Research Article

On Certain Matrices of Bernoulli Numbers

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In this work we compute the determinant and inverse matrices for a certain symmetric matrix of Rayleigh sums. As a special case we also obtain the determinants and inverses for the matrices of the Bernoulli numbers and related numbers.

1. Introduction

The sequence of Bernoulli numbers B_n is one of the most important sequences in mathematics. It has deep connections to number theory, for instance, the Bernoulli numbers are used to express the values of $\zeta(2n)$, where $\zeta(s)$ is the Riemann zeta function and n is a positive integer [1, 2]. The Bernoulli numbers are also very important in analysis, for example, they appear in the Euler-Maclaurin formula [1], which is very important in mathematics and physics. The Bernoulli numbers are also very important in asymptotics of q -special functions; for example, in [3] we proved a complete asymptotic expansion of q -Gamma function $\Gamma_q(z)$ on the complex plane in terms of Bernoulli polynomials and Bernoulli polynomials. The applications of Bernoulli numbers in applied mathematics are just too many to list all of them; just to name a few, for example, see [4–6]. The Rayleigh sums $\sigma_\nu^{(n)}$ generalize $\zeta(2n)$ and it is known that $\sigma_{1/2}^{(n)}$ is a rational multiple of B_{2n} [7]. In this work we first derive the inverse and determinant of a certain symmetric matrix defined by $\sigma_\nu^{(n)}$ and then specialize the result to the matrices defined by Bernoulli numbers B_n and related numbers S_n .

But we have to emphasize that the present work demonstrated a method to compute inverses of certain Hankel matrices, not just determinants. In fact there are many known methods to compute determinants; for example, see [1, 8–11].

2. Preliminaries

For $\nu > -1$ the Bessel function of first kind is defined by [1, 7, 11, 12]:

$$J_\nu(z) = \frac{1}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\nu+1)_k} \left(\frac{z}{2}\right)^{2k+\nu}, \quad (1)$$

where

$$\frac{1}{\Gamma(a)} = a \prod_{j=1}^{\infty} \left(1 + \frac{a}{j}\right) \left(1 + \frac{1}{j}\right)^{-a}, \quad (2)$$

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}, \quad n \in \mathbb{Z}, \quad a \in \mathbb{C}.$$

As a special case we have

$$J_{1/2}(z) = \sqrt{\frac{2}{\pi z}} \sin z. \quad (3)$$

It is known that the even entire function $J_\nu(z)z^{-\nu}$ has infinitely many zeros, all of which are real. Let

$$0 < j_{\nu,1} < j_{\nu,2} < \cdots \quad (4)$$

be all its positive zeros; then the Rayleigh sum is defined by [7]

$$\sigma_\nu^{(n)} = \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^{2n}}, \quad n \in \mathbb{N}. \quad (5)$$

Clearly [1],

$$\sigma_{1/2}^{(n)} = \sum_{n=1}^{\infty} \frac{1}{(n\pi)^{2k}} = \frac{(-1)^{n+1} 2^{2n-1} B_{2n}}{(2n)!}, \quad (6)$$

where the Bernoulli numbers B_n are defined by [1, 2, 12]

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}, \quad |z| < 2\pi. \quad (7)$$

The related numbers $\{S_n\}_{n=1}^{\infty}$ are defined by [2, 13]

$$S_n = 2 \left(\frac{2}{\pi} \right)^n \sum_{k=-\infty}^{\infty} \frac{1}{(4k+1)^n} \quad (8)$$

for $n = 2, 3, \dots$ and $S_1 = 1$; it is known that

$$\frac{(-1)^{n+1} B_{2n}}{(2n)!} = \frac{S_{2n}}{2^{2n} (2^{2n} - 1)}, \quad n \in \mathbb{N}. \quad (9)$$

