Maps Preserving Schatten $p$-Norms of Convex Combinations

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Abstract and Applied Analysis

We study maps $\phi$ of positive operators of the Schatten $p$-classes ($1 < p < +\infty$), which preserve the $p$-norms of convex combinations, that is, $\|t\rho + (1-t)\sigma\|_p = \|t\phi(\rho) + (1-t)\phi(\sigma)\|_p$, $\forall \rho, \sigma \in S^+_p(H)$, $t \in [0,1]$. They are exactly those carrying the form $\phi(\rho) = U\rho U^*$ for a unitary or antiunitary $U$. In the case $p = 2$, we have the same conclusion whenever it just holds $\|\rho + \sigma\|_2 = \|\phi(\rho) + \phi(\sigma)\|_2$ for all the positive Hilbert-Schmidt class operators $\rho, \sigma$ of norm 1. Some examples are demonstrated.

1. Introduction

The Mazur-Ulam theorem states that every bijective distance preserving map $\Phi$ from a Banach space onto another is affine; that is,

$$\Phi(tx + (1-t)y) = t\Phi(x) + (1-t)\Phi(y),$$

$$\forall x, y, 0 \leq t \leq 1.$$  

(1)

After translation, we can assume that $\Phi(0) = 0$ and $\Phi$ is indeed a surjective real linear isometry. Let us consider another version of this statement. Suppose that $\Phi$ is a bijective map from a Hilbert space $H$ onto $\tilde{H}$ and $\Phi$ preserves norm of convex combinations:

$$\|t\Phi(x) + (1-t)\Phi(y)\| = \|tx + (1-t)y\|,$$

$$\forall x, y \in H, 0 \leq t \leq 1.$$  

(2)

Let us further relax the assumption that (2) holds for just one fixed $t$ in $(0,1)$. By letting $y = x$ in (2), we see that $\|\Phi(x)\| = \|x\|$ for all $x$ in $H$. Squaring both sides of (2), we will see that the real parts of the inner products coincide; that is,

$$\Re \langle x, y \rangle = \Re \langle \Phi(x), \Phi(y) \rangle, \quad \forall x, y \in H.$$  

(3)

Then the classical Wigner theorem (see, e.g., [1, Theorem 3]) ensures that there is a surjective real linear isometry $U : H \to H$ such that $\Phi(x) = Ux$ for all $x$ in $H$.

Characterizing isometries, linear or not, of spaces of operators under various norms has been a fruitful area of research for a long time. See, for example, [2, 3] for good surveys. In particular, the spaces $S^+_p(H)$ of the Schatten $p$-class operators on a (complex) Hilbert space $H$ ($1 \leq p < +\infty$) are important objects in both analysis and physics. They are widely used in operator theory and quantum mechanics, for example.

Let $\delta^+_p(H)$ be the set of all positive operators in $S^+_p(H)$, and let $\delta^+_p(H)_1$ be the set of all positive operators in $S^+_p(H)$ of $p$-norm one. Recall that an affine automorphism (or S-automorphism in [4] or Kadison automorphism in [5]) is a bijective affine map $\phi : \delta^+_1(H)_1 \to \delta^+_1(H)_1$; that is,

$$\phi(tp + (1-t)\sigma) = t\phi(p) + (1-t)\phi(\sigma),$$

$$\forall p, \sigma \in \delta^+_1(H)_1, \ t \in [0,1].$$  

(4)

It is known (see, e.g., [6]) that affine automorphisms are exactly those carrying the form $\phi(\rho) = U\rho U^*$ for a unitary or antiunitary $U$ on $H$.

