_{1i} g_1 \\
&\quad + g_2^T V_{1i} F_2 g_1 + g_1^T F_2 V_{1i} g_2 - g_2^T V_{1i} g_2 \\
&\quad - g_1^T G C_i g_1 - g_1^T C_i^T G^T g_1 \\
&\quad + g_1^T G W_{0i} g_1 + g_1^T W_{0i}^T G^T g_1, \\
\Sigma_{12} &= g_1^T Z_1 g_1, \qquad \Sigma_{13i} = g_1^T G W_{1i} g_2, \\
\Sigma_{15i} &= g_1^T G W_{2i}, \\
\Sigma_{16i} &= g_1^T(X + X_1) - g_1^T H_{1i}^T + g_1^T H_{2i}^T + g_1^T L, \\
\Sigma_{17i} &= g_1^T P_i + g_1^T H_{3i} - g_1^T G \\
&\quad - \gamma g_1^T C_i^T G^T + \gamma g_2^T W_{0i}^T G^T, \\
\Sigma_{22i} &= -Q_1 - g_1^T Z_1 g_1 - g_1^T Z_2 g_1 - g_1^T F_1 V_{2i} g_1 \\
&\quad + g_2^T V_{2i} F_2 g_1 + g_1^T F_2 V_{2i} g_2 - g_2^T V_{2i} g_2, \\
\Sigma_{23} &= g_1^T(Z_2 - S)g_1, \qquad \Sigma_{24} = g_1^T S g_1, \\
\Sigma_{33i} &= -(1-\mu)Q_2 + g_1^T\left(-2Z_2 + S + S^T\right)g_1 \\
&\quad - g_1^T F_1 V_{3i} g_1 + g_2^T V_{3i} F_2 g_1 \\
&\quad + g_1^T F_2 V_{3i} g_2 - g_2^T V_{3i} g_2, \\
\Sigma_{34} &= g_1^T(-S + Z_2)g_1, \qquad \Sigma_{37i} = \gamma g_2^T W_{1i}^T G^T, \\
\Sigma_{44} &= -Q_3 - g_1^T Z_3 g_1 - g_1^T F_1 V_{4i} g_1 \\
&\quad + g_2^T V_{4i} F_2 g_1 + g_1^T F_2 V_{4i} g_2 - g_2^T V_{4i} g_2, \\
\Sigma_{55} &= -Z_3, \qquad \Sigma_{57i} = \gamma W_{2i}^T G^T, \\
\Sigma_{66i} &= X_1 + X_1^T - \frac{X+X^T}{2} - H_{2i} - H_{2i}^T, \\
\Sigma_{67i} &= -H_{3i} + \gamma L^T, \\
\Sigma_{77} &= d_1^2 Z_1 + d_{12}^2 Z_2 - \gamma G - \gamma G^T, \\
\Lambda_1(h) &= hg_1^T\frac{X+X^T}{2}, \\
\Lambda_2(h) &= h\left(-X^T + X_1^T\right),
\end{aligned} \tag{25}$$

and then the master system (3) and the slave system (6) are exponentially synchronous. Moreover, the desired controller gain matrix in (9) can be given by $K = G^{-1}L$.

Proof. Denote $\eta(t) = [e^T(t) f^T(e(t))]^T$ and

$$\mathscr{H} = \begin{pmatrix} \dfrac{X+X^T}{2} & -X + X_1 \\ * & -X_1 - X_1^T + \dfrac{X+X^T}{2} \end{pmatrix}, \tag{26}$$

and then consider the following Lyapunov functional for error system (11):

$$V(e(t), t, r(t)) = \sum_{l=1}^{7} V_l(e(t), t, r(t)), \qquad t \in [t_k, t_{k+1}), \tag{27}$$

where

$$\begin{aligned}
&V_1(e(t),t,r(t)) = e^T(t)P(r(t))e(t),\\
&V_2(e(t),t,r(t))\\
&\quad= \int_{t-d_1}^{t}\eta^T(s)Q_1\eta(s)\,ds + \int_{t-d(t)}^{t}\eta^T(s)Q_2\eta(s)\,ds\\
&\qquad+\int_{t-d_2}^{t}\eta^T(s)Q_3\eta(s)\,ds,\\
&V_3(e(t),t,r(t))\\
&\quad= d_1\int_{-d_1}^{0}\int_{t+\theta}^{t}\dot{e}^T(s)Z_1\dot{e}(s)\,ds\,d\theta,\\
&V_4(e(t),t,r(t))\\
&\quad= d_{12}\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}\dot{e}^T(s)Z_2\dot{e}(s)\,ds\,d\theta,\\
&V_5(e(t),t,r(t))\\
&\quad= \tau\int_{-\tau}^{0}\int_{t+\theta}^{t}f^T(e(s))Z_3f(e(s))\,ds\,d\theta,\\
&V_6(e(t),t,r(t))\\
&\quad= (t_{k+1}-t)\int_{t_k}^{t}\dot{e}^T(s)U\dot{e}(s)\,ds,\\
&V_7(e(t),t,r(t))\\
&\quad= (t_{k+1}-t)\begin{pmatrix}e(t)\\ e(t_k)\end{pmatrix}^T\mathscr{H}\begin{pmatrix}e(t)\\ e(t_k)\end{pmatrix}.
\end{aligned}\tag{28}$$

Let $\mathscr{L}$ be the weak infinitesimal generator of the random process $(e(t), t \geq 0, r(t))$ along system (11). Next, we will compute $\mathscr{L}V_1(e(t),t,i)$, $\mathscr{L}V_2(e(t),t,i)$, $\mathscr{L}V_3(e(t),t,i)$, $\mathscr{L}V_4(e(t),t,i)$, $\mathscr{L}V_5(e(t),t,i)$, $\mathscr{L}V_6(e(t),t,i)$, and $\mathscr{L}V_7(e(t),t,i)$ along the trajectories of the error system (11), respectively, for $r(t) = i \in \mathscr{S}$:

$$\begin{aligned}
&\mathscr{L}V_1(e(t),t,i)\\
&\quad= 2e^T(t)P_i\dot{e}(t) + e^T(t)\left(\sum_{j=1}^{s}\pi_{ij}p_j\right)e(t)\\
&\quad= 2\eta^T(t)g_1^TP_i\dot{e}(t)\\
&\qquad+\eta^T(t)g_1^T\left(\sum_{j=1}^{s}\pi_{ij}p_j\right)g_1\eta(t),
\end{aligned}\tag{29}$$

$$\begin{aligned}
&\mathscr{L}V_2(e(t),t,i)\\
&\quad= \eta^T(t)(Q_1+Q_2+Q_3)\eta(t)\\
&\qquad-\eta^T(t-d_1)Q_1\eta(t-d_1)\\
&\qquad-(1-\dot{d}(t))\eta^T(t-d(t))Q_1\eta(t-d(t))\\
&\qquad-\eta^T(t-d_2)Q_3\eta(t-d_2)\\
&\quad\le \eta^T(t)(Q_1+Q_2+Q_3)\eta(t)\\
&\qquad-\eta^T(t-d_1)Q_1\eta(t-d_1)\\
&\qquad-(1-\mu)\eta^T(t-d(t))Q_1\eta(t-d(t))\\
&\qquad-\eta^T(t-d_2)Q_3\eta(t-d_2),
\end{aligned}\tag{30}$$

$$\begin{aligned}
&\mathscr{L}V_3(e(t),t,i)\\
&\quad= d_1^2\dot{e}^T(t)Z_1\dot{e}(t) - d_1\int_{t-d_1}^{t}\dot{e}^T(s)Z_1\dot{e}(s)\,ds,
\end{aligned}\tag{31}$$

$$\begin{aligned}
&\mathscr{L}V_4(e(t),t,i)\\
&\quad= d_{12}^2\dot{e}^T(t)Z_2\dot{e}(t) - d_{12}\int_{t-d_2}^{t-d_1}\dot{e}^T(s)Z_2\dot{e}(s)\,ds,
\end{aligned}\tag{32}$$

