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Research Article

Indefinite Eigenvalue Problems for p-Laplacian Operators with Potential Terms on Networks

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We address some forward and inverse problems involving indefinite eigenvalues for discrete p-Laplacian operators with potential terms. These indefinite eigenvalues are the discrete analogues of p-Laplacians on Riemannian manifolds with potential terms. We first define and discuss some fundamental properties of the indefinite eigenvalue problems for discrete p-Laplacian operators with potential terms with respect to some given weight functions. We then discuss resonance problems, anti-minimum principles, and inverse conductivity problems for the discrete p-Laplacian operators with potential terms involving the smallest indefinite eigenvalues.

1. Introduction

In this paper, we study a generalized version of spectral theory, resonance problems, antiminimum principles, and inverse problems for discrete p-Laplacian operators with potential terms on a network. We define a network as a way of interconnecting any pair of users or nodes by means of some meaningful links. Therefore, we represent a network by a weighted graph $G = G(S \bigcup \partial S, E, \omega)$ with a *weight function*.

The main goal of this paper is to characterize the indefinite eigenvalues and to solve the inverse conductivity problems for the equations

$$\begin{aligned} -\Delta_{p,\omega}\phi(x) + V(x)\left|\phi(x)\right|^{p-2}\phi(x) \\ = \lambda_h h(x)\left|\phi(x)\right|^{p-2}\phi(x), \quad x \in S \\ \phi(z) = 0, \quad z \in \partial S, \end{aligned} \tag{1}$$

where V and h are real valued functions on a network S with boundary ∂S. Here, $\Delta_{p,\omega}$ is the discrete p-Laplacian on a network S with weight ω defined by

$$\begin{aligned} &\Delta_{p,\omega}u(x) \\ &\quad := \sum_{y\in\bar{S}}\left|u(y) - u(x)\right|^{p-2}(u(y) - u(x))\,\omega(x,y), \quad x \in S \end{aligned} \tag{2}$$

for $1 < p < \infty$. To address these problems, many researchers have especially concentrated on spectral graph theory which has been one of the most significant tools used in studying graphs. This has led to noteworthy progress in the study of these questions (see, e.g., [1, 2]). In this paper, we are primarily concerned with *indefinite eigenvalue problems*.

In particular, we deal with these problems under the assumptions that h is positive and that h has both positive and negative values. For each case, we present properties for the smallest indefinite eigenvalue $\lambda_{h,0}$ as follows:

(i) the variationally expressed form of $\lambda_{h,0}$,

(ii) the positivity of eigenfunctions corresponding to $\lambda_{h,0}$,

(iii) the multiplicity of $\lambda_{h,0}$.

Moreover, we also show that $\lambda_{h,0}$ is isolated. Using these properties, we then discuss resonance problems, antiminimum principles, and the inverse conductivity problems. Note that the uniqueness of the conductivity ω is not guaranteed from $\lambda_{h,0}$. This implies that there can be different conductivities ω_1 and ω_2 on edges such that the smallest indefinite eigenvalues of networks for ω_1 and ω_2 are the same. Therefore, to guarantee the uniqueness of the conductivity, we impose the additional constraint, the *monotonicity condition*, on conductivity of the edges. The result for the case that h is

positive is Theorem 10 and the results for the other case of h are Theorems 18 and 19.

Recently, in order to expand the results on spectral graph theory with respect to the above viewpoint, great efforts have been concentrated on studying the properties of graphs involving eigenvalues of operators such as discrete Schrödinger or discrete p-Laplacian operators (see, e.g., [3–8]) which are generalizations of the discrete Laplacian. In [9], in particular, Amghibech introduces the *indefinite eigenvalue problem* for the case where $V \equiv 0$ and $h > 0$ in (1) and gives some characterizations of the smallest indefinite eigenvalue. The author also addresses a resonance problem, an antiminimum principle, and an inverse problem.

This paper is organized as follows. In Section 2, we recall some basic terminology and properties of networks. In Section 3, for the case that h is positive, we give some characterizations of the smallest positive indefinite eigenvalue, and we study the resonance problems, the antiminimum principles, and the inverse conductivity problems. Finally, in Section 4, we discuss the same problems discussed in Section 3 under the assumption that h has both positive and negative values.

2. Preliminaries

In this section, we describe the theoretic graph notations frequently used throughout this paper.

By a *graph* $G = G(\mathscr{V}, E)$ we refer to a finite set $\mathscr{V}$ of *vertices* with a set E of two-element subsets of $\mathscr{V}$ whose elements are called *edges*.

For notational convenience, we denote by $x \in G$ the fact that x is a vertex in $G(\mathscr{V}, E)$. A graph $G' = G'(\mathscr{V}', E')$ is said to be a *subgraph* of $G(\mathscr{V}, E)$ if $V' \subset V$ and $E' \subset E$. If E' consists of all the edges from E which connect the vertices of $\mathscr{V}'$ in G, then G' is called an *induced* subgraph. Throughout this paper, we assume that the graph $G(\mathscr{V}, E)$ is finite, simple, and connected.

A *weight* on a graph $G(\mathscr{V}, E)$ is a function $\omega : \mathscr{V} \times \mathscr{V} \to [0, \infty)$ satisfying

(i) $\omega(x, x) = 0, x \in \mathscr{V}$,

(ii) $\omega(x, y) = \omega(y, x) > 0$ if $\{x, y\} \in E$,

(iii) $\omega(x, y) = 0$ if and only if $\{x, y\} \notin E$,

and a graph $G(\mathscr{V}, E)$ with a weight ω is called a network $G(\mathscr{V}, E, \omega)$. The integration of a function $u : \mathscr{V} \to \mathbb{R}$ is defined by

$$\int_{\mathscr{V}} u := \sum_{x \in \mathscr{V}} u(x). \tag{3}$$

For an induced subgraph S of $G(V, E)$, by $\overline{S} := S \dot{\cup} \partial S$ we denote a graph whose vertices and edges are in S and vertices in $\partial S := \{y \in V \setminus S \mid \omega(x, y) > 0 \text{ for some } x \in S\}$. Here, S and ∂S are called *interiors* and *boundaries*, respectively.

The *p-gradient* $\nabla_{p,\omega}$ of a function $u : \overline{S} \to \mathbb{R}$ is defined as

$$\nabla_{p,\omega} u(x) := \left(D_{p,\omega,y} u(x)\right)_{y \in \overline{S}} \tag{4}$$

for $x \in \overline{S}$. In the case of $p = 2$, we write simply ∇_ω instead of $\nabla_{2,\omega}$.

It has been known that for any pair of functions $u : \overline{S} \to \mathbb{R}$ and $v : \overline{S} \to \mathbb{R}$, we have

$$2 \int_{\overline{S}} v \left(-\Delta_{p,\omega} u\right) = \int_{\overline{S}} \nabla_\omega v \cdot \nabla_{p,\omega} u, \tag{5}$$

where $\mathbb{A} \cdot \mathbb{B} := \sum_{i=1}^{n} a_i b_i$ for $\mathbb{A} = (a_1, \ldots, a_n)$ and $\mathbb{B} = (b_1, \ldots, b_n)$. This fact yields many useful formulas such as the network version of the Green theorem (for details, see [7]).

