Research Article

Global Exponential Stability of Weighted Pseudo-Almost Periodic Solutions of Neutral Type High-Order Hopfield Neural Networks with Distributed Delays

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Some sufficient conditions are obtained for the existence, uniqueness, and global exponential stability of weighted pseudo-almost periodic solutions to a class of neutral type high-order Hopfield neural networks with distributed delays by employing fixed point theorem and differential inequality techniques. The results of this paper are new and they complement previously known results. Moreover, an example is given to show the effectiveness of the proposed method and results.

1. Introduction

Since high-order Hopfield neural networks (HHNNs) have stronger approximation property, faster convergence rate, greater storage capacity, and higher fault tolerance than lower-order Hopfield neural networks, the study of high-order Hopfield neural networks has recently gained a lot of attention and there have been extensive results on the problem of the existence and stability of equilibrium points, periodic solutions, and almost periodic solutions of high-order Hopfield neural networks in the literature. We refer the reader to [1–9] and the references cited therein. Also, since it is natural and important that systems will contain some information about the derivative of the past state to further describe and model the dynamics for such complex neural reactions [10], many authors investigated the dynamical behaviors of neutral type neural networks with delays [11–25]. Moreover, it is well known that, compared with periodic effects, almost periodic effects are more frequent, and many phenomena exhibit great regularity with being pseudo-almost periodic which allow complex repetitive phenomena to be represented as an almost periodic process plus an ergodic component. However, to the best of our knowledge, few authors have considered the exponential convergence on the pseudo-almost periodic solution for neutral type neural networks with delays. Motivated by the above, in this

paper, we consider the following high-order Hopfield neural networks with neutral distributed delays:

$$\begin{aligned}
 x_i'(t) &= -c_i(t)x_i(t) + \sum_{j=1}^n a_{ij}(t) \int_0^\infty d_{ij}(\theta) f_j(x_j(t-\theta)) d\theta \\
 &\quad + \sum_{j=1}^n \alpha_{ij}(t) \int_0^\infty \beta_{ij}(\theta) h_j(x_j'(t-\theta)) d\theta \\
 &\quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \int_0^\infty k_{ij}(\theta) g_j(x_j(t-\theta)) d\theta \\
 &\quad \times \int_0^\infty k_{il}(\theta) g_l(x_l(t-\theta)) d\theta + I_i(t), \\
 i &= 1, 2, \dots, n, \quad t > 0,
 \end{aligned} \tag{1}$$

where n corresponds to the number of units in a neural network, $x_i(t)$ corresponds to the state vector of the i th unit at the time t , $c_i(t)$ represents the rate with which the i th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs, $a_{ij}(t)$ and $b_{ijl}(t)$ are the first- and second-order connection weights of the neural network, d_{ij} , k_{ij} is the kernel, $I_i(t)$ denote the

external inputs at time t , and f_j and g_j are the activation functions of signal transmission.

The initial conditions of (1) are the form

$$\begin{aligned} x_i(s) &= \phi_i(s), & x'_i(s) &= \phi'_i(s), \\ s &\in (-\infty, 0], & i &= 1, 2, \dots, n, \end{aligned} \quad (2)$$

where $\phi_i(\cdot)$ denotes a differential real-value bounded function defined on $(-\infty, 0]$ and satisfies that $\phi'_i(\cdot)$ is bounded on $(-\infty, 0)$.

To the best of our knowledge, there is no paper published on the global exponential stability and existence of weighted pseudo-almost periodic solution to system (1). Our main purpose of this paper is, for fixed $\mu \in \mathbb{U}_{\infty}^{\text{Inv}}$ ($\mathbb{U}_{\infty}^{\text{Inv}}$ will be defined in Section 2), to study the existence, uniqueness, and globally exponential stability of weighted pseudo-almost periodic solution by employing fixed point theorem and differential inequality techniques.

This paper is organized as follows. In Section 2, we introduce some notations and definitions and state some preliminary results which are needed in later sections. In Section 3, we establish some sufficient conditions for the existence of weighted pseudo-almost periodic solutions of (1). In Section 4, we prove that the weighted pseudo-almost periodic solution is globally exponentially stable. In Section 5, we give an example to illustrate the feasibility of our results obtained in previous sections.

2. Assumptions and Preliminaries

Throughout this paper we assume that

(H₁) $c_i(t), a_{ij}(t), \alpha_{ij}(t), b_{ijl}(t)$ are almost periodic functions, and $\inf_{t \in \mathbb{R}} c_i(t) > 0, i, j, l = 1, 2, \dots, n$;

(H₂) there exist positive constants $\epsilon_i > 0, \vartheta_i > 0, \varepsilon_i > 0$, and $G_i > 0$ such that $|f_i(x) - f_i(y)| \leq \epsilon_i|x - y|, |g_i(x) - g_i(y)| \leq \varepsilon_i|x - y|, |h_i(x) - h_i(y)| \leq \vartheta_i|x - y|$, and $|g_i(x)| \leq G_i$, for all $x, y \in \mathbb{R}$, and $f_i(0) = g_i(0) = h_i(0) = 0, i = 1, 2, \dots, n$;

(H₃) for $i, j \in \{1, 2, \dots, n\}$, the delay kernels $d_{ij}, k_{ij}, \beta_{ij} : [0, \infty) \rightarrow \mathbb{R}$ are continuous and integrable with

$$0 \leq \int_0^\infty |k_{ij}(s)| ds \leq k_{ij}^M, \quad 0 \leq \int_0^\infty |d_{ij}(s)| ds \leq d_{ij}^M, \quad (3)$$

$$0 \leq \int_0^\infty |\beta_{ij}(s)| ds \leq \beta_{ij}^M;$$

(H₄) for fixed $\mu \in \mathbb{U}_{\infty}^{\text{Inv}}$, I_i ($i = 1, 2, \dots, n$) is a weighted pseudo-almost periodic function.

Definition 1 (see [26, 27]). Let $u : \mathbb{R} \rightarrow \mathbb{R}^n$ be continuous in t ; $u(t)$ is said to be almost periodic on $\mathbb{R}$ if, for any $\varepsilon > 0$, the set $T(u, \varepsilon) = \{\delta : |u(t + \delta) - u(t)| < \varepsilon, \forall t \in \mathbb{R}\}$ is relatively dense; that is, for all $\varepsilon > 0$, it is possible to find a real number $l = l(\varepsilon) > 0$; for any interval with length $l(\varepsilon)$, there exists a number $\delta = \delta(\varepsilon)$ in this interval such that $|u(t + \delta) - u(t)| < \varepsilon$, for all $t \in \mathbb{R}$.

For $z = (z_1, z_2, \dots, z_n)^T \in \mathbb{R}^n$, we define $\|z\| = \max_{1 \leq i \leq n} |z_i|$.

Definition 2 (see [26, 27]). Let $x \in \mathbb{R}^n$ and let $Q(t)$ be an $n \times n$ continuous matrix defined on $\mathbb{R}$. The linear system

$$x'(t) = Q(t)x(t) \quad (4)$$

is said to admit an exponential dichotomy on $\mathbb{R}^n$ if there exist positive constants k, α ; projection P and the fundamental solution matrix $X(t)$ of (4) satisfy

$$\begin{aligned} \|X(t)PX^{-1}(s)\| &\leq ke^{-\alpha(t-s)} \quad (\forall t \geq s), \\ \|X(t)(I - P)X^{-1}(s)\| &\leq ke^{-\alpha(s-t)} \quad (\forall s \geq t). \end{aligned} \quad (5)$$

Lemma 3 (see [26, 27]). Let $x(t)$ be an almost periodic function on $\mathbb{R}$; then $x(t)$ is bounded on $\mathbb{R}$ and $x(t)$ is uniformly continuous in $t \in \mathbb{R}$.

The collection of all almost periodic functions which go from $\mathbb{R}$ to $\mathbb{R}^n$ will be denoted by $AP(\mathbb{R}, \mathbb{R}^n)$. $AP(\mathbb{R}, \mathbb{R}^n)$ equipped with the sup-norm is a Banach space.

Let $\mathbb{U}$ denote the collection of functions (weights) $\mu : \mathbb{R} \rightarrow (0, +\infty)$, which are locally integrable over $\mathbb{R}$ such that $\mu > 0$ almost everywhere. If $\mu \in \mathbb{U}$ and for $r > 0$, we set $Q_r := [-r, r]$ and

$$\mu(Q_r) := \int_{Q_r} \mu(x) dx. \quad (6)$$

Let $\mathbb{U}_{\infty} = \{\mu \in \mathbb{U} : \inf_{x \in \mathbb{R}} \mu(x) = \mu_0 > 0 \text{ and } \lim_{r \rightarrow \infty} \mu(Q_r) = \infty\}$.

Definition 4 (see [28]). Fix $\mu \in \mathbb{U}_{\infty}$. A continuous function $f : \mathbb{R} \rightarrow \mathbb{X}$ is called weighted pseudo-almost periodic if it can be written as $f = h + \varphi$, with $h \in AP(\mathbb{R}, \mathbb{X})$ and $\varphi \in PAP_0(\mathbb{R}, \mathbb{X}, \mu)$, where the space $PAP_0(\mathbb{R}, \mathbb{X}, \mu)$ is defined by

$$PAP_0(\mathbb{R}, \mathbb{X}, \mu) = \left\{ g \in BC(\mathbb{R}, \mathbb{X}) : \lim_{r \rightarrow \infty} \frac{1}{\mu(Q_r)} \int_{Q_r} \|g(t)\| \mu(t) dt = 0 \right\}. \quad (7)$$

The collection of all weighted pseudo-almost periodic functions $f : \mathbb{R} \rightarrow \mathbb{R}^n$ will be denoted by $PAP(\mathbb{R}, \mathbb{R}^n, \mu)$.