$$\begin{aligned}
&\mathscr{L}V_5(e(t),t,i)\\
&\quad= \tau^2f^T(e(s))Z_3f(e(s))\\
&\qquad-\tau\int_{t-\tau}^{t}f^T(e(s))Z_3f(e(s))\,ds\\
&\quad\le \tau^2f^T(e(s))Z_3f(e(s))\\
&\qquad-\tau\int_{t-\tau(t)}^{t}f^T(e(s))Z_3f(e(s))\,ds\\
&\quad\le \tau^2f^T(e(s))Z_3f(e(s))\\
&\qquad-\left(\int_{t-\tau(t)}^{t}f(e(s))\,ds\right)^TZ_3\int_{t-\tau(t)}^{t}f(e(s))\,ds,
\end{aligned}\tag{33}$$

$$\begin{aligned}
&\mathscr{L}V_6(e(t),t,i)\\
&\quad= -\int_{t_k}^{t}\dot{e}(s)^TU\dot{e}(s)\,ds + (t_{k+1}-t)\dot{e}(t)^TU\dot{e}(t),
\end{aligned}\tag{34}$$

$$\begin{aligned}
&\mathscr{L}V_7(e(t),t,i)\\
&\quad= -\begin{pmatrix}e(t)\\ e(t_k)\end{pmatrix}^T\mathscr{H}\begin{pmatrix}e(t)\\ e(t_k)\end{pmatrix}\\
&\qquad+2(t_{k+1}-t)\begin{pmatrix}e(t)\\ e(t_k)\end{pmatrix}^T\mathscr{H}\begin{pmatrix}\dot{e}(t)\\ 0\end{pmatrix}.
\end{aligned}\tag{35}$$

According to Lemma 4, it follows that

$$\begin{aligned}
&-d_1\int_{t-d_1}^{t}\dot{e}^T(t)Z_1\dot{e}(t)\,ds\\
&\quad\le -\int_{t-d_1}^{t}\dot{e}^T(t)\,dsZ_1\int_{t-d_1}^{t}\dot{e}(t)\,ds\\
&\quad= -\begin{pmatrix}\eta(t)\\ \eta(t-d_1)\end{pmatrix}^T\begin{pmatrix}g_1^TZ_1g_1 & -g_1^TZ_1g_1\\ * & g_1^TZ_1g_1\end{pmatrix}\begin{pmatrix}\eta(t)\\ \eta(t-d_1)\end{pmatrix}.
\end{aligned}\tag{36}$$

On the other hand, denote

$$\mathcal{J}_1(t) = \int_{t-d(t)}^{t-d_1} \dot{e}(s)\,ds,$$
$$\mathcal{J}_2(t) = \int_{t-d_2}^{t-d(t)} \dot{e}(s)\,ds. \tag{37}$$

When $d_1 < d(t) < d_2$, by Lemma 4, we have that

$$\begin{aligned} &d_{12}\int_{t-d_2}^{t-d_1} \dot{e}^T(s) Z_2 \dot{e}(s)\,ds \\ &\quad = d_{12}\int_{t-d(t)}^{t-d_1} \dot{e}^T(s) Z_2 \dot{e}(s)\,ds \\ &\qquad + d_{12}\int_{t-d_{12}}^{t-d(t)} \dot{e}^T(s) Z_2 \dot{e}(s)\,ds \\ &\quad \geq \frac{d_{12}}{d(t)-d_1}\mathcal{J}_1^T(t) Z_2 \mathcal{J}_1(t) \\ &\qquad + \frac{d_{12}}{d_2-d(t)}\mathcal{J}_2^T(t) Z_2 \mathcal{J}_2(t). \end{aligned} \tag{38}$$

It is clear from (21) that

$$\begin{aligned} &\begin{pmatrix} \mathcal{J}_1(t) \\ \mathcal{J}_2(t) \end{pmatrix}^T \begin{pmatrix} Z_2 & S \\ * & Z_2 \end{pmatrix} \begin{pmatrix} \mathcal{J}_1(t) \\ \mathcal{J}_2(t) \end{pmatrix} \\ &\quad = \begin{pmatrix} \mathcal{J}_1^T(t) Z_2 \mathcal{J}_1(t) & \mathcal{J}_1^T(t) S \mathcal{J}_2(t) \\ * & \mathcal{J}_2^T(t) Z_2 \mathcal{J}_2(t) \end{pmatrix} \geq 0. \end{aligned} \tag{39}$$

Moreover,

$$\frac{1}{d_{12}/(d(t)-d_1)} + \frac{1}{d_{12}/(d_1-d(t))} = 1. \tag{40}$$

Based on the lower bounds lemma of [43], we have from (38)–(40) that

$$\begin{aligned} &d_{12}\int_{t-d_2}^{t-d_1} \dot{e}^T(s) Z_2 \dot{e}(s)\,ds \\ &\quad \geq \mathcal{J}_1^T(t) Z_2 \mathcal{J}_1(t) + \mathcal{J}_2^T(t) Z_2 \mathcal{J}_2(t) \\ &\qquad + \mathcal{J}_1^T(t) S \mathcal{J}_2(t) + \mathcal{J}_2^T(t) S \mathcal{J}_1(t) \\ &\quad = \begin{pmatrix} \mathcal{J}_1(t) \\ \mathcal{J}_2(t) \end{pmatrix}^T \begin{pmatrix} Z_2 & S \\ * & Z_2 \end{pmatrix} \begin{pmatrix} \mathcal{J}_1(t) \\ \mathcal{J}_2(t) \end{pmatrix}. \end{aligned} \tag{41}$$

Note that, when $d(t) = d_1$ or $d(t) = d_2$, we have $\mathcal{J}_1(t) = 0$ or $\mathcal{J}_2(t) = 0$, respectively. Thus, (41) still holds.

Therefore,

$$-d_{12}^2\int_{t-d_2}^{t-d_1} \dot{e}^T(t) Z_2 \dot{e}(t) \leq \mathcal{N}^T(t)\mathcal{W}\mathcal{N}(t), \tag{42}$$

where

$$\begin{aligned} &\mathcal{N}(t) \\ &\quad = \left[\eta^T(t-d_1)\ \ \eta^T(t-d(t))\ \ \eta^T(t-d_2)\right]^T, \\ &\mathcal{W} \\ &\quad = \begin{pmatrix} -g_1^T Z_2 g_1 & g_1^T(Z_2-S)g_1 & g_1^T S g_1 \\ * & g_1^T(-2Z_2+S+S^T)g_1 & g_1^T(-S+Z_2)g_1 \\ * & * & -g_1^T Z_2 g_1 \end{pmatrix}. \end{aligned} \tag{43}$$

Inspired by the free-weighting matric approach [44], we can find that, for any appropriately dimensioned matrix H_i, $\begin{pmatrix} H_i^T U^{-1} H_i & H_i^T \\ * & U \end{pmatrix} \geq 0$. Hence, the following inequality holds:

$$\int_{t_k}^{t} \begin{pmatrix} \psi(t) \\ \dot{e}(s) \end{pmatrix}^T \begin{pmatrix} H_i^T U^{-1} H_i & H_i^T \\ * & U \end{pmatrix} \begin{pmatrix} \psi(t) \\ \dot{e}(s) \end{pmatrix} ds \geq 0, \tag{44}$$

where $\psi(t) = [e^T(t)\ e^T(t_k)\ \dot{e}^T(t)]^T$.

From (44), we can immediately get that

$$\begin{aligned} &-\int_{t_k}^{t} \dot{e}^T(s) U \dot{e}(s)\,ds \\ &\quad \leq (t-t_k)\psi^T(t) H_i^T U^{-1} H_i \psi(t) \\ &\qquad + 2\psi^T(t) H_i^T (e(t) - e(t_k)). \end{aligned} \tag{45}$$

Furthermore, according to error system (11), for any appropriately dimensioned matrix G and scalar γ, the following equality is satisfied:

$$\begin{aligned} &2\left[\eta^T(t) g_1^T G + \gamma \dot{e}^T(t) G\right] \\ &\quad \times \Big[-\dot{e}(t) - C_i e(t) + W_{0i} f(e(t)) \\ &\qquad + W_{1i} f(e(t-d(t))) \\ &\qquad + W_{2i}\int_{t-\tau(t)}^{t} f(e(s))\,ds + K e(t_k)\Big] = 0. \end{aligned} \tag{46}$$