For the given functions V and $h : S \to \mathbb{R}$, if $\lambda_h \in \mathbb{R}$ and $\phi : \overline{S} \to \mathbb{R}$ satisfy (1), then λ_h is called the *(Dirichlet) indefinite eigenvalue* for $-\mathscr{L}^V_{p,\omega}$ where $\mathscr{L}^V_{p,\omega} u := \Delta_{p,\omega} u - V|u|^{p-2} u$ and ϕ is called an eigenfunction corresponding to λ_h. Moreover, (λ_h, ϕ) is called an *indefinite eigenpair*.

Finally, we recall some known results on discrete p-Laplacian operators such as the minimum principle and Picone's identity.

Theorem 1 (see [6] minimum principle for $-\Delta_{p,\omega}$ on networks). *Let $u : \overline{S} \to \mathbb{R}$ satisfy the differential inequality $-\Delta_{p,\omega} u(x) \geq 0$ for all $x \in S$. If u attains the minimum at a point in S, then u is constant in $\overline{S}$.*

Theorem 2 (see [9] Picone's identity for $\Delta_{p,\omega}$ on networks). *For an induced subnetwork S of a given weighted network G, let two functions u_1 and u_2 be nonnegative and positive on S, respectively. Then*

$$\left(\nabla_{S,\omega} u_1 \cdot \nabla_{S,p,\omega} u_1 - \nabla_{S,\omega}\left(\frac{u_1^p}{u_2^{p-1}}\right) \cdot \nabla_{S,p,\omega} u_2\right)(x) \geq 0 \tag{6}$$

for all x in S. Moreover, if the induced subnetwork S is connected, then the equality holds if and only if there exists $t > 0$ such that $u(x) = tv(x)$ for all x in S.

3. Indefinite Eigenvalue Problems with Positive Weight Functions

In [9], Amghibech introduces the indefinite eigenvalue problems for $-\Delta_{p,\omega}$ on networks with standard weights. In this paper, we study the indefinite eigenvalue problems under more complicated situations than those of Amghibech. More specifically, we look at the p-Laplacian operator combined with *potential terms* and moreover, we do not impose any restrictions on the weight of the networks, further differentiating this paper from [9].

We now start this section under the assumption that h is positive.

3.1. The Smallest Indefinite Eigenvalue. In this subsection, we prove the existence of the smallest indefinite eigenvalue $\lambda_{h,0}$ for $-\mathscr{L}^V_{p,\omega}$ when h is positive. We also address some fundamental problems such as the multiplicity of $\lambda_{h,0}$ and its isolation.

It will be shown in the next theorem that $\lambda_{h,0}$ exists and can be variationally expressed as

$$\lambda_{h,0} = \inf_{\substack{\phi\in\mathscr{A}\\ \phi\not\equiv 0}} \frac{(1/2)\int_{\overline{S}} \nabla_\omega\phi\cdot\nabla_{p,\omega}\phi + \int_S V|\phi|^p}{\int_S h|\phi|^p}, \tag{7}$$

where

$$\mathscr{A} := \left\{\phi\in\overline{S}\,|\phi|_{\partial S} = 0\right\}. \tag{8}$$

Theorem 3. *There exists a nonzero function $\phi_0\in\mathscr{A}$ such that*

$$\lambda_{h,0} = \frac{(1/2)\int_{\overline{S}} \nabla_\omega\phi_0\cdot\nabla_{p,\omega}\phi_0 + \int_S V|\phi_0|^p}{\int_S h|\phi_0|^p}. \tag{9}$$

Moreover, $\lambda_{h,0}$ is the smallest eigenvalue for $-\mathscr{L}^V_{p,\omega}$ and ϕ_0 is an eigenfunction corresponding to $\lambda_{h,0}$.

Proof. Note that

$$\begin{aligned}
&\inf_{\substack{\phi\in\mathscr{A}\\ \phi\not\equiv 0}} \frac{(1/2)\int_{\overline{S}} \nabla_\omega\phi\cdot\nabla_{p,\omega}\phi + \int_S Vh|\phi|^p}{\int_S h|\phi|^p}\\
&\quad= \inf_{\substack{\phi\in\mathscr{A}\\ \phi\not\equiv 0}} \frac{1}{2}\int_{\overline{S}} \nabla_\omega\left(\frac{\phi}{\int_S h|\phi|^p}\right)\cdot\nabla_{p,\omega}\left(\frac{\phi}{\int_S h|\phi|^p}\right)\\
&\qquad+\int_S Vh\left|\frac{\phi}{\int_S h|\phi|^p}\right|^p\\
&\quad= \inf_{\phi\in S_1}\left(\int_{\overline{S}}\frac{1}{2}\nabla_\omega\phi\cdot\nabla_{p,\omega}\phi + \int_S V|\phi|^p\right),
\end{aligned} \tag{10}$$

where $\mathscr{S}_1 := \{\phi\in\mathscr{A}|\int_{\overline{S}} h|\phi|^p = 1\}$. Here, we note that S_1 is closed and bounded (i.e., compact), since it is a subset of vectors in $\mathbb{R}^n$, for $n = |\overline{S}|$, and since h is positive. Therefore, there exists $\phi_0\in\mathscr{S}_1$ such that

$$\begin{aligned}
&\left(\frac{1}{2}\right)\int_{\overline{S}} \nabla_\omega\phi_0\cdot\nabla_{p,\omega}\phi_0 + \int_S V|\phi_0|^p\\
&\quad= \min_{\phi\in\mathscr{S}_1}\left(\frac{1}{2}\int_{\overline{S}} \nabla_\omega\phi\cdot\nabla_{p,\omega}\phi + \int_S V|\phi|^p\right).
\end{aligned} \tag{11}$$

Since it is easily seen from (1) and (5) that $\lambda_{h,0}\le\lambda_h$ for each eigenvalues λ_h, it suffices to show that $(\lambda_{h,0},\phi_0)$ is an eigenpair. For any $x\in S$, we define a function $\delta_x:\overline{S}\to\mathbb{R}$ as

$$\delta_x(y) = \begin{cases}1, & x = y,\\ 0, & \text{otherwise.}\end{cases} \tag{12}$$

Taking an arbitrary $x_0\in S$, we have

$$\int_{\overline{S}}\left|\phi_1 + t\delta_{x_0}\right|^p \neq 0 \tag{13}$$

for a sufficiently small t and

$$\begin{aligned}
\lambda_{h,0} \le \Bigg(&\frac{1}{2}\int_{\overline{S}} \nabla_\omega\left(\phi_0 + t\delta_{x_0}\right)\cdot\nabla_{p,\omega}\left(\phi_0 + t\delta_{x_0}\right)\\
&+\int_S V\left|\left(\phi_0 + t\delta_{x_0}\right)\right|^p\Bigg)\\
&\times\left(\int_S h\left|\left(\phi_0 + t\delta_{x_0}\right)\right|^p\right)^{-1}.
\end{aligned} \tag{14}$$