Remark 5. If $f, g \in PAP(\mathbb{R}, \mathbb{R}^n, \mu)$, then $f + g, fg \in PAP(\mathbb{R}, \mathbb{R}^n, \mu)$; if $f \in PAP(\mathbb{R}, \mathbb{R}^n, \mu), g \in AP(\mathbb{R}, \mathbb{R}^n)$, then $fg \in PAP(\mathbb{R}, \mathbb{R}^n, \mu)$.

Lemma 6 (see [29]). Fix $\mu \in \mathbb{U}_{\infty}$. Suppose that, for any $s \in \mathbb{R}$,

$$\lim_{|t| \rightarrow \infty} \frac{\mu(t+s)}{\mu(t)} < \infty. \quad (8)$$

Then $PAP_0(\mathbb{R}, \mathbb{X}, \mu)$ is translation-invariant.

Denote $\mathbb{U}_{\infty}^{\text{Inv}} = \{\mu \in \mathbb{U}_{\infty} : \text{for all } s \in \mathbb{R}, \overline{\lim_{|t| \rightarrow \infty} (\mu(t+s)/\mu(t))} < \infty\}$.

Remark 7. Fix $\mu \in \mathbb{U}_{\infty}^{\text{Inv}}$. Let

$$\begin{aligned} PAP^1(\mathbb{R}, \mathbb{R}^n, \mu) &= \left\{ f = (f_1(t), f_2(t), \dots, f_n(t))^T : \right. \\ &\quad \left. f_i, f'_i \in PAP(\mathbb{R}, \mathbb{R}, \mu), i = 1, 2, \dots, n \right\}. \end{aligned} \quad (9)$$

Then $PAP^1(\mathbb{R}, \mathbb{R}^n, \mu)$ is a Banach space with the norm defined by $\|f\|_{\infty}^1 = \max\{\|f\|_{\infty}, \|f'\|_{\infty}\}$, where $\|f\|_{\infty} = \sup_{t \in \mathbb{R}} \|f(t)\|$, $\|f(t)\| = \max_{1 \leq i \leq n} |f_i(t)|$.

3. Existence of Weighted Pseudo-Almost Periodic Solution

To obtain the existence of weighted pseudo-almost periodic solution to system (1), we need the following lemmas.

Lemma 8 (see [26, 27]). *If the linear system (4) admits an exponential dichotomy, then the almost periodic system*

$$x'(t) = Q(t)x(t) + g(t) \quad (10)$$

has a unique almost periodic solution $x(t)$, and

$$\begin{aligned} x(t) &= \int_{-\infty}^t X(t) P X^{-1}(s) g(s) ds \\ &\quad - \int_t^{+\infty} X(t) (I - P) X^{-1}(s) g(s) ds. \end{aligned} \quad (11)$$

Lemma 9 (see [26, 27]). *Let $c_i(t)$ be an almost periodic function on $\mathbb{R}^n$, and*

$$M[c_i] = \lim_{T \rightarrow +\infty} \frac{1}{T} \int_t^{t+T} c_i(s) ds > 0, \quad i = 1, 2, \dots, n. \quad (12)$$

Then the linear system

$$x'(t) = \text{diag}(-c_1(t), -c_2(t), \dots, -c_n(t)) x(t) \quad (13)$$

admits an exponential dichotomy on $\mathbb{R}^n$.

Lemma 10. *Fix $\mu \in \mathbb{U}_{\infty}^{\text{Inv}}$. If $f : \mathbb{R} \rightarrow \mathbb{R}$ satisfies the Lipschitz condition, $k : [0, +\infty) \rightarrow \mathbb{R}$ is continuous and integrable, and satisfying $0 \leq \int_0^{+\infty} |k(u)| du \leq h$ (where h is a positive constant) and $\varphi \in PAP(\mathbb{R}, \mathbb{R}, \mu)$, then $\Pi : t \rightarrow (\int_0^{+\infty} k(u) f(\varphi(t-u)) du)$ belongs to $PAP(\mathbb{R}, \mathbb{R}, \mu)$.*

Proof. Since $\varphi \in PAP(\mathbb{R}, \mathbb{R}, \mu)$, there exist $\varphi_1 \in AP(\mathbb{R}, \mathbb{R})$ and $\varphi_2 \in PAP_0(\mathbb{R}, \mathbb{R}, \mu)$ such that $\varphi = \varphi_1 + \varphi_2$; then

$$\begin{aligned} \Pi(t) &= \int_0^{+\infty} k(u) f(\varphi_1(t-u) + \varphi_2(t)) du \\ &= \int_0^{+\infty} k(u) f(\varphi_1(t-u)) du \\ &\quad + \int_0^{+\infty} k(u) [f(\varphi_1(t-u) + \varphi_2(t-u)) \\ &\quad \quad - f(\varphi_1(t-u))] du \\ &=: \Pi_1(t) + \Pi_2(t). \end{aligned} \quad (14)$$

First, we prove that $\Pi_1 \in AP(\mathbb{R}, \mathbb{R})$. Since f satisfies the Lipschitz condition, there exists a positive constant L , such that $|f(u_1) - f(u_2)| \leq L|u_1 - u_2|$ for all $u_1, u_2 \in \mathbb{R}$. For any $\varepsilon > 0$, since $\varphi_1 \in AP(\mathbb{R}, \mathbb{R})$, it is possible to find a real number $l = l(\varepsilon) > 0$; for any interval with length $l(\varepsilon)$, there exists a number $\tau = \tau(\varepsilon)$ in this interval such that $|\varphi_1(u + \tau) - \varphi_1(u)| < \varepsilon/Lk$ for all $t \in \mathbb{R}$; then

$$\begin{aligned} &|\Pi_1(t + \tau) - \Pi_1(t)| \\ &= \left| \int_0^{+\infty} k(u) f(\varphi_1(t + \tau - u)) du \right. \\ &\quad \left. - \int_0^{+\infty} k(u) f(\varphi_1(t - u)) du \right| \\ &= \left| \int_0^{+\infty} k(u) [f(\varphi_1(t + \tau - u)) - f(\varphi_1(t - u))] du \right| \quad (15) \\ &= \left| \int_{-\infty}^t k(t-u) [f(\varphi_1(u + \tau)) - f(\varphi_1(u))] du \right| \\ &\leq \int_{-\infty}^t |k(u)| L |\varphi_1(u + \tau) - \varphi_1(u)| du < kL \frac{\varepsilon}{kL} = \varepsilon, \end{aligned}$$

which implies that $\Pi_1 \in AP(\mathbb{R}, \mathbb{R})$. Next, we prove that $\Pi_2 \in PAP_0(\mathbb{R}, \mathbb{R}, \mu)$. Consider

$$\begin{aligned} &\lim_{r \rightarrow \infty} \frac{1}{\mu(Q_r)} \int_{Q_r} |\Pi_2(t)| dt \\ &= \lim_{r \rightarrow \infty} \int_{Q_r} \left| \int_0^{+\infty} k(u) [f(\varphi_1(t-u) + \varphi_2(t-u)) \right. \\ &\quad \left. - f(\varphi_1(t-u))] \mu(t) dt \right| \\ &\leq \lim_{r \rightarrow \infty} \frac{L}{\mu(Q_r)} \int_0^{+\infty} |k(u)| \left(\int_{Q_r} |\varphi_2(t-u)| \mu(t) dt \right) du. \end{aligned} \quad (16)$$

Consider the following function:

$$\Gamma_r(u) = \frac{1}{\mu(Q_r)} \int_{Q_r} |\varphi_2(t-u)| \mu(t) dt; \quad (17)$$

obviously, $\Gamma_r(u)$ is bounded, and, by using Lemma 6, we have $\lim_{r \rightarrow \infty} \Gamma_r(u) = 0$. Consequently, by the Lebesgue dominated convergence theorem, we get

$$\begin{aligned} &\lim_{r \rightarrow \infty} \frac{1}{\mu(Q_r)} \int_{Q_r} |\Pi_2(t)| dt \\ &\leq L \lim_{r \rightarrow \infty} \int_0^{+\infty} |k(u)| \Gamma_r(u) du = 0, \end{aligned} \quad (18)$$

which implies that $\Pi_2 \in PAP_0(\mathbb{R}, \mathbb{R}, \mu)$. The proof is complete. $\square$

Lemma 11. Fix $\mu \in \mathbb{U}_{\infty}^{Inv}$. Suppose that assumptions (H_1) – (H_3) hold. For each $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n(t))^T \in PAP^1(\mathbb{R}, \mathbb{R}^n, \mu)$, define a nonlinear operator as follows:

$$\begin{aligned} T_{\varphi}(t) &= (x_{\varphi_1}(t), x_{\varphi_2}(t), \dots, x_{\varphi_n}(t))^T \\ &= \left(\int_{-\infty}^t e^{-\int_s^t c_1(u) du} \Gamma_1(s) ds, \right. \\ &\quad \left. \int_{-\infty}^t e^{-\int_s^t c_2(u) du} \Gamma_2(s) ds, \dots, \int_{-\infty}^t e^{-\int_s^t c_n(u) du} \Gamma_n(s) ds \right)^T, \end{aligned} \quad (19)$$

where, for $i = 1, 2, \dots, n$,

$$\begin{aligned} \Gamma_i(s) &= \sum_{j=1}^n a_{ij}(s) \int_0^{+\infty} d_{ij}(\theta) f_j(\varphi_j(s-\theta)) d\theta \\ &\quad + \sum_{j=1}^n \alpha_{ij}(s) \int_0^{+\infty} \beta_{ij}(\theta) h_j(\varphi'_j(s-\theta)) d\theta \\ &\quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \int_0^{+\infty} k_{ijl}(\theta) g_j(\varphi_j(s-\theta)) d\theta \\ &\quad \times \int_0^{+\infty} k_{il}(\theta) g_l(\varphi_l(s-\theta)) d\theta + I_i(s), \end{aligned} \quad (20)$$

and then T maps $PAP^1(\mathbb{R}, \mathbb{R}^n, \mu)$ into itself.