On the other hand, we have from (8) that, for any $\varepsilon = 1, 2, \ldots, n$,

$$\begin{aligned} &(f_\varepsilon(e_\varepsilon(t)) - F_\varepsilon^- e_\varepsilon(t))(f_\varepsilon(e_\varepsilon(t)) - F_\varepsilon^+ e_\varepsilon(t)) \\ &\quad \leq 0, \end{aligned} \tag{47}$$

which is equivalent to

$$\begin{aligned} &\eta^T(t) \begin{pmatrix} F_\varepsilon^- F_\varepsilon^+ \hat{e}_\varepsilon \hat{e}_\varepsilon^T & -\frac{F_\varepsilon^- + F_\varepsilon^+}{2}\hat{e}_\varepsilon \hat{e}_\varepsilon^T \\ -\frac{F_\varepsilon^- + F_\varepsilon^+}{2}\hat{e}_\varepsilon \hat{e}_\varepsilon^T & \hat{e}_\varepsilon \hat{e}_\varepsilon^T \end{pmatrix} \eta(t) \\ &\quad \leq 0, \end{aligned} \tag{48}$$

where $\hat{e}_\varepsilon$ denotes the unit column vector with one element on its εth row and zeros elsewhere. Thus, for any appropriately dimensioned diagonal matrices $V_{1i} > 0$, the following inequality holds [45]:

$$\eta^T(t)\begin{pmatrix} -F_1V_{1i} & F_2V_{1i} \\ * & -V_{1i}\end{pmatrix}\eta(t) \geq 0, \tag{49}$$

which implies

$$\begin{aligned}&\eta^T(t)\\ &\quad\times\left(-g_1^TF_1V_{1i}g_1+g_2^TV_{1i}F_2+g_1^TF_2V_{1i}g_2-g_2^TV_{1i}g_2\right)\\ &\quad\times\eta(t)\geq 0.\end{aligned} \tag{50}$$

Similarly, for any appropriately dimensioned diagonal matrices $V_{2i} > 0$, $V_{3i} > 0$, and $V_{4i} > 0$, the following inequalities also hold:

$$\begin{aligned}&\eta^T(t-d_1)\\ &\quad\times\left(-g_1^TF_1V_{2i}g_1+g_2^TV_{2i}F_2+g_1^TF_2V_{2i}g_2-g_2^TV_{2i}g_2\right)\\ &\quad\times\eta(t-d_1)\geq 0,\\ &\eta^T(t-d(t))\\ &\quad\times\left(-g_1^TF_1V_{3i}g_1+g_2^TV_{3i}F_2+g_1^TF_2V_{3i}g_2-g_2^TV_{3i}g_2\right)\\ &\quad\times\eta(t-d(t))\geq 0,\\ &\eta^T(t-d_2)\\ &\quad\times\left(-g_1^TF_1V_{4i}g_1+g_2^TV_{4i}F_2+g_1^TF_2V_{4i}g_2-g_2^TV_{4i}g_2\right)\\ &\quad\times\eta(t-d_2)\geq 0.\end{aligned} \tag{51}$$

Adding the left-hand sides of (46)–(51) to $\mathscr{L}V(e(t),t,i)$ and letting $L = GK$, we have from (29)–(36), (42), and (45) that, for $t \in [t_k, t_{k+1})$,

$$\begin{aligned}&\mathscr{L}V(e(t),t,i)\\ &\quad\leq \mathscr{X}^T(t)\left[\frac{t_{k+1}-t}{h_k}\Sigma_{1i}(h_k)+\frac{t-t_k}{h_k}\Gamma_i(h_k)\right]\mathscr{X}(t),\end{aligned} \tag{52}$$

where

$$\Gamma_i(h_k)=\Sigma_{1i}(0)+h_k\begin{pmatrix} g_1^TH_{1i}^T\\ 0\\ 0\\ 0\\ H_{2i}^T\\ H_{3i}^T\end{pmatrix}U^{-1}\begin{pmatrix} g_1^TH_{1i}^T\\ 0\\ 0\\ 0\\ H_{2i}^T\\ H_{3i}^T\end{pmatrix}^T, \tag{53}$$

$$\mathscr{X}(t)=\left[\eta^T(t)\ \ \eta^T(t-d_1)\ \ \eta^T(t-d(t))\ \ \eta^T(t-d_2)\ \ \left(\int_{t-\tau(t)}^{t}f(e(s))\,ds\right)^T\ \ e^T(t_k)\ \ \dot{e}^T(t)\right]^T.$$

According to Lemma 5, (24) is equivalent to

$$\Sigma_{1i}(h_k)+h\begin{pmatrix} g_1^TH_{1i}^T\\ 0\\ 0\\ 0\\ H_{2i}^T\\ H_{3i}^T\end{pmatrix}U^{-1}\begin{pmatrix} g_1^TH_{1i}^T\\ 0\\ 0\\ 0\\ H_{2i}^T\\ H_{3i}^T\end{pmatrix}^T<0, \tag{54}$$

which implies

$$\Gamma_i(h_k)<0,\qquad \Sigma_{1i}(0)<0. \tag{55}$$

From (23) and (55), it can be seen that

$$\Sigma_{1i}(h_k)=\frac{h_k}{h}\Sigma_{1i}(h)+\frac{h-h_k}{h}\Sigma_{1i}(0)<0. \tag{56}$$

Thus, we can show from (52)–(56) that

$$\begin{aligned}&\mathscr{L}V(e(t),t,i)\\ &\quad\leq -\xi\left(\|e(t)\|^2+\|e(t-d(t))\|^2\right.\\ &\qquad\left.+\|e(t-\tau(t))\|^2+\|e(t_k)\|^2\right).\end{aligned} \tag{57}$$

From the definition of $V(e(t),t,r(t))$, $\dot{e}(t)$, and $f(e(t))$, there exist positive scalars δ_0, δ_1, δ_2, δ_3, and δ_4 such that the following inequality holds:

$$\begin{aligned}&V(e(t),t,i)\\ &\quad\leq \delta_0\|e(t_k)\|^2+\delta_1\|e(t)\|^2+\delta_2\int_{t-\omega}^{t}\|e(s)\|^2ds\end{aligned}$$

$$+\delta_3\int_{t-\varpi}^{t}\|e(s-d(s))\|^2ds$$
$$+\delta_4\int_{t-\varpi}^{t}\|e(s-\tau(s))\|^2ds. \tag{58}$$

Define a new function $\widetilde{V}(e(t),t,r(t)) = e^{\epsilon t}V(e(t),t,r(t))$, where $\epsilon > 0$ and $\epsilon\max\{\delta_0,\delta_1+\delta_2\varpi e^{\epsilon\varpi},\delta_3\varpi e^{\epsilon\varpi},\delta_4\varpi e^{\epsilon\varpi}\} \le \xi$. It can be found that

$$\begin{aligned}
&\widetilde{V}(e(t),t,r(t))\\
&\quad\ge \widetilde{V}_1(e(t),t,r(t)) + \widetilde{V}_7(e(t),t,r(t))\\
&\quad= e^{\epsilon t}\begin{pmatrix}e(t)\\ e(t_k)\end{pmatrix}^T\left[\frac{t_{k+1}-t}{h_k}\Sigma_i(h_k)+\frac{t-t_k}{h_k}\Sigma_i(0)\right]\\
&\qquad\times\begin{pmatrix}e(t)\\ e(t_k)\end{pmatrix},
\end{aligned} \tag{59}$$

where

$$\Sigma_i(h_k) = \frac{h_k}{h}\Sigma_i(h)+\frac{h-h_k}{h}\Sigma_i(0). \tag{60}$$

Due to the fact that $P_i > 0$ and $\Sigma_i(h) > 0$, we can find a sufficiently small scalar $\delta > 0$ such that $P_i > \delta I$ and $\Sigma_i(h) > \delta I$, which implies $\widetilde{V}(e(t),t,r(t)) > \delta e^{\epsilon t}\|e(t)\|^2 > 0$.