Hence, we have

$$\begin{aligned}
0 \le &\frac{1}{2}\int_{\overline{S}} \nabla_\omega\left(\phi_0 + t\delta_{x_0}\right)\cdot\nabla_{p,\omega}\left(\phi_0 + t\delta_{x_0}\right)\\
&+\int_S\left(V - \lambda_{h,0}h\right)\left|\left(\phi_0 + t\delta_{x_0}\right)\right|^p
\end{aligned} \tag{15}$$

for a sufficiently small t. Note that the right-hand side is continuously differentiable with respect to t and equals zero at $t = 0$. Thus, we have

$$\begin{aligned}
0 = &\frac{d}{dt}\Bigg[\frac{1}{2}\int_{\overline{S}} \nabla_\omega\left(\phi_0 + t\delta_{x_0}\right)\cdot\nabla_{p,\omega}\left(\phi_0 + t\delta_{x_0}\right)\\
&+\int_S\left(V - \lambda_{h,0}h\right)\left|\left(\phi_0 + t\delta_{x_0}\right)\right|^p\Bigg]_{t=0}\\
= &\, p\left(-\Delta_{p,\omega}\phi_1(x_0) + \left(V(x_0) - \lambda_{h,0}h\right)|\phi_0(x_0)|^{p-2}\right.\\
&\left.\times\phi_0(x_0)\right)d_\omega x_0.
\end{aligned} \tag{16}$$

Since x_0 is chosen arbitrary in S, we have

$$\begin{aligned}
&-\Delta_{p,\omega}\phi_0(x) + V(x)|\phi_0(x)|^{p-2}\phi_0(x)\\
&\quad= \lambda_{h,0}h|\phi_0(x)|^{p-2}\phi_0(x), \quad x\in S,
\end{aligned} \tag{17}$$

which completes the proof. □

We now prove the simplicity of $\lambda_{h,0}$. To achieve this goal, we first prove a theorem which asserts that there always exists an eigenfunction ϕ_0 corresponding to $\lambda_{h,0}$ which is positive in S.

Theorem 4. *There exists $\phi_0\in\mathscr{A}$ with $\phi_0 > 0$ in S such that $(\lambda_{h,0},\phi_0)$ is an indefinite eigenpair for $-\mathscr{L}^V_{p,\omega}$.*

Proof. It follows from Theorem 3 that there exists an eigenfunction ϕ_0 corresponding to $\lambda_{h,0}$ satisfying

$$\begin{aligned}
&-\Delta_{p,\omega}\phi_0(x) + V(x)|\phi_0(x)|^{p-2}\phi_0(x)\\
&\quad= \lambda_{h,0}h(x)|\phi_0(x)|^{p-2}\phi_0(x), \quad x\in S.
\end{aligned} \tag{18}$$

Let $\psi(x) := |\phi_0(x)|$, $x\in\overline{S}$. Then

$$\begin{aligned}
&\int_{\overline{S}}|\psi|^p = \int_{\overline{S}}|\phi_0|^p,\\
&\frac{1}{2}\int_{\overline{S}} \nabla_\omega\phi_0\cdot\nabla_{p,\omega}\phi_0 + \int_S V|\phi_0|^p\\
&\quad\ge \frac{1}{2}\int_{\overline{S}} \nabla_\omega\psi\cdot\nabla_{p,\omega}\psi + \int_S V|\psi|^p.
\end{aligned} \tag{19}$$

Thus, we have

$$\lambda_{h,0} = \frac{(1/2)\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 + \int_S V|\phi_0|^p}{\int_S h|\phi_0|^p} \geq \frac{(1/2)\int_S \nabla_\omega \psi \cdot \nabla_{p,\omega}\psi + \int_S V|\psi|^p}{\int_S h|\psi|^p}. \tag{20}$$

Otherwise, by the definition of $\lambda_{h,0}$,

$$\lambda_{h,0} \leq \frac{(1/2)\int_{\overline{S}} \nabla_\omega \psi \cdot \nabla_{p,\omega}\psi + \int_S V|\psi|^p}{\int_S h|\psi|^p}. \tag{21}$$

Thus,

$$\lambda_{h,0} = \frac{(1/2)\int_{\overline{S}} \nabla_\omega \psi \cdot \nabla_{p,\omega}\psi + \int_S V|\psi|^p}{\int_{\overline{S}} h|\psi|^p}. \tag{22}$$

It follows from Theorem 3 that $(\lambda_{h,0}, \psi)$ is an indefinite eigenpair. Now it suffices to show that $\psi > 0$ in $\overline{S}$. Suppose, to the contrary, that $\psi(x_0) = 0$ for some $x_0 \in S$. It will be shown that $\psi \equiv 0$. Since ψ is an eigenvalue, it follows from (1) that

$$\sum_{y\in S} |\psi(y)|^{p-2}\psi(y)\,\omega(x_0, y) = 0 \tag{23}$$

and thus $\psi(y) = 0$ for all $y \sim x_0$ where $y \sim x$ means that two vertices x and y are connected by an edge. By repeating the above process for $y \sim x_0$, we conclude that $\psi(z) = 0$ for each $z \sim y$. Since the network $\overline{S}$ is assumed to be connected, $\psi(x) = 0$ for all $x \in \overline{S}$. □

Using the above theorem, we prove the simplicity of $\lambda_{h,0}$ as follows.

Theorem 5. *If $(\lambda_{h,0}, \phi_0)$ is an indefinite eigenpair for $-\mathscr{L}^V_{p,\omega}$, then*

$$\operatorname{sgn}\phi_0(x) = \operatorname{sgn}\phi_0(y), \quad x, y \in S. \tag{24}$$

Proof. As shown in the proof for Theorem 4, if $(\lambda_{h,0}, \phi_0)$ is indefinite, then $(\lambda_{h,0}, |\phi_0|)$ is also an indefinite eigenpair. Let $\overline{\phi_0}(x) := |\phi_0(x)|$ for all x in $\overline{S}$. Then we have

$$\frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 + \int_S V|\phi_0|^p = \frac{1}{2}\int_{\overline{S}} \nabla_\omega \overline{\phi}_0 \cdot \nabla_{p,\omega}\overline{\phi}_0 + \int_S V|\overline{\phi}_0|^p, \tag{25}$$

which implies that

$$\sum_{x,y\in S} |\phi_0(y) - \phi_0(x)|^p w(x, y) = \sum_{x,y\in S} \left|\overline{\phi_0}(y) - \overline{\phi_0}(x)\right|^p w(x, y). \tag{26}$$

Since

$$|\phi_0(y) - \phi_0(x)| \geq \left|\overline{\phi_0}(y) - \overline{\phi_0}(x)\right| \tag{27}$$

for all $x \sim y$ in S, we have

$$|\phi_0(y) - \phi_0(x)| = \left|\overline{\phi_0}(y) - \overline{\phi_0}(x)\right| \tag{28}$$

for all x, y in S. Hence, either $\phi_0(x) = \overline{\phi_0}(x)$ or $\phi_0(x) = -\overline{\phi_0}(x)$ for all x in S. □

The above theorem shows that the dimension of the eigenspace corresponding to $\lambda_{h,0}$ is one. Thus, we have the following.