3. Main Results

Theorem 1. Given a nonnegative integer n , one has

$$\begin{aligned} \det \left(\sigma_v^{(j+k+1)} \right)_{j,k=1}^n &= \frac{2^{(n+1)(2n+1)}}{((v+1)/2)_{n+1} \prod_{k=0}^n (v+1)_{2k}^2}, \\ \left\{ \left(\sigma_v^{(j+k+1)} \right)_{j,k=1}^n \right\}^{-1} &= \left(\sum_{m=0}^n \left((-1)^{j+k} (2m+v+1) (m+j)! (m+k)! \right. \right. \\ &\quad \times (v+1)_{m+j} (v+1)_{m+k} \\ &\quad \times \left. \left. \left(4^{j+k+1} (2j)! (2k)! (m-j)! (m-k)! \right. \right. \right. \\ &\quad \times \left. \left. \left. (v+1)_{m-j} (v+1)_{m-k} \right)^{-1} \right) \right)_{j,k=0}^n, \end{aligned} \quad (10)$$

for $v > -1$.

Corollary 2. For any nonnegative integer n , one has

$$\begin{aligned} \det \left(\frac{B_{2j+2k+2}}{(2j+2k+2)!} \right)_{j,k=1}^n &= \frac{1}{(3/4)_{n+1} \prod_{k=0}^n (3/2)_{2k}^2}, \\ \left\{ \left(\frac{B_{2(j+k+1)}}{(2j+2k+2)!} \right)_{j,k=0}^n \right\}^{-1} &= \left(\sum_{m=0}^n \left(\left(m + \frac{3}{4} \right) (m+j)! (m+k)! \left(\frac{3}{2} \right)_{m+j} \left(\frac{3}{2} \right)_{m+k} \right. \right. \\ &\quad \times \left((2j)! (2k)! (m-j)! (m-k)! \left(\frac{3}{2} \right)_{m-j} \right. \\ &\quad \times \left. \left. \left. \left(\frac{3}{2} \right)_{m-k} \right)^{-1} \right) \right)_{j,k=0}^n, \end{aligned} \quad (11)$$

or, equivalently,

$$\det \left(\frac{S_{2j+2k+2}}{4^{j+k+1} - 1} \right)_{j,k=0}^n = \frac{4^{(n+1)^2}}{(3/4)_{n+1} \prod_{k=0}^n (3/2)_{2k}^2}, \quad (12)$$

$$\begin{aligned} &\left\{ \left(\frac{S_{2j+2k+2}}{4^{j+k+1} - 1} \right)_{j,k=0}^n \right\}^{-1} \\ &= \left(\sum_{m=0}^n \left(\left(m + \frac{3}{4} \right) (m+j)! (m+k)! \left(\frac{3}{2} \right)_{m+j} \left(\frac{3}{2} \right)_{m+k} \right. \right. \\ &\quad \times \left((-4)^{j+k} (2j)! (2k)! (m-j)! (m-k)! \right. \\ &\quad \times \left. \left. \left. \left(\frac{3}{2} \right)_{m-j} \left(\frac{3}{2} \right)_{m-k} \right)^{-1} \right) \right)_{j,k=0}^n. \end{aligned} \quad (13)$$

4. Proofs

Given a probability measure $d\mu(x)$ on $\mathbb{R}$ such that $\int_{\mathbb{R}} x^{2n} d\mu(x) < \infty$ for all $n \in \mathbb{N}$, we define the inner product for $d\mu(x)$ square integrable functions $f(x)$ and $g(x)$ by

$$(f, g) = \int_{-\infty}^{\infty} f(x) g(x) d\mu(x). \quad (14)$$

For each $n \in \mathbb{N} \cup \{0\}$, let $G_n = (m_{j,k})_{j,k=0}^n$ with $m_{j,k} = (u_j, u_k)$ for $j, k = 0, 1, \dots, n$ where $\{u_k(x)\}_{k=0}^{\infty}$ is a sequence of polynomials with $u_0(x) = 1$ such that, for each n , $\{u_k(x)\}_{k=0}^n$ are linearly independent. Then there is a unique orthonormal system $\{p_k(x)\}_{k=0}^{\infty}$ [1, 10, 11]:

$$\begin{aligned} p_n(x) &= \frac{1}{\sqrt{\det G_n \det G_{n-1}}} \\ &\times \det \begin{pmatrix} m_{0,0} & m_{0,1} & m_{0,2} & \cdots & m_{0,n} \\ m_{1,0} & m_{1,1} & m_{1,2} & \cdots & m_{1,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m_{n-1,0} & m_{n-1,1} & m_{n-1,2} & \cdots & m_{n-1,n} \\ u_0(x) & u_1(x) & u_2(x) & \cdots & u_n(x) \end{pmatrix}, \end{aligned} \quad (15)$$

with positive leading coefficient in $u_n(x)$. Clearly we have $p_n(x) = \sum_{j=0}^n a_{n,j} u_j(x)$ for some real numbers $a_{j,k}$ for $j, k = 0, 1, \dots, n$ and $a_{j,k} = 0$ for $k > j$.