Recently, Nagy [7] established a Mazur-Ulam-type result for the Schatten $p$-class operators. Suppose that $\phi : \delta^+_p(H)_1 \to \delta^+_p(H)_1$ ($1 < p < +\infty$) is a bijective map preserving the distance induced by the norm $\|\cdot\|_p$. Then $\phi$ is implemented by a unitary or an antiunitary operator $U$ such that $\phi(\rho) = U\rho U^*$. In this paper, we will establish a
counterpart of Nagy’s result similar to the one demonstrated in the first paragraph. More precisely, we will characterize those maps \( \phi : \delta_p^\ast(H)_1 \to \delta_p^\ast(H)_1 \) satisfying
\[
\| tp + (1 - t) \sigma \|_p = \| t\phi(\rho) + (1 - t)\phi(\sigma) \|_p,
\]
\[
\forall \rho, \sigma \in \delta_p^\ast(H)_1, \ t \in [0, 1].
\]
We will show that they are implemented by a unitary or an antiunitary operator.

Our main theorem follows.

**Theorem 1.** Let \( H \) be a separable complex Hilbert space of finite or infinite dimension. Let \( 1 < p < +\infty \). Suppose that \( \phi \) is a map from \( \delta_p^\ast(H)_1 \) into \( \delta_p^\ast(H)_1 \), which will be assumed to be surjective when \( \dim H = +\infty \). The following conditions are equivalent.

1. \( \phi \) preserves the Schatten \( p \)-norms of convex combinations; that is,
\[
\| tp + (1 - t) \sigma \|_p = \| t\phi(\rho) + (1 - t)\phi(\sigma) \|_p,
\]
\[
\forall \rho, \sigma \in \delta_p^\ast(H)_1, \ t \in [0, 1].
\]
2. \( \phi \) preserves the pairings; that is, for all \( \rho, \sigma \in \delta_p^\ast(H)_1 \), one has \( \| \rho \|_p^{p-1} \rho \in \delta_1(H) \), and
\[
\text{tr} (\rho^p)^{-\frac{1}{p}} = \| t\phi(\rho) + (1 - t)\phi(\sigma) \|_p.
\]
3. There exists a unitary or antiunitary operator \( U \) on \( H \) such that
\[
\phi(\rho) = U\rho U^\ast, \ \forall \rho \in \delta_p^\ast(H)_1.
\]

We note that condition (6) becomes a tautology when \( p = 1 \). On the other hand, the conclusion of Theorem 1 holds again if we replace \( \delta_p^\ast(H)_1 \) by \( \delta_p^\ast(H) \) everywhere. In this case, setting \( \sigma = \rho \) in (6), we see that \( \phi \) does map \( \delta_p^\ast(H)_1 \) into \( \delta_p^\ast(H) \).

The proof of Theorem 1 is given in Section 2. When \( p = 2 \), we see in Section 3 that for \( \phi \) carrying the expected form stated in Theorem 1(3) it suffices to say that condition (6) held for only one fixed \( t \) in \( (0, 1) \). Finally, we demonstrate some examples in Section 4.

## 2. Proof of the Main Theorem

In what follows, we fix some notation and definitions used throughout the paper. Let \( H \) stand for a separable complex Hilbert space of finite dimension or infinite dimension. Let \( B(H) \) denote the algebra of all bounded linear operators on \( H \). For a compact operator \( T \) in \( B(H) \), let \( s_1(T) \geq s_2(T) \geq \cdots \geq 0 \) denote the singular values of \( T \), that is, the eigenvalues of \( |T| = (TT^\ast)^{1/2} \) arranged in their decreasing order (repeating according to multiplicity). A compact operator \( T \) belongs to the Schatten \( p \)–classes \( \delta_p^\ast(H) \) (\( 1 \leq p < +\infty \)) if
\[
\| T \|_p := \left( \sum_{i=1}^{\infty} s_i(T)^p \right)^{1/p} = \left( \text{tr} |T|^p \right)^{1/p} < +\infty,
\]
where \( \text{tr} \) denotes the trace functional. We call \( \| T \|_p \) the Schatten \( p \)-norm of \( T \). In particular, \( \delta_1^\ast(H) \) is the trace class and \( \delta_p^\ast(H) \) is the Hilbert–Schmidt class. The cone of positive operators in \( \delta_p^\ast(H) \) is denoted by \( \delta_p^\ast(H) \) and the set of rank one projections in \( \delta_p^\ast(H) \) is denoted by \( P_p(H) \).