Corollary 6. *The multiplicity of $\lambda_{h,0}$ is one.*

For linear operators such as $-\Delta_\omega$ on finite networks, it is clear that the number of eigenvalues (including multiplicity) is the same as the number of vertices. However, when we consider nonlinear operators such as $-\mathscr{L}^V_{p,\omega}$, it becomes significantly more complicated to count the number of eigenvalues. It is not sufficient to simply prove whether the number of eigenvalues is finite of infinite. However, by applying Picone's identity, it is possible to show that the smallest indefinite eigenvalue $\lambda_{h,0}$ is isolated for a set of indefinite eigenvalues.

Theorem 7. *The smallest eigenvalue $\lambda_{h,0}$ is isolated.*

Proof. We proceed by contradiction. Suppose that for each $\epsilon > 0$, there exists u_ϵ satisfying $\int_S |u_\epsilon|^p = 1$ and

$$\begin{aligned} -\Delta_{p,\omega}u_\epsilon + V|u_\epsilon|^{p-2}u_\epsilon &= (\lambda_h + \epsilon)\,h|u_\epsilon|^{p-2}u_\epsilon, \quad \text{in } S \\ u_\epsilon &= 0, \quad \text{on } \partial S. \end{aligned} \tag{29}$$

Since the multiplicity of $\lambda_{h,0}$ is one, there exists an eigenfunction ϕ_0 corresponding to $\lambda_{h,0}$ with ϕ_0 in S such that $u_\epsilon \to \phi_0 > 0$ in S as $\epsilon \to 0$. Hence, for sufficiently small $\epsilon > 0$ we have $u_\epsilon > 0$ in S. Since

$$V = \frac{\Delta_p u\epsilon}{u_\epsilon^{p-1}} + (\lambda_{h,0} + \epsilon)\,h \quad \text{in } S, \tag{30}$$

we have

$$\begin{aligned} -\Delta_p\phi_0 + \frac{\phi_0^{p-1}}{u_\epsilon^{p-1}}\Delta_p u_\epsilon + \lambda_{h,0}h\phi_0^{p-1} + \epsilon h\phi_0^{p-1} \\ = \lambda_{h,0}h\phi_0^{p-1} \quad \text{on } \overline{S}. \end{aligned} \tag{31}$$

That is,

$$-\epsilon\phi_0^{p-1} = -\Delta_{p,\omega}\phi_0 - \frac{\phi_0^{p-1}}{u_\epsilon^{p-1}}\left(-\Delta_{p,\omega}u_\epsilon\right) \quad \text{on } \overline{S}. \tag{32}$$

Multiplying ϕ_0 and integrating over $\overline{S}$ on both sides (32) and using Picone's identity, we have a contradiction. □

3.2. Resonance Problems, Antiminimum Principle, and Inverse Problems. In this subsection, we deal with some interesting problems such as the resonance problems, the antiminimum principles, and the inverse conductivity problems with regard

to indefinite eigenvalues. We remind the reader that during this section, we assume the weight function h is a positive valued function.

For a given function $V : S \to \mathbb{R}$ and a nonnegative source term $g : S \to [0, \infty)$, we consider the following equation:

$$\begin{aligned} -\Delta_{p,\omega} u + V|u|^{p-2}u - \lambda_{h,0} h|u|^{p-2}u &= g \quad \text{in } S, \\ u &= 0 \quad \text{on } \partial S. \end{aligned} \tag{33}$$

It is clear that the above equation has a solution (in fact, an eigenvalue) if $g \equiv 0$ in S. The next result shows that if $\lambda_{h,0} > 0$, then the converse of the statement also holds. Thus, there is no solution of the above equation if g is nonzero in S.

Theorem 8 (resonance problem). *Suppose that a function V satisfies the condition that the smallest indefinite eigenvalue $\lambda_{h,0}$ is positive. Then* (33) *has a solution if and only if $g \equiv 0$.*

Proof. Suppose that a function u_0 is a solution to the equation and we define a function $\overline{u_0}$ as

$$\overline{u_0}(x) := \max\{-u_0(x), 0\}, \quad x \in \overline{S}. \tag{34}$$

Since it is obvious that if $u_0 \equiv 0$, then $g \equiv 0$; we assume that $u_0 \not\equiv 0$. Then we have

$$\begin{aligned} 0 \le \int_S g(\overline{u_0}) &= \int_{\overline{S}} \left(-\Delta_p u_0 + V|u_0|^{p-2}u_0 - \lambda_{h,0} h|u_0|^{p-2}u_0\right)\overline{u_0} \\ &\le -\frac{1}{2}\sum_{x,y\in\overline{S}} \left|(\overline{u_0}(y)) - (\overline{u_0}(x))\right|^p \\ &\quad - \sum_{x\in S} v(x)\left|\overline{u_0}(x)\right|^p \\ &\quad + \lambda_{h,0}\sum_{x\in S} h(x)\left|\overline{u_0}(x)\right|^p, \end{aligned} \tag{35}$$

which implies that $\overline{u_0} \equiv k\phi_0$ for some $k \ge 0$. If $k > 0$ then u_0 is an eigenfunction corresponding to $\lambda_{h,0}$ so that $g \equiv 0$. Now suppose that $k = 0$ so $u_0 \ge 0$. Since $u_0 \not\equiv 0$ and $g \ge 0$, we have

$$\begin{aligned} &-\Delta_{p,\omega} u_0 + v|u_0|^{p-2}u_0 \ge 0, \quad \text{but} \\ &-\Delta_{p,\omega} u_0 + v|u_0|^{p-2}u_0 \not\equiv 0 \\ &\qquad \text{in } S. \end{aligned} \tag{36}$$

Thus, by using a similar method that we used in the proof for Theorem 4, it is easy to we show that the solution u_0 is positive in S. Using Picone's identity, we have

$$\begin{aligned} 0 &\le \frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 - \nabla_\omega\left(\frac{\phi_0^p}{u_0^{p-1}}\right)\cdot \nabla_{p,\omega} u_0 \\ &= \frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 - \int_S \frac{\phi_0^p}{u_0^{p-1}}\left(-vu_0^{p-1} + \lambda_0 h u_0^{p-1} + g\right) \\ &= \frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 + \int_S v\phi_0^p - \lambda_0\int_S h\phi_0^p - \int_S \frac{\phi_0^p}{u_0^{p-1}} g \end{aligned} \tag{37}$$

which implies that $g \equiv 0$. □

The next theorem is the antiminimum principle. From it, we see that each (nonconstant) solution for the following equation

$$\begin{aligned} -\Delta_{p,\omega} u + V|u|^{p-2}u - \lambda h|u|^{p-2}u &\ge 0 \quad \text{in } S \\ u &= 0 \quad \text{on } \partial S. \end{aligned} \tag{38}$$

has its minimum in S if $\lambda > \lambda_{h,0}$.