Proof. Let $\varphi \in PAP^1(\mathbb{R}, \mathbb{R}^n, \mu)$; by Lemma 10 and in view of (H_1) – (H_4) , we have $\Gamma_i \in PAP(\mathbb{R}, \mathbb{R}, \mu)$ ($i = 1, 2, \dots, n$); that is, Γ_i can be rewritten as $\Gamma_i = \Gamma_{i1} + \Gamma_{i2}$, where $\Gamma_{i1} \in AP(\mathbb{R}, \mathbb{R})$ and $\Gamma_{i2} \in PAP_0(\mathbb{R}, \mathbb{R}, \mu)$. Hence,

$$\begin{aligned} x_{\varphi_i}(t) &= \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \Gamma_i(s) ds \\ &= \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \Gamma_{i1}(s) ds + \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \Gamma_{i2}(s) ds \\ &=: T_{i1}(t) + T_{i2}(t), \quad i = 1, 2, \dots, n. \end{aligned} \quad (21)$$

Consider the following almost periodic system:

$$x'_i(t) = -c_i(t) x_i(t) + \Gamma_{i1}(t), \quad i = 1, 2, \dots, n. \quad (22)$$

Since $M[c_i] > 0$ ($i = 1, 2, \dots, n$), from Lemmas 8 and 9, system (22) has an almost periodic solution which can be expressed as follows:

$$\begin{aligned} z(t) &= \left(\int_{-\infty}^t e^{-\int_s^t c_1(u) du} \Gamma_{11}(s) ds, \right. \\ &\quad \left. \int_{-\infty}^t e^{-\int_s^t c_2(u) du} \Gamma_{21}(s) ds, \dots, \int_{-\infty}^t e^{-\int_s^t c_n(u) du} \Gamma_{n1}(s) ds \right)^T; \end{aligned} \quad (23)$$

that is, $T_{i1} \in AP(\mathbb{R}, \mathbb{R})$ ($i = 1, 2, \dots, n$). Let

$$J_i = \frac{1}{\mu(Q_r)} \int_{Q_r} \left| \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \Gamma_{i2}(s) ds \right| \mu(t) dt; \quad (24)$$

in order to prove that $T_{i2} \in PAP_0(\mathbb{R}, \mathbb{R}, \mu)$, we will prove $\lim_{r \rightarrow \infty} J_i = 0$ ($i = 1, 2, \dots, n$). Notice that

$$\begin{aligned} J_i &= \frac{1}{\mu(Q_r)} \int_{Q_r} \left| \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \Gamma_{i2}(s) ds \right| \mu(t) dt \\ &\leq \frac{1}{\mu(Q_r)} \int_{Q_r} \left| \int_{-\infty}^t e^{-c_i^m(t-s)} \Gamma_{i2}(s) ds \right| \mu(t) dt \\ &= \frac{1}{\mu(Q_r)} \int_{Q_r} \left| \int_0^{+\infty} e^{-c_i^m s} \Gamma_{i2}(t-s) ds \right| \mu(t) dt \\ &\leq \frac{1}{\mu(Q_r)} \int_{Q_r} \left(\int_0^{+\infty} e^{-c_i^m s} |\Gamma_{i2}(t-s)| ds \right) \mu(t) dt \\ &= \int_0^{+\infty} e^{-c_i^m s} \left(\frac{1}{\mu(Q_r)} \int_{Q_r} |\Gamma_{i2}(t-s)| \mu(t) dt \right) ds, \end{aligned} \quad (25)$$

where $c_i^m = \inf_{t \in \mathbb{R}} c_i(t)$ ($i = 1, 2, \dots, n$). Since the function $\Gamma_{i2} \in PAP_0(\mathbb{R}, \mathbb{R}, \mu)$, then the functions $\Pi_i(s)$ defined by

$$\Pi_i(s) = \frac{1}{\mu(Q_r)} \int_{Q_r} |\Gamma_{i2}(t-s)| \mu(t) dt \quad (26)$$

are bounded and satisfy $\lim_{r \rightarrow \infty} \Pi_i(s) = 0$. Consequently, by the Lebesgue dominated convergence theorem, we have

$$\lim_{r \rightarrow \infty} J_i = 0, \quad i = 1, 2, \dots, n; \quad (27)$$

that is, $T_{i2} \in PAP_0(\mathbb{R}, \mathbb{R}, \mu)$ ($i = 1, 2, \dots, n$). Now, we can get that $x_{\varphi_i} \in PAP(\mathbb{R}, \mathbb{R}, \mu)$, and

$$x'_{\varphi_i}(t) = -c_i(t) x_{\varphi_i}(t) + \Gamma_i(t); \quad (28)$$

obviously, $x'_{\varphi_i}(t) \in PAP(\mathbb{R}, \mathbb{R}, \mu)$; that is, T maps $PAP^1(\mathbb{R}, \mathbb{R}^n, \mu)$ into itself. $\square$

For the sake of convenience, we introduce the following notations:

$$\begin{aligned} c_i^m &= \inf_{t \in \mathbb{R}} c_i(t), & c_i^M &= \sup_{t \in \mathbb{R}} c_i(t), \\ \overline{I_i} &= \sup_{t \in \mathbb{R}} |I_i(t)|, & \overline{a_{ij}} &= \sup_{t \in \mathbb{R}} |a_{ij}(t)|, \\ \overline{\alpha_{ij}} &= \sup_{t \in \mathbb{R}} |\alpha_{ij}(t)|, & \overline{b_{ijl}} &= \sup_{t \in \mathbb{R}} |b_{ijl}(t)|, \\ & & i, j, l &= 1, 2, \dots, n, \end{aligned}$$

$$A = \max \left\{ \max_{1 \leq i \leq n} \left\{ \frac{\bar{I}_i}{c_i^m} \right\}, \max_{1 \leq i \leq n} \left\{ \bar{I}_i + c_i^M \frac{\bar{I}_i}{c_i^m} \right\} \right\},$$

$$B = \max_{1 \leq i \leq n} \left\{ \max \left\{ \frac{1}{c_i^m}, 1 + \frac{c_i^M}{c_i^m} \right\} \right. \\ \left. \times \sum_{j=1}^n \left[\bar{a}_{ij} d_{ij}^M |f_j(0)| + \bar{\alpha}_{ij} \beta_{ij}^M |h_j(0)| \right. \right. \\ \left. \left. + \sum_{l=1}^n \bar{b}_{ijl} G_l G_j k_{ij}^M k_{il}^M \right] \right\},$$

$$\begin{aligned} \phi_0(t) &= \left(\int_{-\infty}^t e^{-\int_s^t c_1(u) du} I_1(s) ds, \right. \\ &\quad \left. \int_{-\infty}^t e^{-\int_s^t c_2(u) du} I_2(s) ds, \dots, \int_{-\infty}^t e^{-\int_s^t c_n(u) du} I_n(s) ds \right)^T. \end{aligned} \quad (29)$$

Theorem 12. Suppose that (H_1) – (H_4) and

$$\begin{aligned} &\xi \\ &= \max_{1 \leq i \leq n} \left\{ \max \left\{ \frac{1}{c_i^m}, 1 + \frac{c_i^M}{c_i^m} \right\} \right. \\ &\quad \left. \times \sum_{j=1}^n \left[\bar{a}_{ij} d_{ij}^M \epsilon_j + \bar{\alpha}_{ij} \beta_{ij}^M \vartheta_j + 2\epsilon_j \sum_{l=1}^n \bar{b}_{ijl} G_l k_{ij}^M k_{il}^M \right] \right\} \\ &< 1 \end{aligned} \quad (30)$$

hold; then there exists a unique continuously differentiable weighted pseudo-almost periodic solution of system (1) in the region $E = \{\phi \mid \phi \in PAP^1(\mathbb{R}, \mathbb{R}^n, \mu), \|\phi - \phi_0\|_\infty^1 \leq (A\xi + B)/(1 - \xi)\}$.

Proof. For all $\phi \in PAP^1(\mathbb{R}, \mathbb{R}^n, \mu)$, from Lemma 11, T maps $PAP^1(\mathbb{R}, \mathbb{R}^n, \mu)$ into itself. By the definition of the norm of Banach space $PAP^1(\mathbb{R}, \mathbb{R}^n, \mu)$, we have

$$\begin{aligned} \|\phi_0\|_\infty^1 &= \max \{ \|\phi_0\|_\infty, \|\phi_0'\|_\infty \} \\ &= \max \left\{ \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left| \int_{-\infty}^t I_i(s) e^{-\int_s^t c_i(u) du} ds \right|, \right. \\ &\quad \left. \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left| I_i(t) - \int_{-\infty}^t I_i(s) c_i(t) e^{-\int_s^t c_i(u) du} ds \right| \right\} \\ &\leq \max \left\{ \max_{1 \leq i \leq n} \left\{ \frac{\bar{I}_i}{c_i^m} \right\}, \max_{1 \leq i \leq n} \left\{ \bar{I}_i + c_i^M \frac{\bar{I}_i}{c_i^m} \right\} \right\} = A. \end{aligned} \quad (31)$$

Hence, for all $\phi \in E = \{\phi \mid \phi \in PAP^1(\mathbb{R}, \mathbb{R}, \mu), \|\phi - \phi_0\|_\infty^1 \leq (A\xi + B)/(1 - \xi)\}$, we obtain

$$\|\phi_0\|_\infty^1 \leq \|\phi - \phi_0\|_\infty^1 + \|\phi_0\|_\infty^1 \leq \frac{A\xi + B}{1 - \xi} + A = \frac{A + B}{1 - \xi}. \quad (32)$$