On the other hand, from (57) and (58) we have

$$\begin{aligned}
&\mathscr{L}\widetilde{V}(e(t),t,i)\\
&\quad\le e^{\epsilon t}\Big[(\epsilon\delta_0-\xi)\|e(t_k)\|^2+(\epsilon\delta_1-\xi)\|e(t)\|^2\\
&\qquad-\xi\|e(t-d(t))\|^2-\xi\|e(t-\tau(t))\|^2\\
&\qquad+\epsilon\delta_2\int_{t-\varpi}^{t}\|e(s)\|^2ds\\
&\qquad+\epsilon\delta_3\int_{t-\varpi}^{t}\|e(t-d(s))\|^2ds\\
&\qquad+\epsilon\delta_4\int_{t-\varpi}^{t}\|e(t-\tau(s))\|^2ds\Big].
\end{aligned} \tag{61}$$

By using Dynkin's formula, for $T > 0$, we have

$$\begin{aligned}
&\mathbb{E}\left(\widetilde{V}(e(t),t,i)\right)\\
&\quad\le J_1+(\epsilon\delta_0-\xi)\,\mathbb{E}\left\{\int_0^T e^{\epsilon t}\|e(t_k)\|^2dt\right\}\\
&\qquad+(\epsilon\delta_1-\xi)\,\mathbb{E}\left\{\int_0^T e^{\epsilon t}\|e(t)\|^2dt\right\}\\
&\qquad-\xi\mathbb{E}\left\{\int_0^T e^{\epsilon t}\|e(t-d(t))\|^2dt\right\}\\
&\qquad-\xi\mathbb{E}\left\{\int_0^T e^{\epsilon t}\|e(t-\tau(t))\|^2dt\right\}\\
&\qquad+\epsilon\delta_2\mathbb{E}\left\{\int_0^T\int_{t-\varpi}^{t}e^{\epsilon t}\|e(s)\|^2ds\,dt\right\}\\
&\qquad+\epsilon\delta_3\mathbb{E}\left\{\int_0^T\int_{t-\varpi}^{t}e^{\epsilon t}\|e(s-d(s))\|^2ds\,dt\right\}\\
&\qquad+\epsilon\delta_4\mathbb{E}\left\{\int_0^T\int_{t-\varpi}^{t}e^{\epsilon t}\|e(s-\tau(s))\|^2ds\,dt\right\},
\end{aligned} \tag{62}$$

where $J_1 = [\delta_0+\delta_1+\varpi\delta_2+\varpi\delta_3+\varpi\delta_4]\sup_{-\varpi\le s\le 0}\mathbb{E}\|\varphi(s)\|^2$.

Consequently, by changing the integration sequence, the following inequalities hold:

$$\begin{aligned}
&\int_0^T\int_{t-\varpi}^{t}e^{\epsilon t}\|e(s)\|^2ds\,dt\\
&\quad\le\int_{-\varpi}^{T}\left(\int_{\theta}^{\theta+\varpi}e^{\epsilon t}dt\right)\|e(s)\|^2ds\\
&\quad\le\int_{-\varpi}^{T}\varpi e^{\epsilon(\theta+\varpi)}\|e(s)\|^2ds\\
&\quad\le\varpi e^{\epsilon\varpi}\int_0^T e^{\epsilon t}\|e(t)\|^2dt+\varpi e^{\epsilon\varpi}\int_{-\varpi}^{0}\|\varphi(s)\|^2ds\\
&\quad\le\varpi e^{\epsilon\varpi}\int_0^T e^{\epsilon t}\|e(t)\|^2dt+\varpi e^{\epsilon\varpi}\sup_{-\varpi\le s\le 0}\|\varphi(s)\|^2,\\
&\int_0^T\int_{t-\varpi}^{t}e^{\epsilon t}\|e(s-d(s))\|^2dsdt\\
&\quad\le\varpi e^{\epsilon\varpi}\int_0^T e^{\epsilon t}\|e(t-d(t))\|^2dt\\
&\qquad+\varpi e^{\epsilon\varpi}\sup_{-\varpi\le s\le 0}\|\varphi(s)\|^2,\\
&\int_0^T\int_{t-\varpi}^{t}e^{\epsilon t}\|e(s-\tau(s))\|^2dsdt\\
&\quad\le\varpi e^{\epsilon\varpi}\int_0^T e^{\epsilon t}\|e(t-\tau(t))\|^2dt\\
&\qquad+\varpi e^{\epsilon\varpi}\sup_{-\varpi\le s\le 0}\|\varphi(s)\|^2.
\end{aligned} \tag{63}$$

After substituting (63) into the right side of (62) and then using $\epsilon\max\{\delta_0,\delta_1+\delta_2\varpi e^{\epsilon\varpi},\delta_3\varpi e^{\epsilon\varpi},\delta_4\varpi e^{\epsilon\varpi}\} \le \xi$, we can obtain

$$\mathbb{E}\left(\widetilde{V}(e(t),t,i)\right)\le J_1+J_2, \tag{64}$$

where $J_2 = (\epsilon\delta_2\varpi^2e^{\epsilon\varpi}+\epsilon\delta_3\varpi^2e^{\epsilon\varpi}+\epsilon\delta_4\varpi^2e^{\epsilon\varpi})\sup_{-\varpi\le s\le 0}\mathbb{E}\|\varphi(s)\|^2$.

So,

$$\mathbb{E}\|e(T)\|^2 \le \frac{J_1 + J_2}{\delta} e^{-\epsilon t}. \tag{65}$$

Then it can be shown that, for any $t > 0$,

$$\mathbb{E}\|e(t)\|^2 \le \zeta e^{-\epsilon t} \mathbb{E}\left\{\sup_{-\varpi \le s \le 0} \|\varphi(s)\|^2\right\}, \tag{66}$$

where $\zeta = (1/\delta)[\delta_0+\delta_1+\tau\delta_2+\tau\delta_3+\tau\delta_4+\epsilon\delta_2\varpi^2 e^{\epsilon\varpi}+\epsilon\delta_3\varpi^2 e^{\epsilon\varpi}+\epsilon\delta_4\varpi^2 e^{\epsilon\varpi}]$.

Consequently, according to the Lyapunov-Krasovskii stability theory and Definition 3, we know that the error system (11) is exponentially stable. This completes the proof. □

4. Exponential Synchronization for Stochastic Neural Networks with Mixed Time Delays and Markovian Jump via Sampled Data

In this section, some sufficient conditions of exponential synchronization for stochastic error system (14) are obtained by employing the Lyapunov-Krasovskii functionals.

Theorem 8. *Under Assumptions $\mathscr{H}_1$, $\mathscr{H}_2$, and $\mathscr{H}_3$, for given scalar γ, the error system (14) is globally exponentially stable, which ensures that the master system (12) and slave system (13) are stochastically synchronized, if there exist positive scalars λ_i, symmetric positive definite matrices P_i, Q_1, Q_2, Q_3, Z_1, Z_2, Z_3, and Z_4, positive definite matrices Y_{1i}, Y_{2i}, Y_{3i}, and Y_{4i}, and matrices $\widetilde{R}_k$, $\widetilde{Q}_k$, $\widetilde{T}_k$, $(k = 1, 2, \ldots, 8)$, G, and L, such that, for any $i \in \mathcal{S}$, the following matrix inequalities hold:*

$$P_i \le \lambda_i I, \tag{67}$$

$$\Omega = \begin{pmatrix} \Pi & \sqrt{d_1}\widetilde{R} & \sqrt{d_{12}}\widetilde{Q} & \sqrt{h}\widetilde{T} \\ * & -Z_1 & 0 & 0 \\ * & * & -Z_2 & 0 \\ * & * & * & -Z_4 \end{pmatrix} < 0, \tag{68}$$

where $\widetilde{R} = col\{\widetilde{R}_1\ \widetilde{R}_2 \cdots \widetilde{R}_8\}$, $\widetilde{Q} = col\{\widetilde{Q}_1\ \widetilde{Q}_2 \cdots \widetilde{Q}_8\}$, $\widetilde{T} = col\{\widetilde{T}_1\ \widetilde{T}_2 \cdots \widetilde{T}_8\}$, and Π is given as follows:

$$\Pi = \begin{pmatrix} \Pi_{11i} & \Pi_{12} & \Pi_{13i} & \Pi_{14} & \Pi_{15} & \Pi_{16} & \Pi_{17i} & \Pi_{18i} \\ * & \Pi_{22} & \Pi_{23} & \Pi_{24} & \Pi_{25} & \Pi_{26} & \Pi_{27} & \Pi_{28} \\ * & * & \Pi_{33} & -\widetilde{Q}_3 & 0 & -\widetilde{T}_3 & \gamma F^T W_{1i}^T G & 0 \\ * & * & * & \Pi_{44} & -\widetilde{Q}_5^T & \Pi_{46} & -\widetilde{Q}_7^T & -\widetilde{Q}_8^T \\ * & * & * & * & \lambda_i Y_{3i} & -\widetilde{T}_5 & 0 & 0 \\ * & * & * & * & * & \Pi_{66} & \Pi_{67i} & -\widetilde{T}_8^T \\ * & * & * & * & * & * & \Pi_{77} & \gamma G W_{2i} \\ * & * & * & * & * & * & * & -Z_3 \end{pmatrix}, \tag{69}$$