Theorem 9 (antiminimum principle). *For a nonnegative source term $g : S \to [0, \infty)$, suppose u_λ is a solution to the following equation:*

$$\begin{aligned} -\Delta_{p,\omega} u + V|u|^{p-2}u - \lambda h|u|^{p-2}u &= g \quad \text{in } S \\ u &= 0 \quad \text{on } \partial S. \end{aligned} \tag{39}$$

If $\lambda > \lambda_{h,0}$, then $u_\lambda(x_0) < 0$ for some $x_0 \in S$.

Proof. By virtue of Theorem 8, it suffices to show that if there exist a nonnegative solution u_λ for (75), then $\lambda < \lambda_{h,0}$. Suppose u_λ is a solution to (75) with $u_\lambda(x) \ge 0$, $x \in S$. Using a similar method that we used in the proof of Theorem 4, we can easily show that if $u_\lambda(x_0) = 0$ for some $x_0 \in S$, then $u_\lambda \equiv 0$. Thus, we may assume that u_λ is positive in S. By Picone's identity, we have

$$\begin{aligned} 0 &\le \frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 - \nabla_\omega\left(\frac{\phi_0^p}{u_\lambda^{p-1}}\right)\cdot \nabla_{p,\omega} u_\lambda \\ &= \frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 - \frac{\phi_0^p}{u_\lambda^{p-1}}\left(-\Delta_{p,\omega} u_\lambda\right) \\ &= \frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 - \frac{\phi^p}{u_\lambda^{p-1}}\left(-Vu_\lambda^{p-1} + \lambda h u_\lambda^{p-1} + g\right), \end{aligned} \tag{40}$$

where ϕ_0 is the positive eigenfunction corresponding to $\lambda_{h,0}$. Thus, we have

$$0 < (\lambda_{h,0} - \lambda)\int_S h\phi_0^p. \tag{41}$$

Since $\int_S h\phi_0^p > 0$, we finally have $\lambda_{h,0} > \lambda$, which completes the proof. □

We now discuss an inverse conductivity problem on networks. The main concern is related to the problem of recovering the conductivity (weight) ω of the network by the smallest indefinite eigenvalue $\lambda_{h,0}$ for $-\mathscr{L}_{p,\omega}$ with respect to h. Note that the uniqueness of the conductivity ω is not guaranteed by $\lambda_{h,0}$. This implies that there can be different conductivities

ω_1 and ω_2 on the edges which induces the same eigenvalue $\lambda_{h,0}$ for the operators $-\mathscr{L}_{p,\omega_i}, i = 1,2$. To guarantee the uniqueness of the conductivity, we need to impose some more assumption on the structure of network or on the conductivity. We impose here the additional constraint, called the *monotonicity condition*, on the conductivity of the edges. The main result of this section shows that there are no different conductivities ω_1 and ω_2 on the edges satisfying $\omega_1 \le \omega_2$ in $\overline{S}\times\overline{S}$ which induce the same smallest indefinite eigenvalue $\lambda_{h,0}$.

Theorem 10 (inverse conductivity problem). *For networks $G(\overline{S}, E_i, \omega_i)$ for $i = 1,2$, let $\lambda_{h,0}^{\omega_i}$ be the smallest indefinite eigenvalue for $-\mathscr{L}_{p,\omega_i}^{V}$. If the weight functions satisfy*

$$\omega_1 \le \omega_2 \quad \text{in } \overline{S}\times\overline{S}, \tag{42}$$

then one has

$$\lambda_{h,0}^{\omega_1} \le \lambda_{h,0}^{\omega_2}. \tag{43}$$

Moreover, $\lambda_{h,0}^{\omega_1} = \lambda_{h,0}^{\omega_2}$ if and only if one has

(i) $\phi_1 = \phi_2$ *on* $\overline{S}$,

(ii) $\omega_1(x,y) = \omega_2(x,y)$ *whenever* $\phi_1(x) \ne \phi_1(y)$ *or* $\phi_2(x) \ne \phi_2(y)$

where ϕ_i is the eigenfunction corresponding to $\lambda_{h,0}^{\omega_i}$, $i = 1,2$.

Proof. By definition of the smallest eigenvalue, we have

$$\lambda_{0,h}^{\omega_1} \le \frac{(1/2)\int_{\overline{S}} \nabla_{p,w_1}\phi_2 \cdot \nabla_{w_1}\phi_2 + \int_S v|\phi_2|^p}{\int_S h|\phi_2|^p}. \tag{44}$$

It follows from $\omega_1 \le \omega_2$ that

$$\begin{aligned} &\frac{(1/2)\int_{\overline{S}} \nabla_{p,w_1}\phi_2 \cdot \nabla_{w_1}\phi_2 + \int_S v|\phi_2|^p}{\int_S h|\phi_2|^p} \\ &\quad \le \frac{(1/2)\int_{\overline{S}} \nabla_{p,w_2}\phi_2 \cdot \nabla_{w_2}\phi_2 + \int_S v|\phi_2|^p}{\int_S h|\phi_2|^p}. \end{aligned} \tag{45}$$

Hence we have $\lambda_{h,0}^{\omega_1} \le \lambda_{h,0}^{\omega_2}$. Now, we suppose that $\lambda_{h,0}^{\omega_1} = \lambda_{h,0}^{\omega_2}$. Then

$$\begin{aligned} \lambda_{h,0}^{\omega_2} &= \frac{(1/2)\int_{\overline{S}} \nabla_{p,w_2}\phi_2 \cdot \nabla_{w_2}\phi_2 + \int_S V|\phi_2|^p}{\int_S h|\phi_2|^p} \\ &\ge \frac{(1/2)\sum_{x,y\in\overline{S}} |\phi_2(y) - \phi_2(x)|^p (\omega_2(x,y) - \omega_1(x,y))}{\int_S h|\phi_2|^p} \\ &\quad + \lambda_{h,0}^{\omega_1}. \end{aligned} \tag{46}$$

Since $\lambda_{h,0}^{\omega_1} = \lambda_{h,0}^{\omega_2}$, we have

$$0 \ge \frac{(1/2)\sum_{x,y\in\overline{S}} |\phi_2(y) - \phi_2(x)|^p (\omega_2(x,y) - \omega_1(x,y))}{\int_S h|\phi_2|^p}. \tag{47}$$

Thus $\omega_1(x,y) = \omega_2(x,y)$ whenever $\phi_2(x) \ne \phi_2(y)$, which implies that

$$\frac{(1/2)\int_{\overline{S}} \nabla_{p,w_1}\phi_2 \cdot \nabla_{w_1}\phi_2 + \int_S V|\phi_2|^p}{\int_S h|\phi_2|^p} \le \lambda_{h,0}^{\omega_1}. \tag{48}$$

Thus $\phi_1 \equiv \phi_2$. Hence $\omega_1(x,y) = \omega_2(x,y)$ whenever $\phi_i(x) \ne \phi_i(y)$, $i = 1,2$. If $\omega_1(x,y) = \omega_2(x,y)$ whenever $\phi_i(x) \ne \phi_i(y)$, $i = 1,2$, then

$$\begin{aligned} \lambda_{h,0}^{\omega_1} &= \frac{(1/2)\sum_{x,y\in\overline{S}} |\phi_1(y) - \phi_1(x)|^p \omega_1(x,y) + \int_S V|\phi_1|^p}{\int_S h|\phi_1|^p} \\ &= \frac{(1/2)\sum_{x,y\in\overline{S}} |\phi_1(y) - \phi_1(x)|^p \omega_2(x,y) + \int_S V|\phi_1|^p}{\int_S h|\phi_1|^p} \\ &\ge \lambda_{h,0}^{\omega_2}. \end{aligned} \tag{49}$$

Thus we have $\lambda_{h,0}^{\omega_1} = \lambda_{h,0}^{\omega_2}$. □

4. Indefinite Eigenvalue Problems with Weight Functions Which Have Both Positive and Negative Values

In this section, we address problems for the other case that h has both positive and negative values. Namely, we now assume that the function $h : S \to \mathbb{R}$ satisfies

$$h^+ \not\equiv 0, \qquad h^- \not\equiv 0, \tag{50}$$

where

$$h^+(x) := \max\{h(x), 0\}, \qquad h^-(x) := -\min\{h(x), 0\} \tag{51}$$

for $x \in S$.