Next, we show that T maps the closed set E into itself. In fact, for any $\phi \in E$, we obtain by (H_2) – (H_3) that

$$\begin{aligned} &\|T\phi - \phi_0\|_\infty \\ &= \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left\{ \left| \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \right. \right. \\ &\quad \times \left[\sum_{j=1}^n a_{ij}(s) \int_0^\infty d_{ij}(\theta) f_j(\phi_j(s - \theta)) d\theta \right. \\ &\quad \left. + \sum_{j=1}^n \alpha_{ij}(s) \int_0^\infty \beta_{ij}(\theta) h_j(\phi_j'(s - \theta)) d\theta \right. \\ &\quad \left. + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \int_0^\infty k_{ij}(\theta) g_j \right. \\ &\quad \left. \times (\phi_j(s - \theta)) d\theta \right. \\ &\quad \left. \times \int_0^\infty k_{il}(\theta) g_l \right. \\ &\quad \left. \times (\phi_l(s - \theta)) d\theta \right] ds \Big| \Big\} \\ &\leq \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left\{ \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \right. \\ &\quad \times \left[\sum_{j=1}^n \left(|a_{ij}(s)| \int_0^\infty |d_{ij}(\theta)| \right. \right. \\ &\quad \times \left[|f_j(\phi_j(s - \theta)) - f_j(0)| \right. \\ &\quad \left. \left. + |f_j(0)| \right] d\theta \right) \right. \\ &\quad \left. + \sum_{j=1}^n |\alpha_{ij}(s)| \right. \\ &\quad \times \int_0^\infty |\beta_{ij}(\theta)| \\ &\quad \times \left[|h_j(\phi_j'(s - \theta)) - h_j(0)| \right. \\ &\quad \left. \left. + |h_j(0)| \right] d\theta \right. \end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^n \sum_{l=1}^n |b_{jl}(s)| \\
& \quad \times \left[\int_0^\infty |k_{ij}(\theta)| \right. \\
& \quad \times |g_j(\phi_j(s-\theta)) - g_j(0)| d\theta \\
& \quad \times \int_0^\infty |k_{il}(\theta)| \\
& \quad \times |g_l(\phi_l(s-\theta))| d\theta \Big] \\
& + \sum_{j=1}^n \sum_{l=1}^n |b_{jl}(s)| \int_0^\infty |k_{ij}(\theta)| |g_j(0)| d\theta \\
& \quad \times \int_0^\infty |k_{il}(\theta)| \\
& \quad \times |g_l(\phi_l(s-\theta)) - g_l(0)| d\theta \\
& + \sum_{j=1}^n \sum_{l=1}^n |b_{jl}(s)| \int_0^\infty |k_{ij}(\theta)| |g_j(0)| d\theta \\
& \quad \times \int_0^\infty |k_{il}(\theta)| |g_l(0)| d\theta \Big] ds \Big\} \\
& \leq \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left\{ \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \right. \\
& \quad \times \left[\sum_{j=1}^n \overline{a_{ij}} \int_0^\infty |d_{ij}(\theta)| \right. \\
& \quad \times (\epsilon_j |\phi_j(s-\theta)| + |f_j(0)|) d\theta \\
& \quad + \sum_{j=1}^n \overline{\alpha_{ij}} \int_0^\infty |\beta_{ij}(\theta)| \\
& \quad \times (\vartheta_j |\phi'_j(s-\theta)| + |h_j(0)|) d\theta \\
& \quad + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{il}^M \int_0^\infty |k_{ij}(\theta)| \\
& \quad \times |\phi_j(s-\theta)| d\theta \\
& \quad + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_j \epsilon_l k_{ij}^M \int_0^\infty |k_{il}(\theta)| \\
& \quad \times |\phi_l(s-\theta)| d\theta \\
& \quad \left. \left. + \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_j G_l k_{ij}^M k_{il}^M \right] ds \right\} \\
& \leq \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left\{ \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \right. \\
& \quad \times \left[\sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j \right. \right. \\
& \quad \left. \left. + 2 \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{ij}^M k_{il}^M \right) \|\phi\|_\infty^1 \right. \\
& \quad \left. + \sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M |f_j(0)| + \overline{\alpha_{ij}} \beta_{ij}^M |h_j(0)| \right. \right. \\
& \quad \left. \left. + \sum_{l=1}^n \overline{b_{ijl}} G_j G_l k_{ij}^M k_{il}^M \right) \right] ds \Big\} \\
& = \max_{1 \leq i \leq n} \left\{ \frac{1}{c_i^m} \right. \\
& \quad \times \left[\sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j \right. \right. \\
& \quad \left. \left. + 2 \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{ij}^M k_{il}^M \right) \|\phi\|_\infty^1 \right. \\
& \quad \left. + \sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M |f_j(0)| + \overline{\alpha_{ij}} \beta_{ij}^M |h_j(0)| \right. \right. \\
& \quad \left. \left. + \sum_{l=1}^n \overline{b_{ijl}} G_j G_l k_{ij}^M k_{il}^M \right) \right] \Big\}. \tag{33}
\end{aligned}$$

Furthermore, we have

$$\begin{aligned}
& \|(T\phi - \phi_0)'\|_\infty \\
& = \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left\{ \left| \sum_{j=1}^n a_{ij}(t) \right| \right.
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^\infty d_{ij}(\theta) f_j(\phi_j(t-\theta)) d\theta \\
& + \sum_{j=1}^n \alpha_{ij}(t) \int_0^\infty \beta_{ij}(\theta) h_j(\phi'_j(t-\theta)) d\theta \\
& + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \int_0^\infty k_{ij}(\theta) g_j(\phi_j(t-\theta)) d\theta \\
& \quad \times \int_0^\infty k_{il}(\theta) g_l(\phi_l(t-\theta)) d\theta \\
& - \int_{-\infty}^t c_i(t) e^{-\int_s^t c_i(u) du} \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(s) \int_0^\infty d_{ij}(\theta) f_j(\phi_j(s-\theta)) d\theta \right. \\
& \quad + \sum_{j=1}^n \alpha_{ij}(s) \int_0^\infty \beta_{ij}(\theta) h_j(\phi'_j(s-\theta)) d\theta \\
& \quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \\
& \quad \quad \times \int_0^\infty k_{ij}(\theta) g_j(\phi_j(s-\theta)) d\theta \\
& \quad \quad \times \int_0^\infty k_{il}(\theta) g_l \\
& \quad \quad \quad \left. \times (\phi_l(s-\theta)) d\theta \right] ds \Bigg\} \\
& \leq \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left\{ \sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j \right. \right. \\
& \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{ij}^M k_{il}^M \right) \|\phi\|_\infty^1 \right. \\
& \quad + \sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M |f_j(0)| + \overline{\alpha_{ij}} \beta_{ij}^M |h_j(0)| \right. \\
& \quad \left. \left. + G_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{ij}^M k_{il}^M \right) \right\} \\
& \quad + \int_{-\infty}^t c_i^M e^{-c_i^m(t-s)} \\
& \quad \times \left[\sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j \right. \right. \\
& \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{ij}^M k_{il}^M \right) \|\phi\|_\infty^1 \right]
\end{aligned}$$

$$\begin{aligned}
& + \sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M |f_j(0)| + \overline{\alpha_{ij}} \beta_{ij}^M |h_j(0)| \right. \\
& \quad \left. + G_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{ij}^M k_{il}^M \right) \Bigg] ds \Bigg\} \\
& \leq \max_{1 \leq i \leq n} \left\{ \left(1 + \frac{c_i^M}{c_i^m} \right) \left[\sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j \right. \right. \right. \right. \\
& \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{ij}^M k_{il}^M \right) \|\phi\|_\infty^1 \right. \right. \\
& \quad \left. \left. + \sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M |f_j(0)| + \overline{\alpha_{ij}} \beta_{ij}^M |h_j(0)| \right. \right. \right. \\
& \quad \left. \left. \left. + G_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{ij}^M k_{il}^M \right) \right] \right\}. \quad (34)
\end{aligned}$$

Thus, it follows from (33) and (34) that

$$\begin{aligned}
& \|T\phi - \phi_0\|_\infty^1 \\
& \leq \max \left\{ \frac{1}{c_i^m}, 1 + \frac{c_i^M}{c_i^m} \right\} \\
& \quad \times \left[\sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{ij}^M k_{il}^M \right) \right. \\
& \quad \left. \times \|\phi\|_\infty^1 \right] + B \\
& \leq \xi \|\phi\|_\infty^1 + B \leq \frac{A\xi + B}{1 - \xi}, \quad (35)
\end{aligned}$$

which implies that $T\phi \in E$. So, the mapping T is a self-mapping from E to E . Finally, we prove that T is a contraction mapping of the E . In fact, in view of (H_2) , for any $\phi, \psi \in E$, we obtain