Recall that the norm of a normed space is Fréchet differentiable at \( x \neq 0 \) if \( \lim_{t \to 0} (\|x + ty\| - \|x\|)/t \) exists and uniform for all norm one vectors \( y \).

**Lemma 2** (see [8, Theorem 2.3]). Let \( 1 < p < +\infty \) and \( \rho \in \delta_p^\ast(H) \) be nonzero. The norm of \( \delta_p^\ast(H) \) is Fréchet differentiable at \( \rho \). For any \( \sigma \in \delta_p^\ast(H) \), one has
\[
\frac{d\| t\rho + (1 - t)\sigma \|_p}{dt} \bigg|_{t=0} = \text{tr} \left( \rho^p \frac{\sigma}{\| \rho \|_p^p} \right).
\]

**Lemma 3.** Suppose \( \rho, \sigma \in \delta_p^\ast(H) \) (\( 1 < p < +\infty \)). The following conditions are equivalent.

1. \( \rho = \sigma \).
2. \( \| tp + (1 - t)\rho \|_p \leq \| t\sigma + (1 - t)\rho \|_p \) for all \( \rho \) in \( P_p(\delta_p^\ast(H)) \) and all \( t \) in \( [0, 1] \).
3. \( \text{tr}(\rho^p) = \text{tr}(\sigma^p) \) for all \( \rho \) in \( P_p(\delta_p^\ast(H)) \).

**Proof.** (1) \( \Rightarrow \) (2) is obvious.

\( (2) \Rightarrow (3) \): Differentiating both sides of \( \| tp + (1 - t)\rho \|_p = \| t\sigma + (1 - t)\rho \|_p \) at \( t = 0^+ \), we have \( \text{tr} \rho^p = \text{tr} P_p \rho^p = \text{tr} P_p \sigma^p, \) and all \( t \) in \( \{0, 1\} \).

(3) \( \Rightarrow \) (1): Since \( \rho \) and \( \sigma \) are positive, \( \rho - \sigma \) is Hermitian. There exists an orthonormal basis \( \{e_i\}_{i=1}^{\infty} \) of \( H \) such that \( \rho - \sigma = \sum_{i=1}^{\infty} \lambda_i e_i \otimes e_i \). Choosing \( P_i = e_i \otimes e_i \), we have \( \lambda_i = \text{tr} \rho - \sigma = 0 \) for all \( i \geq 1 \).

We say that two self-adjoint operators \( \rho, \sigma \) in \( B(H) \) are orthogonal if \( \rho \perp \sigma \), which is equivalent to the property that they have mutually orthogonal ranges.

**Lemma 4.** Suppose that \( \rho, \sigma \in \delta_p^\ast(H) \) for \( 1 < p < +\infty \). The following conditions are equivalent.

1. \( \rho, \sigma \) are orthogonal; that is, \( \rho \perp \sigma \).
2. \( \| (1 - \alpha)\rho \|_p + \| (1 - \alpha)\sigma \|_p = \alpha \| (1 - \alpha)\rho \|_p + \| \sigma \|_p \) for any \( \alpha \in (0, 1) \).
3. \( \text{tr}(\rho \sigma) = 0 \).
4. \( \| \rho + t\sigma \|_p \geq \| \rho \|_p \) for all \( t \) in \( \mathbb{R} \); that is, \( \rho \perp \sigma \) in Birkhoff’s sense.
5. \( \text{tr}(\rho^p \sigma) = 0 \).