4.1. Indefinite Eigenvalue Problems. We now discuss the indefinite eigenvalue problems with the assumption that h has both positive and negative values and two real values $\lambda_{h,0}^+$ and $\lambda_{h,0}^-$ defined by

$$\begin{aligned} \lambda_{h,0}^+ &:= \inf_{\int_S h|u|^p > 0} \frac{(1/2)\int_{\overline{S}} \nabla_\omega u \cdot \nabla_{p,\omega} u + \int_{\overline{S}} V|u|^p}{\int_S h|u|^p}, \\ \lambda_{h,0}^- &:= \sup_{\int_S h|u|^p < 0} \frac{(1/2)\int_{\overline{S}} \nabla_\omega u \cdot \nabla_{p,\omega} u + \int_{\overline{S}} V|u|^p}{\int_S h|u|^p}. \end{aligned} \tag{52}$$

Theorem 11. *If functions $V : S \to \mathbb{R}$ and $h : S \to \mathbb{R}$ satisfy either $V \ge 0$ or $V \ge h$ in S, then there exists $\phi_0 \in A$ such that*

$$\lambda_{h,0}^+ = \frac{(1/2)\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 + \int_{\overline{S}} V|\phi_0|^p}{\int_S h|\phi_0|^p}, \tag{53}$$

where

$$A := \left\{ u : \overline{S} \longrightarrow \mathbb{R} \mid \int_S h|u|^p > 0,\ u|_{\partial S} = 0 \right\}. \tag{54}$$

Moreover $\lambda_{h,0}^{+}$ is the smallest positive eigenvalue for $-\mathscr{L}_{p,\omega}^{V}$ and ϕ_0 is an eigenfunction corresponding to $\lambda_{h,0}^{+}$.

Proof. Define

$$A_1 := \left\{u : \overline{S} \to \mathbb{R} \mid \int_S |u|^p = 1\right\}. \tag{55}$$

We note that $A \cap A_1$ is not compact and its boundary $\partial(A \cap A_1)$ is given by

$$\partial(A \cap A_1) = \left\{u : \overline{S} \longrightarrow \mathbb{R} \mid \int_S |u|^p = 1, \int_S h|u|^p = 0,\ u|_{\partial S} = 0\right\}, \tag{56}$$

so $\overline{A \cap A_1}$ is compact. Now we take $\{u_n\}_{n=1}^{\infty} \subset A \cap A_1$ such that $\{u_n\}$ converges at some point $u_0 \in \partial(A \cap A_1)$. Since $\{u_n\}$ converges, $(1/2)\int_{\overline{S}} \nabla_\omega u_n \cdot \nabla_{p,\omega} u_n + \int_{\overline{S}} V|u_n|^p$ also converges. From $\int_S h|u_n| \to 0$ and the function V which satisfies either $V \geq 0$ or $V \geq h$, we easily show that

$$\frac{(1/2)\int_{\overline{S}} \nabla_\omega u_n \cdot \nabla_p u_n + \int_{\overline{S}} V|u_n|^p}{\int_S h|u_n|^p} \nearrow \infty. \tag{57}$$

Therefore, there exists $\phi_0 \in A \cap A_1$ such that

$$\begin{aligned}&\frac{(1/2)\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega} \phi_0 + \int_{\overline{S}} V|\phi_0|^p}{\int_S h|\phi_0|^p} \\ &\quad = \min_{\phi \in A \cap A_1} \frac{(1/2)\int_{\overline{S}} \nabla_\omega \phi \cdot \nabla_{p,\omega} \phi + \int_{\overline{S}} V|\phi|^p}{\int_S h|\phi|^p}.\end{aligned} \tag{58}$$

Now we take an arbitrary $x_0 \in S$. Since $\int_S h|\phi_0 + t\delta_{x_0}|^p \neq 0$ for sufficiently small $t > 0$, by definition of $\lambda_{h,1}^{+}$, we have

$$\begin{aligned}\lambda_{h,0}^{+} \leq &\left(\frac{1}{2}\int_{\overline{S}} \nabla_\omega\left(\phi_0 + t\delta_{x_0}\right) \cdot \nabla_{p,\omega}\left(\phi_0 + t\delta_{x_0}\right)\right. \\ &\left. + \int_{\overline{S}} V\left|\phi_0 + t\delta_{x_0}\right|^p\right) \\ &\times \left(\int_S h\left|\phi_0 + t\delta_{x_0}\right|^p\right)^{-1}.\end{aligned} \tag{59}$$

Thus,

$$\begin{aligned}0 \leq &\int_S \frac{1}{2}\nabla_\omega\left(\phi_0 + t\delta_{x_0}\right) \cdot \nabla_p\left(\phi_0 + t\delta_{x_0}\right) \\ &+ \int_{\overline{S}} V\left|\phi_0 + t\delta_{x_0}\right|^p - \lambda_{h,0}^{+} h\left|\phi_0 + t\delta_{x_0}\right|^p\end{aligned} \tag{60}$$

for a sufficiently small $t > 0$. The right-hand side is continuously differentiable with respect to t and is equal to zero at $t = 0$. Thus,

$$\begin{aligned}0 = &\frac{d}{dt}\left[\int_S \frac{1}{2}\nabla_\omega\left(\phi_0 + t\delta_{x_0}\right) \cdot \nabla_{p,\omega}\left(\phi_0 + t\delta_{x_0}\right)\right. \\ &\left. + \int_{\overline{S}} V\left|\phi_0 + t\delta_{x_0}\right|^p - \lambda_{h,0}^{+} h\left|\phi_0 + t\delta_{x_0}\right|^p\right]_{t=0} \\ = &-p\sum_{x,y\in\overline{S}} \left|\phi_0(y) - \phi_0(x)\right|^{p-2}\left(\phi_0(y)\right. \\ &\left. -\phi_0(x)\right)\delta_{x_0}(x)\,\omega(x,y) \\ &+ p\sum_{x\in\overline{S}} V(x)\left|\phi_0(x)\right|^{p-2}\phi_0(x)\,\delta_{x_0}(x) \\ &+ p\sum_{x\in\overline{S}} -\lambda_{h,0}^{+} h\left|\phi_0(x)\right|^{p-2}\phi_0(x)\,\delta_{x_0}(x) \\ = &\,p\left(-\Delta_p\phi_0(x_0) + V(x)\left|\phi_0(x)\right|^{p-2}\phi_0(x)\right. \\ &\left. -\lambda_{h,0}^{+} h\left|\phi_0(x_0)\right|^{p-2}\phi_0(x_0)\right)x_0.\end{aligned} \tag{61}$$