where

$$\begin{aligned}
\Pi_{11i} &= \sum_{j=1}^{s} \pi_{ij} P_j + \lambda_i Y_{1i} + Q_1 + Q_2 + Q_3 \\
&\quad + \tau^2 F^T Z_3 F - G C_i - C_i^T G \\
&\quad + G W_{0i} F + F^T W_{0i}^T G + \widetilde{R}_1 + \widetilde{R}_1^T + \widetilde{T}_1 + \widetilde{T}_1^T, \\
\Pi_{12} &= -\widetilde{R}_1 + \widetilde{R}_2^T + \widetilde{Q}_1 + \widetilde{T}_2^T, \\
\Pi_{13i} &= G W_{1i} F + \widetilde{R}_3^T + \widetilde{T}_3^T, \\
\Pi_{14} &= \widetilde{R}_4^T - \widetilde{Q}_1 + \widetilde{T}_4^T, \\
\Pi_{15} &= \widetilde{R}_5^T + \widetilde{T}_5^T, \\
\Pi_{16} &= L + \widetilde{R}_6^T + \widetilde{T}_6^T - \widetilde{T}_1, \\
\Pi_{17i} &= P_i - G - \gamma C_i^T G + \gamma F^T W_{0i}^T G + \widetilde{R}_7^T + \widetilde{T}_7^T, \\
\Pi_{18i} &= G W_{2i} + \widetilde{R}_8^T + \widetilde{T}_8^T, \\
\Pi_{22} &= -Q_1 - \widetilde{R}_2 - \widetilde{R}_1^T + \widetilde{Q}_2 + \widetilde{Q}_2^T, \\
\Pi_{23} &= -\widetilde{R}_3^T + \widetilde{Q}_3^T, \\
\Pi_{24} &= -\widetilde{R}_4^T - \widetilde{Q}_2 + \widetilde{Q}_4^T, \\
\Pi_{25} &= -\widetilde{R}_5^T + \widetilde{Q}_5^T, \qquad \Pi_{26} = -\widetilde{R}_6^T + \widetilde{Q}_6^T - \widetilde{T}_2, \\
\Pi_{27} &= -\widetilde{R}_7^T + \widetilde{Q}_7^T, \qquad \Pi_{28} = -\widetilde{R}_8^T + \widetilde{Q}_8^T, \\
\Pi_{33} &= \lambda_i Y_{2i} - (1-\mu) Q_2,
\end{aligned}$$

$$\begin{aligned}
\Pi_{44} &= -Q_3 - \widetilde{Q}_4 - \widetilde{Q}_4^T,\\
\Pi_{46} &= -\widetilde{Q}_6^T - \widetilde{T}_4,\\
\Pi_{66} &= \lambda_i Y_{3i} - \widetilde{T}_6 - \widetilde{T}_6^T,\\
\Pi_{67i} &= \gamma L^T - \widetilde{T}_7^T + \lambda_i Y_{4i},\\
\Pi_{77} &= d_1 Z_1 + d_{12} Z_2 + h Z_4 - \gamma G - \gamma G^T,\\
F &= \operatorname{diag}\left\{\widetilde{F}_1\ \widetilde{F}_2 \cdots \widetilde{F}_n\right\}, \qquad \widetilde{F}_\varepsilon = \max\left\{\left|F_\varepsilon^-\right|, \left|F_\varepsilon^+\right|\right\},\\
&\qquad \varepsilon = 1, 2, \ldots, n.
\end{aligned} \tag{70}$$

Proof. Let $\beta(e(t)) = -C(r(t))e(t) + W_0(r(t))f(e(t)) + W_1(r(t))f(e(t-d(t))) + W_2(r(t))\int_{t-\tau(t)}^{t} f(e(s))ds + Ke(t_k)$. Then, the system (14) can be written as

$$\begin{aligned}
de(t) &= \beta(e(t))\,dt + \rho(t, e(t), e(t-d(t)),\\
&\qquad e(t-\tau(t)), e(t_k), r(t))\,d\omega(t).
\end{aligned} \tag{71}$$

To analyze the stability of error system (14), we construct the following stochastic Lyapunov functional candidate:

$$\begin{aligned}
V(e(t), t, r(t)) &= \sum_{l=1}^{6} V_l(e(t), t, r(t)),\\
&\qquad t \in [t_k, t_{k+1}),
\end{aligned} \tag{72}$$

where

$$\begin{aligned}
V_1(e(t), t, r(t)) &= e^T(t) P(r(t)) e(t),\\
V_2(e(t), t, r(t)) &= \int_{t-d_1}^{t} e^T(s) Q_1 e(s)\,ds + \int_{t-d(t)}^{t} e^T(s) Q_2 e(s)\,ds\\
&\quad + \int_{t-d_2}^{t} e^T(s) Q_3 e(s)\,ds,\\
V_3(e(t), t, r(t)) &= \int_{-d_1}^{0}\int_{t+\theta}^{t} \beta^T(e(s)) Z_1 \beta(e(s))\,ds\,d\theta,\\
V_4(e(t), t, r(t)) &= \int_{-d_2}^{-d_1}\int_{t+\theta}^{t} \beta^T(e(s)) Z_2 \beta(e(s))\,ds\,d\theta,\\
V_5(e(t), t, r(t)) &= \tau\int_{-\tau}^{0}\int_{t+\theta}^{t} f^T(e(s)) Z_3 f(e(s))\,ds\,d\theta,\\
V_6(e(t), t, r(t)) &= \int_{-h}^{0}\int_{t+\theta}^{t} \beta^T(e(s)) Z_4 \beta(e(s))\,ds\,d\theta.
\end{aligned} \tag{73}$$

Let $\mathscr{L}$ be the weak infinitesimal operator of stochastic process $(e(t), t \geq 0, r(t))$ along the trajectories of error system (14). Then we obtain that

$$\begin{aligned}
dV(e(t), t, i) &= \mathscr{L}V(e(t), t, i)\,dt\\
&\quad + 2e^T(t) P_i \rho(t, e(t), e(t-d(t)),\\
&\qquad e(t-\tau(t)), e(t_k), i)\,d\omega(t),
\end{aligned} \tag{74}$$

where

$$\begin{aligned}
\mathscr{L}V(e(t), t, i) &= V_t(e(t), t, i) + V_x(e(t), t, i)\beta(e(t))\\
&\quad + \frac{1}{2}\operatorname{trace}\Big[\rho^T(t, e(t), e(t-d(t)),\\
&\qquad e(t-\tau(t)), e(t_k), i)\\
&\qquad \times V_{ee}(e(t), t, i)\\
&\qquad \times \rho(t, e(t), e(t-d(t)),\\
&\qquad e(t-\tau(t)), e(t_k), i)\Big]\\
&\quad + \sum_{j=1}^{\mathcal{S}} \pi_{ij} V(e(t), t, j),\\
V_t(e(t), t, i) &= \frac{\partial V(e(t), t, i)}{\partial t},\\
V_e(e(t), t, i) &= \left(\frac{\partial V(e(t), t, i)}{\partial e_1}, \frac{\partial V(e(t), t, i)}{\partial e_2}, \ldots, \frac{\partial V(e(t), t, i)}{\partial e_n}\right),\\
V_{ee}(e(t), t, i) &= \left(\frac{\partial_2 V(e(t), t, i)}{\partial e_i \partial e_j}\right)_{n\times n}.
\end{aligned} \tag{75}$$