$$\begin{aligned}
& \|T\phi - T\psi\|_\infty \\
& = \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left\{ \left| \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \right. \right. \\
& \quad \left. \left. \times \left[\sum_{j=1}^n a_{ij}(s) \right. \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \times \int_0^\infty d_{ij}(\theta) [f_j(\phi_j(s-\theta)) \\
& \quad - f_j(\psi_j(s-\theta))] d\theta \\
& + \sum_{j=1}^n \alpha_{ij}(s) \\
& \quad \times \int_0^\infty \beta_{ij}(\theta) [h_j(\phi'_j(s-\theta)) \\
& \quad - h_j(\psi'_j(s-\theta))] d\theta \\
& + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \\
& \quad \times \int_0^\infty k_{ij}(\theta) [g_j(\phi_j(s-\theta)) \\
& \quad - g_j(\psi_j(s-\theta))] d\theta \\
& \quad \times \int_0^\infty k_{il}(\theta) g_l(\phi_l(s-\theta)) d\theta \\
& + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \int_0^\infty k_{ij}(\theta) g_j(\psi_j(s-\theta)) d\theta \\
& \quad \times \int_0^\infty k_{il}(\theta) \\
& \quad \times [g_l(\phi_l(s-\theta)) \\
& \quad - g_l(\psi_l(s-\theta))] d\theta \Bigg] ds \Bigg\} \\
& \leq \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left\{ \int_{-\infty}^t e^{-\int_s^t c_i(u) du} \right. \\
& \quad \times \left[\sum_{j=1}^n \overline{a_{ij}} d_{ij}^M \epsilon_j \|\phi - \psi\|_\infty \right. \\
& \quad + \sum_{j=1}^n \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j \|(\phi - \psi)'\|_\infty \\
& \quad \left. + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{ij}^M k_{il}^M \|\phi - \psi\|_\infty \right] ds \Bigg\} \\
& \leq \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left\{ \int_{-\infty}^t e^{-c_i^m(t-s)} \right. \\
& \quad \times \sum_{j=1}^n \left[\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j \right. \\
& \quad \left. + 2 \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{ij}^M k_{il}^M \right] ds \|\phi - \psi\|_\infty^1 \Bigg\}
\end{aligned}$$

$$\begin{aligned}
& \leq \max_{1 \leq i \leq n} \left\{ \frac{1}{c_i^m} \sum_{j=1}^n \left[\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j + 2 \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{ij}^M k_{il}^M \right] \right\} \\
& \quad \times \|\phi - \psi\|_\infty^1 \leq \xi \|\phi - \psi\|_\infty^1, \\
& \|(T\phi - T\psi)'\|_\infty \\
& = \sup_{t \in \mathbb{R}} \max_{1 \leq i \leq n} \left\{ \left| \sum_{j=1}^n a_{ij}(t) \right. \right. \\
& \quad \times \int_0^\infty d_{ij}(\theta) \\
& \quad \times [f_j(\phi_j(t-\theta)) - f_j(\psi_j(t-\theta))] d\theta \\
& \quad + \sum_{j=1}^n \alpha_{ij}(t) \\
& \quad \times \int_0^\infty \beta_{ij}(\theta) \\
& \quad \times [h_j(\phi'_j(t-\theta)) - h_j(\psi'_j(t-\theta))] d\theta \\
& \quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \\
& \quad \times \int_0^\infty k_{ij}(\theta) [g_j(\phi_j(t-\theta)) \\
& \quad - g_j(\psi_j(t-\theta))] d\theta \\
& \quad \times \int_0^\infty k_{il}(\theta) g_l(\phi_l(t-\theta)) d\theta \\
& \quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \\
& \quad \times \int_0^\infty k_{ij}(\theta) g_j(\psi_j(t-\theta)) d\theta \\
& \quad \times \int_0^\infty k_{il}(\theta) [g_l(\phi_l(t-\theta)) \\
& \quad - g_l(\psi_l(t-\theta))] d\theta \\
& \quad \left. - \int_{-\infty}^t c_i(t) e^{-\int_s^t c_i(u) du} \right. \\
& \quad \times \left[\sum_{j=1}^n a_{ij}(s) \right. \\
& \quad \times \int_0^\infty d_{ij}(\theta) [f_j(\phi_j(s-\theta)) \\
& \quad - f_j(\psi_j(s-\theta))] d\theta \\
& \quad \left. + \sum_{j=1}^n \alpha_{ij}(s) \right.
\end{aligned}$$

$$\begin{aligned}
 & \times \int_0^\infty \beta_{ij}(\theta) \\
 & \quad \times [h_j(\phi'_j(s-\theta)) \\
 & \quad \quad - h_j(\psi'_j(s-\theta))] d\theta \\
 & + \sum_{j=1}^n \sum_{l=1}^n b_{jil}(s) \\
 & \quad \times \int_0^\infty k_{ij}(\theta) \\
 & \quad \times [g_j(\phi_j(s-\theta)) \\
 & \quad \quad - g_j(\psi_j(s-\theta))] d\theta \\
 & \quad \times \int_0^\infty k_{il}(\theta) g_l(\phi_l(s-\theta)) d\theta \\
 & + \sum_{j=1}^n \sum_{l=1}^n b_{jil}(s) \\
 & \quad \times \int_0^\infty k_{ij}(\theta) g_j(\psi_j(s-\theta)) d\theta \\
 & \quad \times \int_0^\infty k_{il}(\theta) \\
 & \quad \times [g_l(\phi_l(s-\theta)) - g_l \\
 & \quad \quad \times (\psi_l(s-\theta))] d\theta \Bigg] ds \Bigg\} \\
 & \leq \max_{1 \leq i \leq n} \left\{ \left(1 + \frac{c_i^M}{c_i^m} \right) \right. \\
 & \quad \times \left[\sum_{j=1}^n \left(\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j \right. \right. \\
 & \quad \quad \left. \left. + 2 \sum_{l=1}^n \overline{b_{jil}} G_l \epsilon_j k_{ij}^M k_{il}^M \right) \right] \Bigg\} \|\phi - \psi\|_\infty^1 \\
 & \leq \xi \|\phi - \psi\|_\infty^1.
 \end{aligned} \tag{36}$$

Thus,

$$\|T\phi - T\psi\|_\infty^1 \leq \xi \|\phi - \psi\|_\infty^1. \tag{37}$$

Notice that $\xi < 1$; it means that the mapping T is a contraction mapping. By Banach fixed point theorem, there exists a unique fixed point $z \in E$ such that $Tz = z$, which implies that system (1) has a unique weighted pseudo-almost periodic solution. This completes the proof. $\square$

Since $(AP^1(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_\infty^1)$ and $(PAP^1(\mathbb{R}, \mathbb{R}^n), \|\cdot\|_\infty^1)$ are Banach spaces, we can get the following corollary.

Corollary 13. If (H_1) – (H_3) and (H_5) hold, furthermore, assume that $I_i(t)$ ($i = 1, 2, \dots, n$) are almost periodic functions; then there exists a unique continuously differentiable almost periodic solution of system (1) in the region

$$E = \left\{ \phi \mid \phi \in AP^1(\mathbb{R}, \mathbb{R}^n), \|\phi - \phi_0\|_\infty^1 \leq \frac{A\xi + B}{1 - \xi} \right\}, \tag{38}$$

where $AP^1(\mathbb{R}, \mathbb{R}^n) = \{f = (f_1(t), f_2(t), \dots, f_n(t))^T \mid f_i(t), f'_i(t) \in AP(\mathbb{R}, \mathbb{R}), i = 1, 2, \dots, n\}$.

Corollary 14. If (H_1) – (H_3) and (H_5) hold, furthermore, assume that $I_i(t)$ ($i = 1, 2, \dots, n$) are pseudo-almost periodic functions; then there exists a unique continuously differentiable pseudo-almost periodic solution of system (1) in the region

$$E = \left\{ \phi \mid \phi \in PAP^1(\mathbb{R}, \mathbb{R}^n), \|\phi - \phi_0\|_\infty^1 \leq \frac{A\xi + B}{1 - \xi} \right\}, \tag{39}$$

where $PAP^1(\mathbb{R}, \mathbb{R}^n) = \{f = (f_1(t), f_2(t), \dots, f_n(t))^T \mid f_i(t), f'_i(t) \in PAP(\mathbb{R}, \mathbb{R}), i = 1, 2, \dots, n\}$.

4. Global Exponential Stability of Weighted Pseudo-Almost Periodic Solution

Definition 15. Fix $\mu \in \mathbb{U}_\infty^{\text{Inv}}$. The weighted pseudo-almost periodic solution

$$x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T \tag{40}$$

of system (1) with initial value $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \dots, \phi_n^*(t))^T$ is said to be globally exponentially stable. If there exist constants $\lambda > 0$ and $M \geq 1$ such that for every solution $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ of system (1) with any initial value $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^T$ satisfies

$$\begin{aligned}
 & \|x(t) - x^*(t)\|_1 \\
 & = \max \left\{ \|x(t) - x^*(t)\|, \|x'(t) - x'^*(t)\| \right\} \leq M \|\psi\|_\infty^1 e^{-\lambda t}, \\
 & \quad \forall t > 0,
 \end{aligned} \tag{41}$$

where

$$\begin{aligned}
 \|\psi\|_\infty^1 & = \max \left\{ \sup_{t \in (-\infty, 0]} \max_{1 \leq i \leq n} |\phi_i(t) - \phi_i^*(t)|, \right. \\
 & \quad \left. \sup_{t \in (-\infty, 0]} \max_{1 \leq i \leq n} |\phi'_i(t) - (\phi_i^*)'(t)| \right\}.
 \end{aligned} \tag{42}$$

Theorem 16. Fix $\mu \in \mathbb{U}_\infty^{\text{Inv}}$. If conditions (H_1) – (H_5) hold, then system (1) has a unique continuously differentiable weighted

pseudo-almost periodic solution $z(t)$ which is globally exponentially stable.