**Proof.** (1) \( \Rightarrow \) (2): From [9, Lemma 2.6], we know that for any two positive operators \( A, B \) in \( \delta_p^\ast(H) \), we have
\[
\text{tr}(A + B)^p \geq \text{tr} A^p + \text{tr} B^p.
\]
Here, the equality holds if and only if $AB = 0$. Setting $A = \alpha \rho$ and $B = (1 - \alpha)\sigma$, we get

$$\rho \sigma = 0 \iff (\alpha \rho)(1 - \alpha) \sigma = 0$$

$$\iff \text{tr}(\alpha \rho + (1 - \alpha) \sigma)^p = \text{tr}(\alpha \rho)^p + \text{tr}((1 - \alpha) \sigma)^p$$

$$\iff \|\alpha \rho + (1 - \alpha) \sigma\|_p^p = \alpha \|\rho\|_p^p + (1 - \alpha) \|\sigma\|_p^p.$$  \hfill (12)

(1) $\Rightarrow$ (3): One direction is obvious. For the other, because $\rho, \sigma$ are positive,

$$\text{tr} \left[ (\rho^{1/2} \sigma^{1/2})(\rho^{1/2} \sigma^{1/2})^* \right] = \text{tr}(\rho^{1/2} \sigma^{1/2} \rho^{1/2}) = \text{tr}(\rho \sigma) = 0.$$  \hfill (13)

This forces $\rho^{1/2} \sigma^{1/2} = 0$, and thus $\rho \sigma = \rho^{1/2} (\rho^{1/2} \sigma^{1/2}) \rho^{1/2} = 0$.

(1) $\Rightarrow$ (4): Since $\rho \sigma = 0$, there exists an orthonormal basis $\{e_i\}_{i=1}^\infty$ of $H$ such that $\rho = \sum_{i=1}^\infty \lambda_i e_i \otimes e_i$, $\sigma = \sum_{i=1}^\infty \mu_i e_i \otimes e_i$, $\lambda_i \geq 0$, $\mu_i \geq 0$, and $\lambda_i \mu_i = 0$ for all $i = 1, 2, \ldots$. Hence,

$$\|\rho + t \sigma\|_p^p = \text{tr} |\rho + t \sigma|^p = \sum_{i=1}^\infty (\lambda_i + |t| \mu_i)^p \geq \sum_{i=1}^\infty \lambda_i^p = \|\rho\|_p^p.$$  \hfill (14)

(4) $\Rightarrow$ (5): Without loss of generality, we can assume that $\rho \neq 0$. Define $f(t) = \|\rho + t \sigma\|_p \geq \|\rho\|_p$. Then $f(t)$ is differentiable and attains its minimum at $t = 0$. From Lemma 2,

$$0 = \left. \frac{d}{dt} \|\rho + t \sigma\|_p \right|_{t=0} = \text{tr} \left( \frac{\rho^{p-1} \sigma}{\|\rho\|_p^{p-1}} \right),$$  \hfill (15)

and assertion (5) follows.

(5) $\Rightarrow$ (1): As in proving (3) $\Rightarrow$ (1), we have $\rho^{p-1} \sigma = 0$. Then, there exists an orthonormal basis $\{e_i\}_{i=1}^\infty$ of $H$ such that $\rho^{p-1} = \sum_{i=1}^\infty \xi_i e_i \otimes e_i$, $\sigma = \sum_{i=1}^\infty \eta_i e_i \otimes e_i$, with $\xi_i \geq 0$, $\eta_i \geq 0$, and $\xi_i \mu_i = 0$ for all $i = 1, 2, \ldots$. Thus, $\text{tr}(\rho \sigma) = \sum_{i=1}^\infty \xi_i^{p/(p-1)} \eta_i = 0.$  \hfill □

Lemma 5. Let $1 < p < +\infty$. Suppose that $\phi$ is a map from $\mathcal{S}_p^+(H)$ into $\mathcal{S}_p^+(H)$ preserving the Schatten $p$-norms of convex combinations; that is, (6) holds. Then, one has

$$\text{tr} (\sigma^{p-1}) = \text{tr} (\phi(\sigma)^{p-1} \phi(\rho)).$$  \hfill (16)