So, we have that

$$\begin{aligned}
\mathscr{L}V_1(e(t), t, i) &= 2e^T(t) P_i \beta(e(t)) + e^T(t)\left[\sum_{j=1}^{\mathcal{S}} \pi_{ij} P_j\right] e(t)\\
&\quad + \operatorname{trace}\Big[\rho^T(t, e(t), e(t-d(t)), e(t-\tau(t)), e(t_k), i)\\
&\qquad \times P_i \rho(t, e(t), e(t-d(t)),\\
&\qquad e(t-\tau(t)), e(t_k), i)\Big]\\
&\leq 2e^T(t) P_i \beta(e(t)) + e^T(t)\left[\sum_{j=1}^{\mathcal{S}} \pi_{ij} P_j\right] e(t)\\
&\quad + \lambda_i\Big[e^T(t) Y_{1i} e(t) + e^T(t-d(t))
\end{aligned}$$

$$\times Y_{2i}e(t-d(t)) + e^T(t-\tau(t))Y_{3i}e(t-\tau(t)) + e^T(t_k)Y_{4i}e(t_k)\Big],$$

$$\begin{aligned}
\mathscr{L}V_2(e(t),t,i) &= e^T(t)(Q_1+Q_2+Q_3)e(t) \\
&\quad - e^T(t-d_1)Q_1e(t-d_1) \\
&\quad - (1-\mu)e^T(t-d(t))Q_2e(t-d(t)) \\
&\quad - e^T(t-d_2)Q_3e(t-d_2), \\
\mathscr{L}V_3(e(t),t,i) &= d_1\beta^T(e(t))Z_1\beta(e(t)) \\
&\quad - \int_{t-d_1}^{t}\beta^T(e(t))Z_1\beta(e(t))\,ds, \\
\mathscr{L}V_4(e(t),t,i) &= d_{12}\beta^T(e(t))Z_2\beta(e(t)) \\
&\quad - \int_{t-d_1}^{t}\beta^T(e(t))Z_2\beta(e(t))\,ds, \\
\mathscr{L}V_6(e(t),t,i) &= h\beta^T(e(t))Z_4\beta(e(t)) \\
&\quad - \int_{t-h}^{t}\beta^T(e(t))Z_4\beta(e(t))\,ds \\
&\le h\beta^T(e(t))Z_4\beta(e(t)) \\
&\quad - \int_{t_k}^{t}\beta^T(e(t))Z_4\beta(e(t))\,ds.
\end{aligned} \tag{76}$$

It is easy to see from (71) that

$$\begin{aligned}
e(t)-e(t-d_1) &= \int_{t-d_1}^{t}\beta(e(s))\,ds \\
&\quad + \int_{t-d_1}^{t}\rho(t,e(t),e(t-d(t)), \\
&\qquad e(t-\tau(t)),e(t_k),i)\,d\omega(t).
\end{aligned} \tag{77}$$

Taking the mathematical expectation on both sides of (77) yields

$$\begin{aligned}
&\mathbb{E}\left\{\int_{t-d_1}^{t}\beta(e(s))\,ds\right\} \\
&\quad = \mathbb{E}\{e(t)-e(t-d_1)\} = \mathbb{E}\left\{\xi_1^T\mathscr{X}(t)\right\},
\end{aligned} \tag{78}$$

where

$$\begin{aligned}
\mathscr{X}(t) &= \mathrm{col}\left\{e(t)\ \ e(t-d_1)\ \ e(t-d(t))\ \ e(t-d_2)\ \ e(t-\tau(t))\ \ e(t_k)\ \ \beta(t)\int_{t-\tau(t)}^{t}f(e(s))\,ds\right\}, \\
\xi_1 &= \mathrm{col}\{I\ \ -I\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\}.
\end{aligned} \tag{79}$$

Applying Lemma 6, we have that there exists a matrix $\widetilde{R} \in R^{8n\times n}$ such that

$$\begin{aligned}
&\mathbb{E}\left\{-\int_{t-d_1}^{t}\beta^T(e(s))Z_1\beta(e(s))\,ds\right\} \\
&\quad \le \mathbb{E}\left\{\mathscr{X}(t)^T\left[\widetilde{R}\xi_1^T+\xi_1\widetilde{R}^T+d_1\widetilde{R}Z_1^{-1}\widetilde{R}^T\right]\mathscr{X}(t)\right\}.
\end{aligned} \tag{80}$$

Similarly, it can be seen that there exist matrices $\widetilde{Q}$ and $\widetilde{T} \in R^{8n\times n}$ such that

$$\begin{aligned}
&\mathbb{E}\left\{-\int_{t-d_2}^{t-d_1}\beta^T(e(s))Z_2\beta(e(s))\,ds\right\} \\
&\quad \le \mathbb{E}\left\{\mathscr{X}(t)^T\left[\widetilde{Q}\xi_2^T+\xi_2\widetilde{Q}^T+d_{12}\widetilde{Q}Z_2^{-1}\widetilde{Q}^T\right]\mathscr{X}(t)\right\}, \\
&\mathbb{E}\left\{-\int_{t_k}^{t}\beta^T(e(s))Z_4\beta(e(s))\,ds\right\} \\
&\quad \le \mathbb{E}\left\{\mathscr{X}(t)^T\left[\widetilde{T}\xi_3^T+\xi_3\widetilde{T}^T+h\widetilde{T}Z_4^{-1}\widetilde{T}^T\right]\mathscr{X}(t)\right\},
\end{aligned} \tag{81}$$

where

$$\begin{aligned}
\xi_2 &= \mathrm{col}\ \{0\ \ I\ \ 0\ \ -1\ \ 0\ \ 0\ \ 0\ \ 0\}, \\
\xi_3 &= \mathrm{col}\ \{I\ \ 0\ \ 0\ \ 0\ \ 0\ \ -I\ \ 0\ \ 0\}.
\end{aligned} \tag{82}$$

Furthermore, according to the definition of $\beta(e(t))$, for any appropriately dimensioned matrix G and scalar γ, the following equality is satisfied:

$$\begin{aligned}
&2\left[e^T(t)G+\gamma\beta^T(e(t))G\right] \\
&\quad \times\Big[-\beta(e(t))-C_ie(t)+W_{0i}f(e(t)) \\
&\quad + W_{1i}f(e(t-d(t))) \\
&\quad + W_{2i}\int_{t-\tau(t)}^{t}f(e(s))\,ds+Ke(t_k)\Big] = 0.
\end{aligned} \tag{83}$$

Taking the mathematical expectation on both sides of (74), letting $L = GK$, and considering (8), (33), (76), and (80)–(83), we obtain that

$$\begin{aligned}&\mathbb{E}\{\mathscr{L}V(e(t),t,i)\}\\&= \mathbb{E}\left\{\mathcal{X}(t)^T\left(\Pi + d_1\widetilde{R}Z_1^{-1}\widetilde{R}^T\right.\right.\\&\qquad\left.\left.+d_{12}\widetilde{Q}Z_2^{-1}\widetilde{Q}^T + h\widetilde{T}Z_4^{-1}\widetilde{T}^T\right)\mathcal{X}(t)\right\}.\end{aligned} \tag{84}$$

Applying Lemma 5 and (68), we have that

$$\begin{aligned}&\mathbb{E}\{\mathscr{L}V(e(t),t,i)\}\\&\le \mathbb{E}\left\{-\xi\left(\|e(t)\|^2 + \|e(t-d(t))\|^2 + \|e(t-\tau(t))\|^2\right)\right\}.\end{aligned} \tag{85}$$

Following the similar line of the proof of Theorem 7, we can get that, for any $t > 0$,

$$\mathbb{E}\|e(t)\|^2 \le \zeta e^{-\epsilon t}\mathbb{E}\left\{\sup_{-\varpi\le s\le 0}\|\varphi(s)\|^2\right\}, \tag{86}$$

where $\zeta = (1/\delta)[\delta_1 + \tau\delta_2 + \tau\delta_3 + \tau\delta_4 + \epsilon\delta_2\varpi^2 e^{\epsilon\varpi} + \epsilon\delta_3\varpi^2 e^{\epsilon\varpi} + \epsilon\delta_4\varpi^2 e^{\epsilon\varpi}]$.

Thus, the master system and the slave system are exponentially synchronized; the sampled-data feedback control gain is given by $K = G^{-1}L$. This completes the proof. □

5. Illustrative Examples

In this section, two numerical examples are given to demonstrate the effectiveness of the theoretical results.

Example 9. Consider the second-order master system (3) and slave system (6) with the following parameters:

$$\begin{gathered}C_1 = \begin{pmatrix}0.8 & 0\\ 0 & 0.9\end{pmatrix}, \qquad W_{01} = \begin{pmatrix}1.7 & -0.15\\ -5.2 & 3.3\end{pmatrix},\\ W_{11} = \begin{pmatrix}-1.7 & -0.1\\ -0.26 & -2.5\end{pmatrix},\\ C_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \qquad W_{02} = \begin{pmatrix}1.5 & -0.16\\ -5.1 & 3.4\end{pmatrix},\\ W_{12} = \begin{pmatrix}-1.9 & -0.1\\ -0.25 & -2.6\end{pmatrix},\\ W_{21} = \begin{pmatrix}0.7 & 0.15\\ 2 & -0.12\end{pmatrix}, \qquad W_{22} = \begin{pmatrix}0.8 & 0.15\\ 1.5 & -0.12\end{pmatrix},\end{gathered} \tag{87}$$

and the activation functions are taken as $g(\alpha) = (|\alpha+1| - |\alpha-1|)/2$.