Proof. It follows from Theorem 12 that system (1) has a unique weighted pseudo-almost periodic solution

$$x^*(t) = (x_1^*(t), x_2^*(t), \dots, x_n^*(t))^T \in E, \quad (43)$$

with initial value $\phi^*(t) = (\phi_1^*(t), \phi_2^*(t), \dots, \phi_n^*(t))^T$. Let $x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T$ be an arbitrary solution of system (1) with initial value $\phi(t) = (\phi_1(t), \phi_2(t), \dots, \phi_n(t))^T$. Let $y_i(t) = x_i(t) - x_i^*(t)$, $\psi_i(t) = \phi(t) - \phi^*(t)$, $i = 1, 2, \dots, n$; then

$$\begin{aligned} & y_i'(s) + c_i(s) y_i(s) \\ &= \sum_{j=1}^n a_{ij}(s) \\ & \quad \times \int_0^\infty d_{ij}(\theta) [f_j(y_j(s-\theta) + x_j^*(s-\theta)) \\ & \quad - f_j(x_j^*(s-\theta))] d\theta \\ & + \sum_{j=1}^n \alpha_{ij}(s) \\ & \quad \times \int_0^\infty \beta_{ij}(\theta) [h_j(y_j'(s-\theta) + (x_j^*)'(s-\theta)) \\ & \quad - h_j((x_j^*)'(s-\theta))] d\theta \\ & + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \\ & \quad \times \left[\left(\int_0^\infty k_{ij}(\theta) \right. \right. \\ & \quad \times [g_j(y_j(s-\theta) + x_j^*(s-\theta)) \\ & \quad - g_j(x_j^*(s-\theta))] d\theta \\ & \quad \times \int_0^\infty k_{il}(\theta) g_l(y_l(s-\theta) + x_l^*(s-\theta)) d\theta \\ & \quad + \left(\int_0^\infty k_{ij}(\theta) g_j(x_j^*(s-\theta)) d\theta \right. \\ & \quad \times \int_0^\infty k_{il}(\theta) \\ & \quad \times [g_l(y_l(s-\theta) + x_l^*(s-\theta)) \\ & \quad - g_l(x_l^*(s-\theta))] d\theta \Big) \Big], \end{aligned} \quad (44)$$

where $i = 1, 2, \dots, n$. Let H_i and H_i^* be defined by

$$\begin{aligned} & H_i(\eta) \\ &= c_i^m - \eta \\ & \quad - \sum_{j=1}^n \left[\overline{a_{ij}} \epsilon_j \int_0^\infty |d_{ij}(\theta)| e^{\eta\theta} d\theta + \overline{\alpha_{ij}} \vartheta_j \int_0^\infty |\beta_{ij}(\theta)| e^{\eta\theta} d\theta \right. \\ & \quad \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M \int_0^\infty |k_{ij}(\theta)| e^{\eta\theta} d\theta \right], \\ & H_i^*(\eta) \\ &= c_i^m - \eta - (c_i^M + c_i^m) \\ & \quad \times \sum_{j=1}^n \left[\overline{a_{ij}} \epsilon_j \int_0^\infty |d_{ij}(\theta)| e^{\eta\theta} d\theta + \overline{\alpha_{ij}} \vartheta_j \int_0^\infty |\beta_{ij}(\theta)| e^{\eta\theta} d\theta \right. \\ & \quad \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M \int_0^\infty |k_{ij}(\theta)| e^{\eta\theta} d\theta \right], \end{aligned} \quad (45)$$

where $i = 1, 2, \dots, n, \eta \in [0, \infty)$. By (H_4) , we obtain that

$$\begin{aligned} & H_i(0) \\ &= c_i^m \\ & \quad - \sum_{j=1}^n \left[\int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\ & \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M |k_{ij}(\theta)| \right) d\theta \right] \\ & \geq c_i^m - \sum_{j=1}^n \left[\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M k_{ij}^M \right] > 0, \\ & \quad i = 1, 2, \dots, n, \end{aligned}$$

$$\begin{aligned} & H_i^*(0) \\ &= c_i^m - (c_i^M + c_i^m) \\ & \quad \times \sum_{j=1}^n \left[\overline{a_{ij}} \epsilon_j \int_0^\infty |d_{ij}(\theta)| d\theta + \overline{\alpha_{ij}} \vartheta_j \int_0^\infty |\beta_{ij}(\theta)| d\theta \right. \\ & \quad \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M \int_0^\infty |k_{ij}(\theta)| d\theta \right] \\ & \geq c_i^m - (c_i^M + c_i^m) \\ & \quad \times \sum_{j=1}^n \left[\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M k_{ij}^M \right] > 0, \\ & \quad i = 1, 2, \dots, n. \end{aligned} \quad (46)$$

Since H_i, H_i^* are continuous on $[0, \infty)$ and $H_i(\eta), H_i^*(\eta) \rightarrow -\infty$ as $\eta \rightarrow \infty$, there exist $\eta_i^*, \gamma_i^* > 0$ such that $H_i(\eta_i^*) = H_i^*(\gamma_i^*) = 0$, and $H_i(\eta_i) > 0$ for $\eta_i \in (0, \eta_i^*)$, $H_i^*(\gamma_i) > 0$ for $\gamma_i \in (0, \gamma_i^*)$. By choosing $\gamma = \min\{\eta_1^*, \eta_2^*, \dots, \eta_n^*, \gamma_1^*, \gamma_2^*, \dots, \gamma_n^*\}$, we have

$$H_i(\gamma) \geq 0, \quad H_i^*(\gamma) \geq 0, \quad i = 1, 2, \dots, n. \quad (47)$$

So, we can choose a positive constant $0 < \lambda < \min\{\gamma, \lambda_0\}$ such that $H_i(\lambda) > 0$ and $H_i^*(\lambda) > 0$, which implies that

$$\begin{aligned} & \frac{1}{c_i^m - \lambda} \sum_{j=1}^n \left[\int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\ & \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M |k_{ij}(\theta)| \right) e^{\lambda\theta} d\theta \right] < 1, \\ & \left(1 + \frac{c_i^m}{c_i^m - \lambda} \right) \\ & \times \sum_{j=1}^n \left[\int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\ & \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M |k_{ij}(\theta)| \right) e^{\lambda\theta} d\theta \right] < 1, \end{aligned} \quad (48)$$

where $i = 1, 2, \dots, n$.

Multiplying both sides of (44) by $e^{\int_0^s c_i(u) du}$ and integrating it over $[0, t]$, we have

$$\begin{aligned} & y_i(t) \\ &= y_i(0) e^{-\int_0^t c_i(u) du} \\ &+ \int_0^t e^{-\int_s^t c_i(u) du} \\ &\times \left[\sum_{j=1}^n a_{ij}(s) \right. \\ &\quad \times \int_0^\infty d_{ij}(\theta) [f_j(y_j(s-\theta) + x_j^*(s-\theta)) \\ &\quad \quad \left. - f_j(x_j^*(s-\theta))] d\theta \\ &\quad + \sum_{j=1}^n \alpha_{ij}(s) \\ &\quad \times \int_0^\infty \beta_{ij}(\theta) [f_j(y_j'(s-\theta) + (x_j^*)'(s-\theta)) \\ &\quad \quad \left. - f_j(x_j^*(s-\theta))] d\theta \\ &\quad \left. + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \right. \right. \end{aligned}$$

$$\begin{aligned} & \times \left(\int_0^\infty k_{ij}(\theta) \right. \\ & \quad \times [g_j(y_j(s-\theta) + x_j^*(s-\theta)) \\ & \quad \quad \left. - g_j(x_j^*(s-\theta))] d\theta \\ & \quad \times \int_0^\infty k_{il}(\theta) \\ & \quad \quad \times g_l(y_l(s-\theta) + x_l^*(s-\theta)) d\theta \Big) \\ & + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \\ & \times \left(\int_0^\infty k_{ij}(\theta) g_j(x_j^*(s-\theta)) d\theta \right. \\ & \quad \times \int_0^\infty k_{il}(\theta) \\ & \quad \quad \times [g_j(y_j(s-\theta) + x_j^*(s-\theta)) \\ & \quad \quad \quad \left. - g_j(x_j^*(s-\theta))] d\theta \Big) ds. \end{aligned} \quad (49)$$

Let

$$\begin{aligned} & M \\ &= \max_{1 \leq i \leq n} \left\{ c_i^m \left(\sum_{j=1}^n \left[\int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \right. \right. \\ & \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M \right. \right. \\ & \quad \quad \left. \left. \times |k_{ij}(\theta)| \right) d\theta \right] \Big)^{-1} \right\}. \end{aligned} \quad (50)$$

By (H₄), we have $M > 1$. Thus,

$$\begin{aligned} & \frac{1}{M} - \frac{1}{c_i^m - \lambda} \\ & \times \left[\sum_{j=1}^n \int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\ & \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M |k_{ij}(\theta)| \right) e^{\lambda\theta} d\theta \right] \leq 0, \end{aligned} \quad (51)$$

$$\begin{aligned} & \|y(t)\|_1 = \|\psi(t)\|_1 \leq \|\psi\|_\infty^1 \leq M \|\psi\|_\infty^1 e^{-\lambda t}, \\ & \quad \forall t \in (-\infty, 0], \end{aligned} \quad (52)$$

where $\lambda > 0$ as in (48). We claim that

$$\|y(t)\|_1 \leq M \|\psi\|_\infty^1 e^{-\lambda t}, \quad t > 0. \quad (53)$$

To prove (53), we first show that, for any $p > 1$, the following inequality holds:

$$\|y(t)\|_1 \leq pM \|\psi\|_\infty^1 e^{-\lambda t}, \quad t > 0. \quad (54)$$

If (54) is not true, then there must be some $t_1 > 0$ and some $i, k \in \{1, 2, \dots, n\}$ such that

$$\begin{aligned} \|y(t_1)\|_1 &= \max \{ \|y(t_1)\|, \|y'(t_1)\| \} \\ &= \max \{ |y_i(t_1)|, |y'_i(t_1)| \} = pM \|\psi\|_\infty^1 e^{-\lambda t_1}, \end{aligned} \quad (55)$$

$$\|y(t)\|_1 \leq pM \|\psi\|_\infty^1 e^{-\lambda t}, \quad t \in (-\infty, t_1]. \quad (56)$$