Since the above equations hold for an arbitrary $x_0 \in \overline{S}$, we have

$$\begin{aligned}&-\Delta_{p,\omega}\phi_0^{+}(x) + V(x)\left|\phi_0(x)\right|^{p-2}\phi_0(x) \\ &\quad = \lambda_{h,0}^{+} h\left|\phi_0(x)\right|^{p-2}\phi_0(x), \qquad x \in \overline{S}.\end{aligned} \tag{62}$$

□

Theorem 12. *For a function $V : S \to \mathbb{R}$ and $h : S \to \mathbb{R}$ satisfying either $V \geq 0$ or $V \leq h$ in S, there exists $\phi_0 \in B$ such that*

$$\lambda_{h,0}^{-} = \frac{(1/2)\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega} \phi_0 + \int_{\overline{S}} V|\phi_0|^p}{\int_S h|\phi_0|^p}. \tag{63}$$

where

$$B := \left\{u : \overline{S} \to \mathbb{R} \mid \int_S h|u|^p < 0,\ u|_{\partial S} = 0\right\}. \tag{64}$$

Moreover $\lambda_{h,0}^{-}$ is the largest negative eigenvalue for $-\mathscr{L}_{p,\omega}^{V}$ and ϕ_0 is an eigenfunction corresponding to $\lambda_{h,0}^{-}$.

Proof. Since the proof is similar to that of the previous theorem, we omit it. □

We note that it follows from the two above results that if either $V \geq 0$ or $V \equiv h$, then there exist $\lambda_{h,0}^{+}$ and $\lambda_{h,0}^{-}$ at the same time. The specific case of $V \equiv 0$ was dealt with in [9].

In the following results, we give some properties of $\lambda_{h,0}^{+}$ and its eigenfunction. One also can get similar results for $\lambda_{h,0}^{-}$ and its eigenfunction, assuming that the function V satisfies either $V \geq 0$ or $V \leq h$.

Theorem 13. *For a function $V : S \to \mathbb{R}$ and a weight function h satisfying either $V \geq 0$ or $V \geq h$ in S, there exists a positive eigenfunction ϕ_0 corresponding to the indefinite eigenvalue $\lambda_{h,0}^{+}$ for $-\mathscr{L}_{p,\omega}^{V}$.*

Proof. It follows from Theorem 11 that there exists an indefinite eigenfunction ϕ_0 satisfying

$$\begin{aligned}&-\Delta_{p,\omega}\phi_0(x) + V(x)\left|\phi_0(x)\right|^{p-2}\phi_0(x) \\ &\quad = \lambda_{h,0}^{+} h\left|\phi_0(x)\right|^{p-2}\phi_0(x)\end{aligned} \tag{65}$$

for $x \in S$. Let $\psi(x) := |\phi_0(x)|$ for all x in $\overline{S}$. Then $\int_{\overline{S}} |\psi|^p = \int_{\overline{S}} |\phi_0|^p$ and

$$\begin{aligned} &\frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 + \int_S V|\phi_0|^p \\ &\quad \geq \frac{1}{2}\int_{\overline{S}} \nabla_\omega \psi \cdot \nabla_{p,\omega}\psi + \int_S V|\psi|^p. \end{aligned} \tag{66}$$

Thus, we have

$$\begin{aligned} \lambda_{h,0}^+ &= \frac{(1/2)\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 + \int_S V|\phi_0|^p}{\int_{\overline{S}} h|\phi_0|^p} \\ &\geq \frac{(1/2)\int_{\overline{S}} \nabla_\omega \psi \cdot \nabla_{p,\omega}\psi + \int_S V|\psi|^p}{\int_{\overline{S}} h|\psi|^p}. \end{aligned} \tag{67}$$

Otherwise, by definition of $\lambda_{h,1}^+$,

$$\lambda_{h,0}^+ \leq \frac{(1/2)\int_{\overline{S}} \nabla_\omega \psi \cdot \nabla_{p,\omega}\psi + \int_S V|\psi|^p}{\int_{\overline{S}} h|\psi|^p}. \tag{68}$$

Thus,

$$\lambda_{h,0}^+ = \frac{(1/2)\int_{\overline{S}} \nabla_\omega \psi \cdot \nabla_{p,\omega}\psi + \int_S V|\psi|^p}{\int_{\overline{S}} h|\psi|^p}. \tag{69}$$

It follows from Theorem 11 that $(\lambda_{h,0}^+, \psi)$ is an indefinite Dirichlet eigenpair. Next we show that $\psi(x) > 0$ for all x in $\overline{S}$. It is sufficient to prove that if there exists x_0 in S such that $\psi(x_0) = 0$, then $\psi \equiv 0$. Since $(\lambda_{h,0}^+, \psi)$ is an indefinite Dirichlet eigenpair, it satisfies (1). This implies that

$$\sum_{y\in S} |\psi(y)|^{p-2}\psi(y)\,\omega(x_0, y) = 0. \tag{70}$$

Hence $\psi(y) = 0$ for all $y \sim x_0$. By repeating the above process for $y \sim x_0$, we conclude that $\psi(z) = 0$ for each $z \sim y$. Since G is a connected network, $\psi(x) = 0$ for all $x \in G$. However, this contradicts the fact that $\phi \not\equiv 0$. Thus, $\psi(x) > 0$ for all x in G. □

Corollary 14. *For a function $V : S \to \mathbb{R}$ with $V \leq 0$ in S or $V \leq h$ in S, if ϕ_0 is an eigenfunction corresponding to the eigenvalue $\lambda_{h,0}^+$ for $-\mathscr{L}_{p,\omega}^V$ with respect to h, then* $\operatorname{sgn}\phi_0(x) = \operatorname{sgn}\phi_0(y)$ *for all x, y in S.*

Proof. Let ϕ_0 be an eigenfunction corresponding to $\lambda_{h,0}^+$. Then by Theorem 13, $|\phi_0|$ is also an eigenfunction corresponding to $\lambda_{h,0}^+$. Therefore, we have

$$\begin{aligned} &\frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 + \int_S V|\phi_0|^p \\ &\quad = \frac{1}{2}\int_{\overline{S}} \nabla_\omega |\phi_0| \cdot \nabla_{p,\omega}|\phi_0| + \int_S V|\phi_0|^p. \end{aligned} \tag{71}$$

This implies that $|\phi_0(y) - \phi_0(x)| = ||\phi_0(y)| - |\phi_0(x)||$ for all $x, y \in S$. Thus, we have $\phi_0(x) = |\phi_0(x)|$, $x \in S$. □

Corollary 15. *The multiplicity of $\lambda_{h,0}^+$ is one.*

4.2. Resonance Problems, Antiminimum Principle, and Inverse Problems. As previously mentioned, throughout this section the weight function h is assumed to have both positive and negative values in S. The next theorem shows that even in this case, we can solve a resonance problem similar to that in Theorem 8.