It can be verified that $F_1^- = F_2^- = 0$ and $F_1^+ = F_2^+ = 1$. Thus, $F_1 = \left(\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right)$ and $F_2 = \left(\begin{smallmatrix}0.5 & 0\\ 0 & 0.5\end{smallmatrix}\right)$.

It is assumed that $I(t) = 0$, discrete delay $d(t) = e^t/e^t + 1$, and distributed delay $\tau(t) = 0.5\sin^2(t)$. Hence, a straightforward calculation gives $d_1 = 0.5$, $d_2 = 1$, $\mu = 0.25$, and $\tau = 0.5$. Moreover, the transition probability matrix is chosen as $\Upsilon = \left(\begin{smallmatrix}-5 & 5\\ 5 & -5\end{smallmatrix}\right)$.

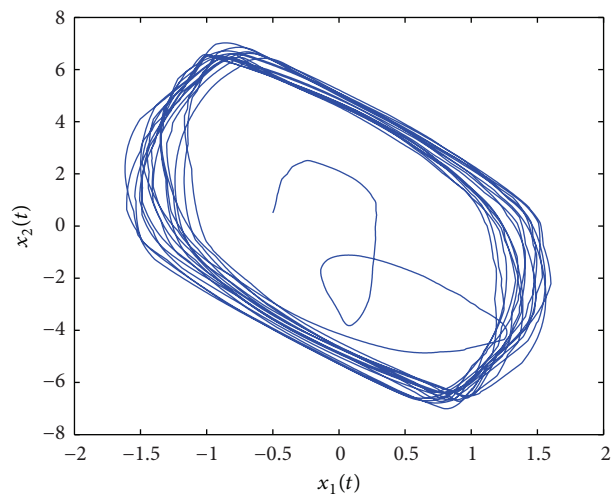

Figure 1: Chaotic behavior of master system (3) with $u(t) = 0$.

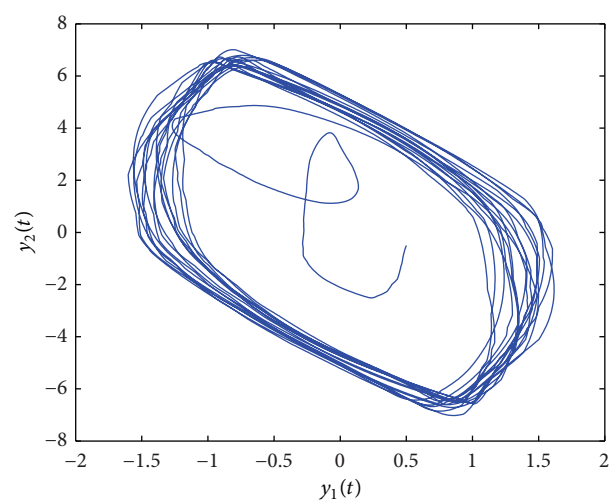

Figure 2: Chaotic behavior of slave system (6) with $u(t) = 0$.

The chaotic behaviors of the master system (3) and slave system (6) with $u(t) = 0$ are given in Figures 1 and 2, respectively, with the initial states chosen as $x(t) = [-0.5\ \ 0.5]^T$ and $y(t) = [0.5\ \ -0.5]^T$, $t \in [-1\ \ 0]$.

Choosing $\gamma = 0.7$ and applying Theorem 7, we can find that the upper bound on the samplings, which preserves that master system (3) and slave system (6) are exponentially synchronous, is 0.11. And using the Matlab LMI Control Toolbox to solve LMIs (21)–(24), we can also obtain the following matrices:

$$\begin{gathered}G = \begin{pmatrix}1.7892 & 0.1326\\ 0.1326 & 0.2696\end{pmatrix},\\ L = \begin{pmatrix}-3.4741 & 0.0513\\ 0.0513 & 0.1577\end{pmatrix}.\end{gathered} \tag{88}$$

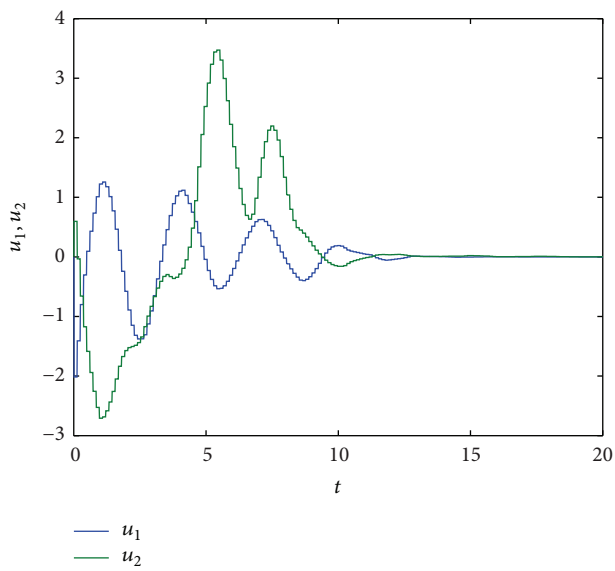

FIGURE 3: Control input $u(t)$.

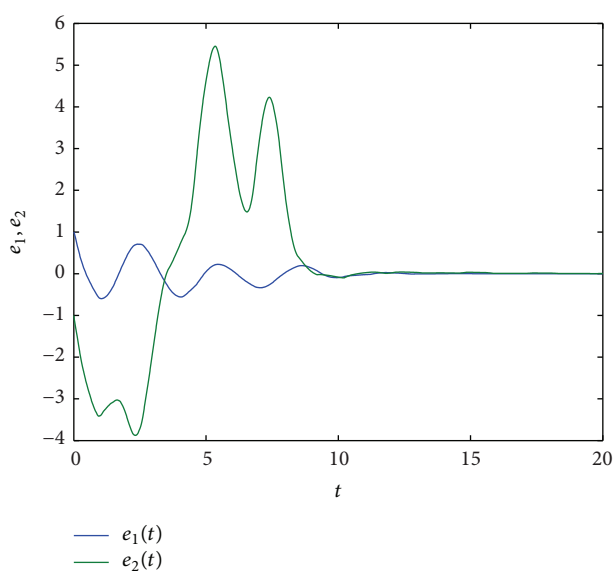

FIGURE 4: State responses of error system (10).

Thus, the corresponding gain matrix in (9) is given by

$$K = \begin{pmatrix} -2.0298 & -0.0152 \\ 1.1882 & 0.5924 \end{pmatrix}. \tag{89}$$

Under the obtained gain matrix in (89), the response curves of control input (9) and error system (11) are exhibited in Figures 3 and 4, respectively. It is obvious from Figure 4 that the slave system (3) exponentially synchronizes with the master system (6).

Example 10. In the following, we consider the second-order stochastic master system (12) and slave system (13) with $I(t) = (I_1(t), I_2(t))^T$; $\omega(t)$ is a second-order Brownian motion and $r(t)$ is a right-continuous Markovian chain taking values in $S = \{1, 2\}$ with generator $\Upsilon = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix}$.