Proof. Differentiating both sides of (6) with respect to $t$ and evaluating at $t = 0$, we have

$$\left. \frac{d}{dt} \|\rho + (t - 1) \rho\|_p \right|_{t=0} = \left. \frac{d}{dt} \|\sigma + t (\rho - \sigma)\|_p \right|_{t=0}$$

$$= \text{tr} \left( \frac{\sigma^{p-1} (\rho - \sigma)}{\|\sigma\|_p^{p-1}} \right) - \|\sigma\|_p$$

$$= \text{tr} (\sigma^{p-1} \rho - 1),$$

and

$$\left. \frac{d}{dt} \|\phi(\rho) + (1-t) \phi(\sigma)\|_p \right|_{t=0} = \left. \frac{d}{dt} \|\phi(\sigma)\|_p \right|_{t=0}$$

$$= \text{tr} (\phi(\sigma)^{p-1} - \|\phi(\sigma)\|_p)$$

$$= \text{tr} (\phi(\sigma)^{p-1} \rho - 1).$$  \hfill (17)

Since (6) holds for $t$ in $(0, 1]$, these derivatives agree. Therefore, $\text{tr}(\sigma^{p-1} \rho) = \text{tr}(\phi(\sigma)^{p-1} \phi(\rho)).$  \hfill □

Proposition 6. Suppose that $\phi : \mathcal{S}_p^+(H) \rightarrow \mathcal{S}_p^+(H)$ satisfies

$$\text{tr} (\sigma^{p-1} \rho) = \text{tr} (\phi(\sigma)^{p-1} \phi(\rho)), \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H).$$  \hfill (18)

Then the following assertions hold.

(1) $\phi$ preserves orthogonality in both directions; that is

$$\rho \sigma = 0 \iff \phi(\rho) \phi(\sigma) = 0, \quad \forall \rho, \sigma \in \mathcal{S}_p^+(H).$$  \hfill (19)

(2) When $\dim H < +\infty$, $\phi$ maps rank-one projections to rank-one projections. This also holds when $\dim H = +\infty$ and $\phi$ is surjective.

(3) When $\dim H < +\infty$, one has

$$\text{tr} P Q = \text{tr} \phi(P) \phi(Q), \quad \forall P, Q \in P_1(H).$$  \hfill (20)

This also holds when $\dim H = +\infty$ and $\phi$ is surjective.

Proof. (1) follows from Lemma 4.

(2) First, we assume that $\dim H = n < +\infty$. Suppose $\rho$ is a rank-one projection. We can find $n - 1$ pairwise orthogonal rank-one projections $\rho_1, \ldots, \rho_{n-1}$ such that $\rho \rho_i = 0$ for $1 \leq i \leq n - 1$. From (1), we know that $\phi(\rho), \phi(\rho_1), \ldots, \phi(\rho_{n-1})$ are nonzero and pairwise orthogonal. This forces that $\phi(\rho)$ has rank one since $\dim H = n$. By (18), taking $\sigma = \rho$, we see that $\text{tr}(\phi(\rho)^p) = \text{tr} \rho^p = \text{tr} \rho = 1$. Therefore, the rank-one positive operator $\phi(\rho)$ is a projection.

Next, we consider the case $\dim H = +\infty$ and $\phi$ is surjective. Suppose that there exists a rank-one projection $\rho$ in $\mathcal{S}_p^+(H)$ such that $\phi(\rho)$ has rank greater than one. Then, there are two nonzero orthogonal operators $T_1$ and $T_2$ in $\mathcal{S}_p^+(H)$ such that $\phi(\rho) = T_1 + T_2$. Since $\phi$ is surjective and preserves orthogonality in both directions, there are two
nonzero orthogonal operators $\rho_1$ and $\rho_2$ in $\mathcal{S}^\ast_p(H)$ such that $\phi(\rho_1) = T_1/\|T_1\|_p$ and $\phi(\rho_2) = T_1/\|T_2\|_p$. For any $\sigma$ in $\mathcal{S}^\ast_p(H)$ with $\sigma \rho = 0$, we have
\[
\phi(\sigma) \left( \|T_1\|_p \phi(\rho_1) + \|T_2\|_p \phi(\rho_2) \right) = \phi(\sigma) \left( T_1 + T_2 \right) = \phi(\sigma) \phi(\rho) = 0.
\]
(21)