Lemma 3. For each nonnegative integer n , let $G_n = (m_{j,k})_{j,k=0}^n$ and $A_n = (a_{j,k})_{j,k=0}^n$. Then

$$\det G_n = \prod_{j=0}^n a_{j,j}^{-2}, \quad G_n^{-1} = A_n^T A_n. \quad (16)$$

Proof. From (15) and $p_n(x) = \sum_{j=0}^n a_{n,j} u_j(x)$ it is clear that

$$a_{n,n} = \sqrt{\frac{\det G_{n-1}}{\det G_n}}, \quad \det G_n = \prod_{j=0}^n a_{j,j}^{-2}. \quad (17)$$

For each n , since both $\{p_k(x)\}_{k=0}^n$ and $\{u_k(x)\}_{k=0}^n$ are a basis for the same set of polynomials, A_n must be invertible for each $n \in \mathbb{N} \cup \{0\}$. We denote $A_n^{-1} = (s_{j,k})_{j,k=0}^n$; then $u_j(x) = \sum_{\ell=0}^n s_{j,\ell} p_\ell(x)$ for $j = 0, 1, \dots, n$. Clearly, $s_{j,\ell} = 0$ for $\ell > j$. Thus,

$$m_{j,k} = (u_j(x), u_k(x)) = \sum_{m=0}^n s_{j,m} s_{k,m}, \quad (18)$$

for $j, k = 0, 1, \dots, n$, which is

$$G_n = A_n^{-1} (A_n^{-1})^T = A_n^{-1} (A_n^T)^{-1}, \quad (19)$$

and hence $G_n^{-1} = A_n^T A_n$. $\square$

4.1. Proof of Theorem 1. The normalized even order Lommel polynomials are defined by [11]

$$\begin{aligned} h_n(x) &= \frac{\sqrt{2n+\nu+1}}{2} h_{2n,\nu+1}(x) \\ &= \sum_{k=0}^n \frac{(-1)^{n-k} \sqrt{2n+\nu+1} (n+k)! (\nu+1)_{n+k}}{2^{2k+1} (2k)! (n-k)! (\nu+1)_{n-k}} (x^2)^k, \end{aligned} \quad (20)$$

for $n \in \mathbb{N}$ and $h_0(x) = \sqrt{\nu+1}/2$. They satisfy the orthogonal relation

$$\sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^2} h_m\left(\frac{1}{j_{\nu,k}}\right) h_n\left(\frac{1}{j_{\nu,k}}\right) = \delta_{m,n}. \quad (21)$$

For $n = 0, 1, \dots$, it is clear that the n th moment with respect to the measure of orthogonality is

$$m_n = \sum_{k=1}^{\infty} \frac{1}{j_{\nu,k}^{2+2n}} = \sigma_{\nu}^{(n+1)}. \quad (22)$$

Let $u_n(x) = x^{2n}$ for $n = 0, 1, \dots$; then

$$a_{n,k} = \frac{(-1)^{n-k} \sqrt{2n+\nu+1} (n+k)! (\nu+1)_{n+k}}{2^{2k+1} (2k)! (n-k)! (\nu+1)_{n-k}}. \quad (23)$$