For the two operating conditions (modes), the associated data are

$$\begin{gathered} C_1 = \begin{pmatrix} 0.8 & 0 \\ 0 & 0.9 \end{pmatrix}, \qquad W_{01} = \begin{pmatrix} 0.1 & 5.1 \\ 3.2 & 0.1 \end{pmatrix}, \\ W_{11} = \begin{pmatrix} -0.1 & 3.2 \\ 5.1 & -0.1 \end{pmatrix}, \\ C_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad W_{02} = \begin{pmatrix} 0 & 5.2 \\ 3.1 & 0 \end{pmatrix}, \\ W_{12} = \begin{pmatrix} -0.1 & 3.1 \\ 5.2 & -0.1 \end{pmatrix}, \\ W_{21} = \begin{pmatrix} 0.7 & 0.15 \\ 2 & -0.12 \end{pmatrix}, \qquad W_{22} = \begin{pmatrix} 0.8 & 0.15 \\ 1.5 & -0.12 \end{pmatrix}, \\ \rho(t, e(t), e(t-d(t)), e(t-\tau(t)), e(t_k), 1) \\ = \begin{pmatrix} z_1(t) & 0 \\ 0 & z_2(t) \end{pmatrix}, \\ \rho(t, e(t), e(t-d(t)), e(t-\tau(t)), e(t_k), 2) \\ = \begin{pmatrix} z_3(t) & 0 \\ 0 & z_4(t) \end{pmatrix}, \end{gathered} \tag{90}$$

where

$$\begin{aligned} z_1(t) &= 0.2e_1(t) + 0.3e_1(t-d(t)) \\ &\quad + 0.1e_1(t-\tau(t)) + 0.4e_1(t_k), \\ z_2(t) &= 0.1e_2(t) + 0.2e_2(t-d(t)) \\ &\quad + 0.2e_2(t-\tau(t)) + 0.3e_2(t_k), \\ z_3(t) &= 0.2e_1(t) + 0.3e_1(t-d(t)) \\ &\quad + 0.1e_1(t-\tau(t)) + 0.2e_1(t_k), \\ z_4(t) &= 0.2e_2(t) + 0.1e_2(t-d(t)) \\ &\quad + 0.1e_2(t-\tau(t)) + 0.3e_2(t_k) \end{aligned} \tag{91}$$

and the activation functions are taken as $g(\alpha) = (1/2)\sin(\alpha)$.

It can be verified that $F_1^- = F_2^- = -0.5$ and $F_1^+ = F_2^+ = 0.5$. Thus, $F = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$.

In this example, $I(t) = 0$, discrete delay $d(t) = 1 + 0.3\sin(2t)$, and distributed delay $\tau(t) = 0.5\sin^2(t)$. Then, a straightforward calculation gives $d_1 = 0.7$, $d_2 = 1.3$, $\tau = 0.5$, and $\dot{d}(t) = 0.6\cos(2t) \le 0.6$, which implies $\mu = 0.6$.

Figures 5 and 6 show the chaotic behavior of the master system (12) and slave system (13) with $u(t) = 0$, respectively,

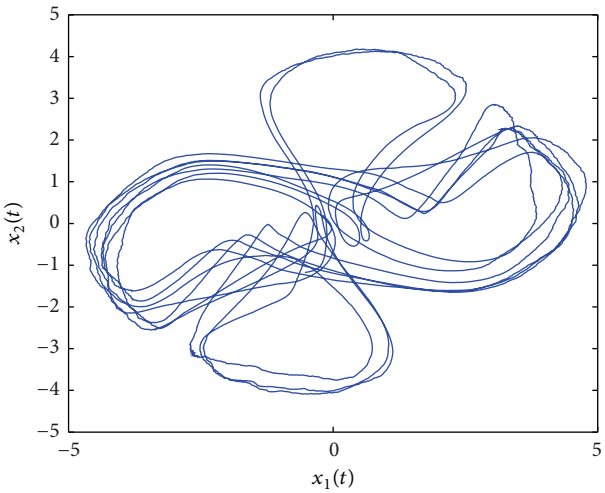

FIGURE 5: Chaotic behavior of master system (12) with $u(t) = 0$.

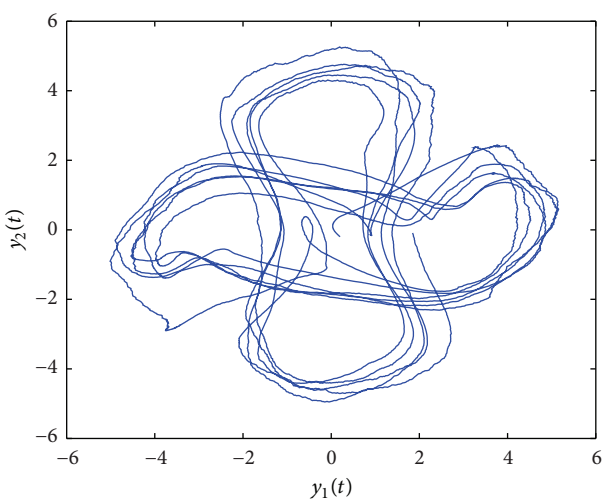

FIGURE 6: Chaotic behavior of slave system (13) with $u(t) = 0$.

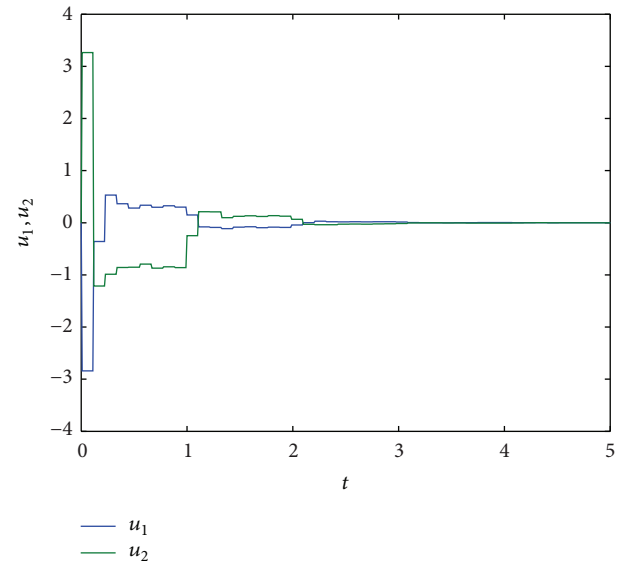

FIGURE 7: Control input $u(t)$.

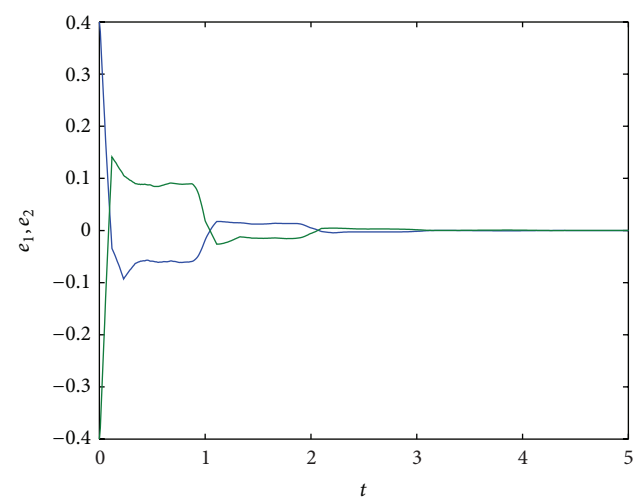

FIGURE 8: State responses of error system (14).

under the initial states $x(t) = [-0.2 \;\; 0.2]^T$ and $y(t) = [0.2 \;\; -0.2]^T$, $t \in [-1.3 \;\; 0]$.

By using Matlab LMI Control Toolbox to solve the LMIs given in Theorem 8 with $\gamma = 0.2$, we can find that the upper bound on the samplings, which preserves that master system (12) and slave system (13) are exponentially synchronous, is 0.1. Moreover, we can also get the following matrices:

$$G = \begin{pmatrix} 0.1798 & -0.0507 \\ -0.0507 & 0.1495 \end{pmatrix}, \qquad L = \begin{pmatrix} -1.7755 & -0.0854 \\ -0.0854 & -1.6652 \end{pmatrix}. \tag{92}$$

Thus, the corresponding gain matrix in (9) is given by

$$K = G^{-1}L = \begin{pmatrix} -11.1000 & -3.9995 \\ -4.3360 & -12.4944 \end{pmatrix}. \tag{93}$$

Under the above given gain matrix, Figures 7 and 8 show the response curves of control input (9) and error system (14), respectively. It is clear from Figures 7 and 8 that the obtained sampled-data controller achieves the exponential synchronization of master system (12) and slave system (13).

6. Conclusion

In this paper, the exponential synchronization issue for stochastic neural networks (SNNs) with mixed time delays and Markovian jump parameters under sampled-data control

has been addressed. New delay-dependent conditions have been presented in terms of LMIs to ensure the exponential stability of the considered error systems, and, thus, the master systems exponentially synchronize with the slave systems. The results obtained in this paper are a little conservative comparing the previous results in the literature. The methods of this paper can be applied to other classes of neural networks such as complex neural networks and impulsive neural networks.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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