It forces that
\[
\|T_1\|_p \phi(\sigma) \phi(\rho_1) \phi(\sigma) = -\|T_2\|_p \phi(\sigma) \phi(\rho_2) \phi(\sigma) = 0.
\]
(22)

and hence $\phi(\sigma) \phi(\rho_1) = \phi(\sigma) \phi(\rho_2) = 0$, because $\phi(\sigma)$, $\phi(\rho_1)$, and $\phi(\rho_2)$ are all positive. This implies $\sigma \rho_1 = \sigma \rho_2 = 0$. Therefore, $\rho_1 = \lambda_1 \rho$ and $\rho_2 = \lambda_2 \rho$ for some nonzero $\lambda_1, \lambda_2$. This contradicts the fact that $\rho_1 \rho_2 = 0$.

(3) From (2), we know that $\phi(\rho), \phi(\sigma)$ are rank-one projections in $P_1(H)$. Therefore, $P^{p-1} = \rho, \phi(P)^{p-1} = \phi(P)$. Using (18) with $\sigma = P, \rho = Q$, we have
\[
\text{tr} \, PQ = \text{tr} \left( P^{p-1}Q \right) = \text{tr} \left( \phi(P)^{p-1} \phi(Q) \right) = \text{tr} \left( \phi(P) \phi(Q) \right).
\]
(23)

Proof of Theorem 1. (1) $\Rightarrow$ (2) follows from Lemma 5.
(2) $\Rightarrow$ (3): From Proposition 6, we obtain that $\phi|_{P_1(H)} : P_1(H) \rightarrow P_1(H)$ satisfies $\text{tr} \, PQ = \text{tr} \phi(P) \phi(Q)$ for all rank-one projections $P, Q$ in $P_1(H)$. From a nonsurjective version of Wigner's theorem, cf. [6, Theorem 2.1.4], there exists an isometry or anti-isometry $U$ on $H$ such that
\[
\phi(P) = UPU^*, \quad \forall P \in P_1(H).
\]
(24)

Note that $U$ is indeed surjective even when $H$ is of infinite dimension, since $\phi$ is assumed to be surjective in this case.

For any rank-one projection $P$ in $P_1(H)$, setting $\sigma = P$ in (7), we have
\[
\text{tr} \left( P^P \right) = \text{tr} \left( \phi(P)^{p-1} \phi(\rho) \right) = \text{tr} \left( \phi(\rho) \phi(P) \right) = \text{tr} \left( UPU^* \phi(\rho) U \right) = \text{tr} \left( UP^\ast \phi(\rho) U \right).
\]
(25)

We have $U^* \phi(\rho) U = \rho$ by Lemma 3. This gives $\phi(\rho) = U \rho U^*$.  

\[\Box\]

3. Maps Preserving Norms of Just a Special Convex Combination

A careful look at the proof of Lemma 5 tells us that the condition $\|\rho + (1 - t) \sigma\|_p = \|\phi(\rho) + (1 - t) \phi(\sigma)\|_p$, suffices to hold for the members of any sequence in $[0, 1]$ converging to 0 rather than for any point $t$ in $[0, 1]$. Indeed, in order to get some good properties of $\phi$ stated in the previous section, we only need to assume that $\phi$ preserves the Schatten $p$-norm of convex combination with a given system of coefficients.

\[\Box\]

Proposition 7. Let $\phi : \mathcal{S}^\ast_p(H)_1 \rightarrow \mathcal{S}^\ast_p(H)_1$ ($1 < p < +\infty$). Let $\alpha$ in $(0, 1)$ be arbitrary but fixed. Suppose
\[
\|\alpha \rho + (1 - \alpha) \sigma\|_p = \|\alpha \phi(\rho) + (1 - \alpha) \phi(\sigma)\|_p,
\]
\[\forall \rho, \sigma \in \mathcal{S}^\ast_p(H)_1.\]
(26)

The following properties are satisfied.
(1) $\phi$ is injective.
(2) $\phi$ preserves orthogonality in both directions.
(3) When $\dim H < +\infty$, $\phi$ maps rank-one projections to rank-one projections. This also holds when $\dim H = +\infty$ and $\phi$ is surjective.