By (48)–(51), (56), and (H₂), we get

$$\begin{aligned} &|y_i(t_1)| \\ &\leq e^{-\int_0^{t_1} c_i(u) du} \|\psi\|_\infty^1 \\ &\quad + \int_0^{t_1} e^{-\int_s^{t_1} c_i(u) du} \\ &\quad \times \left[\sum_{j=1}^n \overline{a_{ij}} \epsilon_j \int_0^\infty |d_{ij}(\theta)| \|y_j(s-\theta)\| d\theta \right. \\ &\quad + \sum_{j=1}^n \overline{\alpha_{ij}} \vartheta_j \int_0^\infty |\beta_{ij}(\theta)| |y'_j(s-\theta)| d\theta \\ &\quad + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M \epsilon_j \\ &\quad \times \int_0^\infty |k_{ij}(\theta)| |y_j(s-\theta)| d\theta \Big] ds \\ &\leq e^{-c_i^m t_1} \|\psi\|_\infty^1 \\ &\quad + \int_0^{t_1} e^{-\int_s^{t_1} c_i(u) du} \end{aligned}$$

$$\begin{aligned} &\times \left[\sum_{j=1}^n \overline{a_{ij}} \epsilon_j pM \|\psi\|_\infty^1 e^{-\lambda s} \int_0^\infty |d_{ij}(\theta)| e^{\lambda \theta} d\theta \right. \\ &\quad + \sum_{j=1}^n \overline{\alpha_{ij}} \vartheta_j pM \|\psi\|_\infty^1 e^{-\lambda s} \int_0^\infty |\beta_{ij}(\theta)| e^{\lambda \theta} d\theta \\ &\quad + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{il}^M pM \|\psi\|_\infty^1 e^{-\lambda s} \\ &\quad \times \int_0^\infty |k_{ij}(\theta)| e^{\lambda \theta} d\theta \Big] ds \\ &\leq e^{-c_i^m t_1} \|\psi\|_\infty^1 + pM \|\psi\|_\infty^1 \\ &\quad \times \left[\sum_{j=1}^n \int_0^\infty (\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)|) e^{\lambda \theta} d\theta \right. \\ &\quad + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{il}^M \int_0^\infty |k_{ij}(\theta)| e^{\lambda \theta} d\theta \Big] \\ &\quad \times \int_0^{t_1} e^{-c_i^m(t_1-s)-\lambda s} ds \\ &= e^{-c_i^m t_1} \|\psi\|_\infty^1 + pM \|\psi\|_\infty \\ &\quad \times \sum_{j=1}^n \left[\int_0^\infty (\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)|) e^{\lambda \theta} d\theta \right. \\ &\quad + 2 \epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M \int_0^\infty |k_{ij}(\theta)| e^{\lambda \theta} d\theta \Big] \\ &\quad \times \frac{e^{-\lambda t_1}}{c_i^m - \lambda} (1 - e^{(\lambda - c_i^m)t_1}) \\ &= pM \|\psi\|_\infty^1 \\ &\quad \times e^{-\lambda t_1} \left\{ \frac{1}{pM} e^{(\lambda - c_i^m)t_1} + \frac{1}{c_i^m - \lambda} (1 - e^{(\lambda - c_i^m)t_1}) \right. \\ &\quad \times \left[\sum_{j=1}^n \int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\ &\quad \left. \left. + 2 \epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M |k_{ij}(\theta)| \right) \right. \\ &\quad \left. \times e^{\lambda \theta} d\theta \right\} \\ &< pM \|\psi\|_\infty^1 \\ &\quad \times e^{-\lambda t_1} \left\{ \frac{1}{M} e^{(\lambda - c_i^m)t_1} + \frac{1}{c_i^m - \lambda} (1 - e^{(\lambda - c_i^m)t_1}) \right\} \end{aligned}$$

$$\begin{aligned}
 & \times \left[\sum_{j=1}^n \int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\
 & \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M |k_{ij}(\theta)| \right) \right. \\
 & \quad \left. \times e^{\lambda\theta} d\theta \right] \Bigg\} \\
 & = pM \|\psi\|_\infty^1 \\
 & \quad \times e^{-\lambda t_1} \left(\frac{1}{M} - \frac{1}{c_i^m - \lambda} \right. \\
 & \quad \times \sum_{j=1}^n \left[\int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right) \right. \\
 & \quad \times e^{\lambda\theta} d\theta \\
 & \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M \int_0^\infty |k_{ij}(\theta)| e^{\lambda\theta} d\theta \right] \right) \\
 & \quad \times e^{(\lambda - c_i^m)t_1} \\
 & \quad + \frac{1}{c_i^m - \lambda} \sum_{j=1}^n \left[\int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\
 & \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M |k_{ij}(\theta)| \right) d\theta \right] \\
 & \leq pM \|\psi\|_\infty^1 e^{-\lambda t_1} \frac{1}{c_i^m - \lambda} \\
 & \quad \times \sum_{j=1}^n \left[\int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\
 & \quad \left. \left. + 2\epsilon_j \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M |k_{ij}(\theta)| \right) d\theta \right] \\
 & < pM \|\psi\|_\infty^1 e^{-\lambda t_1}.
 \end{aligned} \tag{57}$$

Direct differentiation of (49) gives

$$\begin{aligned}
 & y_i'(t) \\
 & = -c_i(t) \phi_i(0) e^{-\int_0^t c_i(u) du} \\
 & \quad + \sum_{j=1}^n a_{ij}(t) \int_0^\infty d_{ij}(\theta) [f_j(y_j(t-\theta) + x_j^*(t-\theta)) \\
 & \quad - f_j(x_j^*(t-\theta))] d\theta
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{j=1}^n \alpha_{ij}(t) \int_0^\infty \beta_{ij}(\theta) [h_j(y_j'(t-\theta) + (x_j^*)'(t-\theta)) \\
 & \quad - h_j((x_j^*)'(t-\theta))] d\theta \\
 & + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \left(\int_0^\infty k_{ij}(\theta) \right. \\
 & \quad \times [g_j(y_j(t-\theta) + x_j^*(t-\theta)) \\
 & \quad - g_j(x_j^*(t-\theta))] d\theta \\
 & \quad \times \int_0^\infty k_{il}(\theta) g_l \\
 & \quad \times (y_l(t-\theta) + x_l^*(t-\theta)) d\theta \Bigg) \\
 & + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(t) \left(\int_0^\infty k_{ij}(\theta) g_j(x_j^*(t-\theta)) d\theta \right. \\
 & \quad \times \int_0^\infty k_{il}(\theta) [g_l(y_l(t-\theta) + x_l^*(t-\theta)) \\
 & \quad - g_l(x_l^*(t-\theta))] d\theta \Bigg) \\
 & - \int_0^t c_i(t) e^{-\int_s^t c_i(u) du} \\
 & \quad \times \left[\sum_{j=1}^n a_{ij}(s) \int_0^\infty d_{ij}(\theta) [f_j(y_j(s-\theta) + x_j^*(s-\theta)) \right. \\
 & \quad \left. - f_j(x_j^*(s-\theta))] d\theta \right. \\
 & \quad + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \left(\int_0^\infty k_{ij}(\theta) \right. \\
 & \quad \times [g_j(y_j(s-\theta) + x_j^*(s-\theta)) \\
 & \quad - g_j(x_j^*(s-\theta))] d\theta \\
 & \quad \times \int_0^\infty k_{il}(\theta) g_l \\
 & \quad \times (y_l(s-\theta) + x_l^*(s-\theta)) d\theta \Bigg) \\
 & \quad \left. + \sum_{j=1}^n \sum_{l=1}^n b_{ijl}(s) \left(\int_0^\infty k_{ij}(\theta) g_j(x_j^*(s-\theta)) d\theta \right. \right. \\
 & \quad \times \int_0^\infty k_{il}(\theta) \\
 & \quad \times [g_l(y_l(s-\theta) + x_l^*(s-\theta)) \\
 & \quad - g_l(x_l^*(s-\theta))] d\theta \Bigg) \Bigg] ds,
 \end{aligned} \tag{58}$$

where $i = 1, 2, \dots, n$. Thus, we have by (48) and (58) and (H_2) - (H_3) that

$$\begin{aligned}
 & |y'_i(t_1)| \\
 & \leq c_i^M \|\psi\|_\infty^1 e^{-\int_0^{t_1} c_i(u) du} \\
 & \quad + \sum_{j=1}^n \overline{a_{ij}} \int_0^\infty |d_{ij}(\theta)| \epsilon_j |y_j(t_1 - \theta)| d\theta \\
 & \quad + \sum_{j=1}^n \overline{\alpha_{ij}} \int_0^\infty |\beta_{ij}(\theta)| \vartheta_j |y'_j(t_1 - \theta)| d\theta \\
 & \quad + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M \int_0^\infty |k_{ij}(\theta)| \\
 & \quad \quad \times \epsilon_j |y_j(t_1 - \theta)| d\theta \\
 & \quad + \int_0^{t_1} c_i^M e^{-c_i^m(t_1-s)} \\
 & \quad \times \left[\sum_{j=1}^n \overline{a_{ij}} pM \|\psi\|_\infty^1 e^{-\lambda t_1} \right. \\
 & \quad \quad \times \int_0^\infty |d_{ij}(\theta)| \epsilon_j |y_j(s - \theta)| d\theta \\
 & \quad \quad + \sum_{j=1}^n \overline{\alpha_{ij}} pM \|\psi\|_\infty^1 e^{-\lambda t_1} \\
 & \quad \quad \times \int_0^\infty |\beta_{ij}(\theta)| \vartheta_j |y'_j(s - \theta)| d\theta \\
 & \quad \quad + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l k_{il}^M pM \|\psi\|_\infty e^{-\lambda t_1} \\
 & \quad \quad \times \left. \int_0^\infty |k_{ij}(\theta)| \epsilon_j |y_j(s - \theta)| d\theta \right] ds \\
 & = pM \|\psi\|_\infty^1 e^{-\lambda t_1} \\
 & \quad \times \left\{ \frac{c_i^M}{pM} e^{(\lambda - c_i^m)t_1} + \left(1 + c_i^M \int_0^{t_1} e^{(t_1-s)(\lambda - c_i^m)} ds \right) \right. \\
 & \quad \times \left[\sum_{j=1}^n \int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\
 & \quad \quad + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{il}^M |k_{ij}(\theta)| \\
 & \quad \quad \times e^{\lambda \theta} d\theta \left. \right] \left. \right\} \\
 & < pM \|\psi\|_\infty^1 e^{-\lambda t_1}
 \end{aligned}$$