By Lemma 3, the matrix $(\sigma_{\nu}^{(j+k+1)})_{j,k=0}^n$ has determinant

$$\prod_{k=0}^n \frac{2^{4k+2}}{(\nu+1+2k)(\nu+1)_{2k}^2} = \frac{2^{(n+1)(2n+1)}}{((\nu+1)/2)_{n+1} \prod_{k=0}^n (\nu+1)_{2k}^2} \quad (24)$$

and its inverse $(\gamma_{j,k})_{j,k=0}^n$ has elements

$$\begin{aligned} \gamma_{j,k} &= \sum_{m=0}^n \left((-1)^{j+k} (2m+\nu+1) (m+j)! (m+k)! \right. \\ &\quad \times (\nu+1)_{m+j} (\nu+1)_{m+k} \\ &\quad \times \left(4^{j+k+1} (2j)! (2k)! (m-j)! (m-k)! \right. \\ &\quad \times (\nu+1)_{m-j} (\nu+1)_{m-k} \left. \right)^{-1} \Big). \end{aligned} \quad (25)$$

4.2. Proof of Corollary 2. From (24), (25), and (6), we get

$$\begin{aligned} \det \left(\frac{(-1)^{j+k} 2^{2j+2k+1} B_{2(j+k+1)}}{(2j+2k+2)!} \right)_{j,k=0}^n &= \frac{2^{(n+1)(2n+1)}}{(3/4)_{n+1} \prod_{k=0}^n (3/2)_{2k}^2}, \\ &\left\{ \left(\frac{(-1)^{j+k} 2^{2j+2k+1} B_{2(j+k+1)}}{(2j+2k+2)!} \right)_{j,k=0}^n \right\}^{-1} \\ &= \left(\sum_{m=0}^n \left((-1)^{j+k} \left(2m + \frac{3}{2} \right) (m+j)! (m+k)! \right. \right. \\ &\quad \times \left(\frac{3}{2} \right)_{m+j} \left(\frac{3}{2} \right)_{m+k} \\ &\quad \times \left(4^{j+k+1} (2j)! (2k)! (m-j)! (m-k)! \left(\frac{3}{2} \right)_{m-j} \right. \\ &\quad \times \left. \left. \left. \left(\frac{3}{2} \right)_{m-k} \right)^{-1} \right) \right)_{j,k=0}^n \right)^{-1}. \end{aligned} \quad (26)$$

They are simplified to

$$\begin{aligned} \det \left(\frac{B_{2(j+k+1)}}{(2j+2k+2)!} \right) &= \frac{1}{(3/4)_{n+1} \prod_{k=0}^n (3/2)_{2k}^2}, \\ &\left\{ \left(\frac{B_{2(j+k+1)}}{(2j+2k+2)!} \right)_{j,k=0}^n \right\}^{-1} \\ &= \left(\sum_{m=0}^n \left(\left(m + \frac{3}{4} \right) (m+j)! (m+k)! \left(\frac{3}{2} \right)_{m+j} \left(\frac{3}{2} \right)_{m+k} \right. \right. \\ &\quad \times \left((2j)! (2k)! (m-j)! (m-k)! \left(\frac{3}{2} \right)_{m-j} \right. \\ &\quad \times \left. \left. \left. \left(\frac{3}{2} \right)_{m-k} \right)^{-1} \right) \right)_{j,k=0}^n \right)^{-1}. \end{aligned} \quad (27)$$

By (9) we get

$$\begin{aligned}
 & \det \left(\frac{S_{2j+2k+2}}{2(4^{j+k+1} - 1)} \right)_{j,k=0}^n = \frac{2^{(n+1)(2n+1)}}{(3/4)_{n+1} \prod_{k=0}^n (3/2)_{2k}^2}, \\
 & \left\{ \left(\frac{S_{2j+2k+2}}{2(4^{j+k+1} - 1)} \right)_{j,k=0}^n \right\}^{-1} \\
 & = \left(\sum_{m=0}^n \left((-1)^{j+k} \left(2m + \frac{3}{2} \right) (m+j)! (m+k)! \left(\frac{3}{2} \right)_{m+j} \right. \right. \\
 & \quad \times \left(\frac{3}{2} \right)_{m+k} \\
 & \quad \times \left(4^{j+k+1} (2j)! (2k)! (m-j)! (m-k)! \left(\frac{3}{2} \right)_{m-j} \right. \\
 & \quad \times \left. \left. \left. \left(\frac{3}{2} \right)_{m-k} \right)^{-1} \right) \right)_{j,k=0}^n, \tag{28}
 \end{aligned}$$

which are simplified to (12) and (13), respectively.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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