Proof. (1) Assume $\phi(\rho) = \phi(\sigma)$. We have $\|\alpha \phi(\rho) + (1 - \alpha) \phi(\sigma)\|_p = 1$. From (26), we get $\|\alpha \rho + (1 - \alpha) \sigma\|_p = 1$. Hence,
\[
\|\alpha \rho + (1 - \alpha) \sigma\|_p = \|\alpha \rho\|_p + (1 - \alpha) \|\sigma\|_p.
\]
(27)

This forces $\rho = \sigma$ since the norm $\|\cdot\|_p$ is strictly convex for $1 < p < +\infty$.

(2) Assume $\rho \sigma = 0$. From Lemma 4, we have
\[
\|\alpha \rho + (1 - \alpha) \sigma\|_p^p = \alpha \rho\|\rho\|_p^p + (1 - \alpha) \|\sigma\|_p^p
\]
\[= \alpha \|\phi(\rho)\|_p^p + (1 - \alpha) \|\phi(\sigma)\|_p^p.
\]
(28)

Together with (26), we have
\[
\|\alpha \phi(\rho) + (1 - \alpha) \phi(\sigma)\|_p^p
\]
\[= \alpha \|\phi(\rho)\|_p^p + (1 - \alpha) \|\phi(\sigma)\|_p^p.
\]
(29)

Hence, we have $\phi(\rho) \phi(\sigma) = 0$ from Lemma 4 again. The other implication follows similarly.

(3) The proof is similar to that of Proposition 6(2).  

\[\Box\]

When $p = 2$, we get an improvement of Theorem 1.

Theorem 8. Let $H$ be a separable complex Hilbert space. Suppose that $\phi : \mathcal{S}^\ast_p(H)_1 \rightarrow \mathcal{S}^\ast_p(H)_1$, which needs to be surjective when $\dim H = +\infty$. The following conditions are equivalent.

(1) $\phi$ preserves the Hilbert-Schmidt norms of all convex combinations; that is,
\[
\|t \rho + (1 - t) \sigma\|_2 = \|t \phi(\rho) + (1 - t) \phi(\sigma)\|_2,
\]
\[\forall \rho, \sigma \in \mathcal{S}^\ast_p(H)_1, \quad t \in [0, 1].\]
(30)

(2) For any (and thus all) $\alpha$ in $(0, 1)$ one has
\[
\|\alpha \rho + (1 - \alpha) \sigma\|_2 = \|\alpha \phi(\rho) + (1 - \alpha) \phi(\sigma)\|_2,
\]
\[\forall \rho, \sigma \in \mathcal{S}^\ast_p(H)_1.\]
(31)

A special case states that
\[
\|\rho + \sigma\|_2 = \|\phi(\rho) + \phi(\sigma)\|_2, \quad \forall \rho, \sigma \in \mathcal{S}^\ast_p(H)_1.
\]
(32)
(3) \( \text{tr}(\rho \sigma) = \text{tr}(\phi(\rho)\phi(\sigma)) \) for all \( \rho, \sigma \) in \( \mathcal{S}_+^1(H) \).

(4) There exists a unitary or antiunitary operator \( U \) such that \( \phi(\rho) = U\rho U^*, \ \forall \rho \in \mathcal{S}_+^1(H) \).