$$\begin{aligned}
 & \times \left\{ \frac{c_i^M}{M} e^{(\lambda - c_i^m)t_1} + \left(1 + \frac{c_i^M}{c_i^m - \lambda} (1 - e^{(\lambda - c_i^m)t_1}) \right) \right. \\
 & \quad \times \left[\sum_{j=1}^n \int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\
 & \quad \quad + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{il}^M |k_{ij}(\theta)| \\
 & \quad \quad \times e^{\lambda \theta} d\theta \left. \right] \left. \right\} \\
 & = pM \|\psi\|_\infty^1 e^{-\lambda t_1} \\
 & \quad \times \left\{ \left(\frac{1}{M} - \frac{1}{c_i^m - \lambda} \right) \right. \\
 & \quad \times \left[\sum_{j=1}^n \int_0^\infty (\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)|) e^{\lambda \theta} d\theta \right. \\
 & \quad \quad + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{il}^M \\
 & \quad \quad \times \left. \int_0^\infty |k_{ij}(\theta)| e^{\lambda \theta} d\theta \right] \left. \right\} \\
 & \quad \times c_i^M e^{(\lambda - c_i^m)t_1} \\
 & \quad + \left(1 + \frac{c_i^M}{c_i^m - \lambda} \right) \\
 & \quad \times \left[\sum_{j=1}^n \int_0^\infty \left(\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)| \right. \right. \\
 & \quad \quad + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{il}^M |k_{ij}(\theta)| \\
 & \quad \quad \times e^{\lambda \theta} d\theta \left. \right] \left. \right\} \\
 & < pM \|\psi\|_\infty^1 e^{-\lambda t_1} \left(1 + \frac{c_i^M}{c_i^m - \lambda} \right) \\
 & \quad \times \left[\sum_{j=1}^n \int_0^\infty (\overline{a_{ij}} \epsilon_j |d_{ij}(\theta)| + \overline{\alpha_{ij}} \vartheta_j |\beta_{ij}(\theta)|) e^{\lambda \theta} d\theta \right. \\
 & \quad \quad + 2 \sum_{j=1}^n \sum_{l=1}^n \overline{b_{ijl}} G_l \epsilon_j k_{il}^M \int_0^\infty |k_{ij}(\theta)| d\theta \left. \right] \\
 & < pM \|\psi\|_\infty^1 e^{-\lambda t_1}.
 \end{aligned}$$

In view of (57) and (59), we obtain

$$\|y(t_1)\|_1 < pM \|\psi\|_\infty^1 e^{-\lambda t_1}, \quad (60)$$

which contradicts the equality (55), and so (54) holds. Letting $p \rightarrow 1$, (53) holds. Hence, the weighted pseudo-almost periodic solution of system (1) is globally exponentially stable. $\square$

Corollary 17. *If conditions (H_1) – (H_3) and (H_5) hold, furthermore, assume that $I_i(t)$ ($i = 1, 2, \dots, n$) are almost periodic functions; then system (1) has a unique continuously differentiable almost periodic solution which is globally exponentially stable.*

Corollary 18. *If conditions (H_1) – (H_3) and (H_5) hold, furthermore, assume that $I_i(t)$ ($i = 1, 2, \dots, n$) are pseudo-almost periodic functions; then system (1) has a unique continuously differentiable pseudo-almost periodic solution which is globally exponentially stable.*

5. An Example

In this section, we give one example to illustrate our result. Consider the weight $\mu(t) = e^{|t|}$ and let

$$\begin{aligned} f_1(x) &= f_2(x) = \frac{\sin\left(\left(1/2^{1/2}\right)x\right)}{10}, \\ h_1(x) &= h_2(x) = \frac{\sin\left(\left(1/2^{3/2}\right)x\right)}{20}, \\ g_1(x) &= g_2(x) = \frac{|\arctan\left(\left(1/2^{1/2}\right)x\right)|}{20}, \\ a_{11}(t) &= 1 + \cos(2\pi t), & a_{12}(t) &= 2 + \cos(2\pi t), \\ a_{21}(t) &= 2 + \cos(2\pi t), & a_{22}(t) &= 3 + \cos(2\pi t), \\ \alpha_{11}(t) &= 1 + \sin(\pi t), & \alpha_{12}(t) &= 2 + \sin^5(2\pi t), \\ \alpha_{21}(t) &= 2 + \cos^3(2\pi t), & \alpha_{22}(t) &= 3 + \cos(\pi t), \\ c_1(t) &= 25 + 5 \sin(2\pi t), & c_2(t) &= 35 + 5 \sin(2\pi t), \\ I_1(t) &= \sin(\pi t) + \sin t, & I_2(t) &= \cos(\pi t) + \sin t, \end{aligned} \quad (61)$$

$$b_{111}(t) = b_{211}(t) = \frac{1}{4} + \frac{1}{4} \sin(\pi t),$$

$$b_{112}(t) = b_{212}(t) = \frac{1}{3} + \frac{1}{3} \cos(\pi t),$$

$$b_{121}(t) = b_{221}(t) = \frac{1}{5} + \frac{1}{5} \cos(\pi t),$$

$$b_{122}(t) = b_{222}(t) = \frac{1}{6} + \frac{1}{6} \sin(\pi t),$$

$$\begin{aligned} (d_{ij}(\theta))_{2 \times 2} &= (k_{ij}(\theta))_{2 \times 2} = (\beta_{ij}(\theta))_{2 \times 2} \\ &= \begin{pmatrix} e^{-10\theta} & 0 \\ 0 & e^{-50\theta} \end{pmatrix}. \end{aligned}$$

Then system (1) has exactly one continuously differentiable weighted pseudo-almost periodic solution, which is globally exponentially stable.

Proof. By calculating,

$$\begin{aligned} c_1^m &= 20, & c_1^M &= 30, & c_2^m &= 30, & c_2^M &= 40, \\ \overline{a_{11}} &= \overline{\alpha_{11}} = 2, & \overline{a_{12}} &= \overline{\alpha_{12}} = 3, & \overline{a_{21}} &= \overline{\alpha_{21}} = 3, \\ \overline{a_{22}} &= \overline{\alpha_{22}} = 4, & \overline{b_{111}} &= \overline{b_{211}} = \frac{1}{2}, & \overline{b_{112}} &= \overline{b_{212}} = \frac{2}{3}, \\ \overline{b_{121}} &= \overline{b_{221}} = \frac{2}{5}, & \overline{b_{122}} &= \overline{b_{222}} = \frac{1}{3}, & \epsilon_1 &= \epsilon_2 = \frac{1}{10}, \\ \vartheta_1 &= \vartheta_2 = \frac{1}{20}, & \epsilon_1 &= \epsilon_2 = \frac{1}{20}, & G_1 &= G_2 = \frac{\pi}{40}, \\ d_{11}^M &= k_{11}^M = \beta_{11}^M = \frac{1}{10}, & d_{12}^M &= k_{12}^M = \beta_{12}^M = 0, \\ d_{21}^M &= k_{21}^M = \beta_{21}^M = 0, & d_{22}^M &= k_{22}^M = \beta_{22}^M = \frac{1}{50}, \end{aligned} \quad (62)$$

hence we have

$$\begin{aligned} \xi &= \max_{1 \leq i \leq 2} \left\{ \max \left\{ \frac{1}{c_i^m}, 1 + \frac{c_i^M}{c_i^m} \right\} \right. \\ &\quad \times \sum_{j=1}^2 \left[\overline{a_{ij}} d_{ij}^M \epsilon_j + \overline{\alpha_{ij}} \beta_{ij}^M \vartheta_j + 2\epsilon_j \sum_{l=1}^2 \overline{b_{ijl}} G_l k_{ij}^M k_{il}^M \right] \left. \right\} \\ &= \max \left\{ \frac{3}{40} + \frac{\pi}{16000}, \frac{7}{100} + \frac{7\pi}{240000} \right\} < 1. \end{aligned} \quad (63)$$

It is obvious that (H_1) – (H_5) are satisfied. By Theorems 12 and 16, system (1) has exactly one continuously differentiable weighted pseudo-almost periodic solution, which is globally exponentially stable (see Figures 1, 2, and 3). $\square$

6. Conclusion

In this paper, we employ fixed point theorem and differential inequality techniques to study the existence, uniqueness, and global exponential stability of weighted pseudo-almost periodic solutions to a class of neutral type high-order Hopfield neural networks with infinitely distributed delays. Our results of this paper are new and complement previously known results, and our methods used in this paper can be used to investigate other types of neural networks such as neutral type BAM neural networks, neutral type Cohen-Grossberg neural networks, and neutral type high-order Hopfield neural networks with delays in the leakage term and so on.

FIGURE 1: Responds of x_1, x_2 with time t .

FIGURE 2: Responds of x_1, x_2 .

FIGURE 3: Responds of x_1, x_2 with time t .

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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