Proof. We prove (2) \( \Rightarrow \) (3) only. Observe
\[
\|a\rho + (1 - a)\sigma\|^2 = \text{tr}(a\rho + (1 - a)\sigma)^2
= a^2 \text{tr} \rho^2 + 2a(1 - a) \text{tr} (\rho \sigma)
+ (1 - a)^2 \text{tr} \sigma^2,
\]
\[
\|a\phi(\rho) + (1 - a)\phi(\sigma)\|^2 = a^2 \text{tr} \phi(\rho)^2
+ 2a(1 - a) \text{tr} (\phi(\rho)\phi(\sigma))
+ (1 - a)^2 \text{tr} \phi(\sigma)^2.
\]
We have \( \text{tr}(\rho \sigma) = \text{tr}(\phi(\rho)\phi(\sigma)) \).

4. Examples

We remark that all results in previous sections hold for a map \( \phi : \mathcal{S}_+^p(H) \rightarrow \mathcal{S}_+^1(H) \) which satisfies instead of (6) the condition
\[
\|t\rho + (1 - t)\sigma\|_p = \|t\phi(\rho) + (1 - t)\phi(\sigma)\|_p,
\]
\[
\forall \rho, \sigma \in \mathcal{S}_+^p(H), \ t \in [0, 1].
\]

The proofs go in exactly the same ways.

The following example shows that a norm preserver of \( \mathcal{S}_+^p(H) \) might not be affine.

Example 1. Let \( H \) be a finite dimensional Hilbert space with an orthonormal basis \( \{e_i\}_{i=1}^n \). Let \( 1 < p < +\infty \). Define a map \( \phi \) from \( \mathcal{S}_+^p(H) \) into itself by
\[
\phi(\rho) = \begin{cases} 0, & \text{if } \rho = 0, \\
\|\rho\|_p \sum_{i=1}^n P_i \rho P_i, & \text{if } \rho \neq 0,
\end{cases}
\]
where \( P_i = e_i \otimes e_i \) is a rank-one projection for \( i = 1, \ldots, n \). Obviously, \( \phi(\rho) \) is positive and \( \|\phi(\rho)\|_p = \|\rho\|_p \) for all \( \rho \) in \( \mathcal{S}_+^p(H) \). However, \( \phi \) does not preserve the Schatten \( p \)-norms of convex combinations, as the eigenvalues of \( \rho \) and \( \phi(\rho) \) can be different from each other.

Our theorems are about the Schatten \( p \)-norms for \( 1 < p < +\infty \). Here is an example of a map of \( \mathcal{S}_+^1(H) \) which preserves trace norms of convex combinations. However, it is not implemented by a unitary or antiunitary.

Example 2. Consider Example 1 in the case where \( p = 1 \). In this case,
\[
\phi(\rho) = \sum_{i=1}^n P_i \rho P_i.
\]
It is easy to see that the map \( \phi \) satisfies the condition
\[
\|t\rho + (1 - t)\sigma\|_1 = \|t\phi(\rho) + (1 - t)\phi(\sigma)\|_1,
\]
\[
\forall \rho, \sigma \in \mathcal{S}_+^1(H), \ t \in [0, 1].
\]

But there does not exist a unitary or antiunitary \( U \) such that \( \phi(\rho) = U\rho U^* \) for all \( \rho \) in \( \mathcal{S}_+^1(H) \).

Example 3. Let \( H \) be a separable Hilbert space of infinite dimension, and let \( \{e_n : n = 1, 2, \ldots\} \) be a basis of \( H \). Let \( S \) be the unilateral shift on \( H \) defined by \( Se_n = e_{n+1} \) for \( n = 1, 2, \ldots \). Let \( \phi \) be defined by \( \phi(\rho) = S\rho S^* \) for \( \rho \) in \( \mathcal{S}_+^p(H) \). The map \( \phi \) is not surjective, as \( e_i \otimes e_i \) is not in its range. It is easy to see that \( \|t\rho + (1 - t)\sigma\|_p = \|t\phi(\rho) + (1 - t)\phi(\sigma)\|_p \) holds for all \( \rho, \sigma \) in \( \mathcal{S}_+^p(H) \) and \( t \in [0, 1] \). However, \( \phi \) is not implemented by a unitary or antiunitary.

Conflicts of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References


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