Theorem 16 (resonance problems). *Suppose that a function V satisfies $\lambda_{h,0}^+ > 0$. For $g : S \to [0, \infty)$, the equation*

$$\begin{aligned} -\Delta_{p,\omega}u + V|u|^{p-2}u - \lambda_{h,0}^+ h|u|^{p-2}u &= g \quad \text{in } S, \\ u &= 0 \quad \text{on } \partial S \end{aligned} \tag{72}$$

has a solution if and only if $g \equiv 0$. Moreover, the solutions are eigenfunctions corresponding to $\lambda_{h,0}^+$.

Proof. Suppose that a function u_0 is a solution to (72). If $u_0 \equiv 0$, then we have $g \equiv 0$. Suppose $u_0 \not\equiv 0$ and set a function $\overline{u_0}$ as $\overline{u_0}(x) := \max\{-u_0(x), 0\}$ for all $x \in \overline{S}$. Since $u(x) = 0$ for all $x \in \partial S$, $\overline{u_0}(x) = 0$ for all $x \in \partial S$. Since u_0 is a solution of (72), we have

$$\begin{aligned} 0 \leq \int_S g(\overline{u_0}) &= \int_{\overline{S}} \left(-\Delta_p u_0 + V|u_0|^{p-2}u_0 - \lambda_0 h|u_0|^{p-2}u_0\right)\overline{u_0} \\ &\leq -\frac{1}{2}\sum_{x,y\in\overline{S}} \left|(-\overline{u_0}(y)) - (-\overline{u_0}(x))\right|^p - \sum_{x\in S} V(x)\left|-\overline{u_0}(x)\right|^p \\ &\quad + \lambda_{h,0}^+ \sum_{x\in S} h(x)\left|-\overline{u_0}(x)\right|^p \end{aligned} \tag{73}$$

which implies that $\overline{u_0} \equiv k\phi_0$ for some $k \geq 0$. Assume $k > 0$; then u_0 is an eigenfunction corresponding to $\lambda_{h,0}^+$ so that $g \equiv 0$. Now, assume $k = 0$. Then $u_0 \geq 0$. Suppose $u_0(x_0) = 0$ for some $x_0 \in S$. Then we have $-\Delta_{p,\omega}u_0(x_0) = g(x_0)$. Since g is a nonnegative function, we have $u_0(y) = 0$ for $y \sim x_0$, so $u_0 \equiv 0$. This presents a contradiction. Thus, $u_0(x) > 0$, $x \in S$. Let ϕ_0 be a positive eigenfunction corresponding to $\lambda_{h,0}^+$. By Picone's identity,

$$\begin{aligned} 0 &\leq \frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 - \nabla_\omega\left(\frac{\phi_0^p}{u_0^{p-1}}\right)\cdot \nabla_{p,\omega}u_0 \\ &= \int_{\overline{S}} \left(-\Delta_p\phi_0\right)\phi_0 - \left(-\Delta_p\phi_0\right)\frac{\phi_0^p}{u_0^{p-1}} \\ &= \int_{\overline{S}} \left(-\Delta_p\phi_0\right)\phi_0 - \left(\left(\lambda_{h,0}^+ - V\right)u_0^{p-1} + g\right)\frac{\phi_0^p}{u_0^{p-1}} \\ &= \int_{\overline{S}} \left(-\Delta_p\phi_0 + V\phi_0^p\right) - \int_{\overline{S}} \lambda_{h,0}^+ h\phi_0^p - \int_S g\frac{\phi_0^p}{u_0^{p-1}} \end{aligned} \tag{74}$$

which implies that $g \equiv 0$. □

The next result that we will discuss is the parallel version of the antiminimum principle discussed in Theorem 9 where the weight function h is assumed to have both positive and negative values in S.

Theorem 17 (antiminimum principle). *Let a function $V : S \to \mathbb{R}$ and a weight function h with $V \geq 0$ in S or $V \geq h$ in S be given. For a nonnegative source term $g : S \to [0, \infty)$, suppose u_λ is a solution to the following equation:*

$$\begin{gathered} -\Delta_{p,\omega} u + V|u|^{p-2}u - \lambda h|u|^{p-2}u = g \quad \text{in } S \\ u = 0 \quad \text{on } \partial S. \end{gathered} \tag{75}$$

If $\lambda > \lambda^+_{h,0}$, then $u_\lambda(x_0) < 0$ for some $x_0 \in S$.

Proof. By virtue of Theorem 16, it suffices to show that if there exists a nonnegative solution u_λ of (75) then $\lambda < \lambda^+_{h,0}$. Suppose $u_\lambda(x_0) = 0$ for some $x_0 \in S$. Then we have $-\Delta_{p,\omega} u_0(x_0) = g(x_0)$. Since g is a nonnegative function, we have $u_0(y) = 0$ for $y \sim x_0$, so $u_0 \equiv 0$. This is a contradiction to the assumption. Thus, we have $u_\lambda > 0$ in S. Let ϕ_0 be a positive eigenfunction corresponding to $\lambda^+_{h,0}$. By Picone's identity,

$$\begin{aligned} 0 &\leq \frac{1}{2}\int_{\overline{S}} \nabla_\omega \phi_0 \cdot \nabla_{p,\omega}\phi_0 - \nabla_\omega\left(\frac{\phi_0^p}{u_\lambda^{p-1}}\right)\cdot \nabla_{p,\omega} u_\lambda \\ &= \int_{\overline{S}} \left(-\Delta_p \phi_0\right)\phi_0 - \left(-\Delta_p \phi_0\right)\frac{\phi_0^p}{u_\lambda^{p-1}} \\ &= \int_{\overline{S}} \left(-\Delta_p \phi_0\right)\phi_0 - \int_{\overline{S}} \left(-V u_\lambda^{p-1} + \lambda h u_\lambda^{p-1} + g\right)\frac{\phi_0^p}{u_\lambda^{p-1}} \\ &= \int_S \lambda^+_{h,0} h\phi_0^p - \int_S \lambda h \phi_0^p - \int_S g\frac{\phi_0^p}{u_0^{p-1}}. \end{aligned} \tag{76}$$

Since $\int_S g(\phi_0^p/u_0^{p-1}) > 0$, we have

$$0 < \left(\lambda^+_{h,0} - \lambda\right)\int_S h\phi_0^p. \tag{77}$$

□

Finally, we deal with inverse conductivity problems for $\lambda^+_{h,0}$ and $\lambda^-_{h,0}$.

Theorem 18 (inverse conductivity problem). *For networks $G(\overline{S}, E_i, \omega_i)$, $i = 1, 2$, let λ^+_{0,ω_i} be the smallest positive indefinite eigenvalue for $-\mathscr{L}^V_{p,\omega_i}$. One can suppose that the given functions V and h satisfy either $V \geq 0$ or $V \geq h$. If the weight functions satisfy*

$$\omega_1 \leq \omega_2 \quad \text{in } \overline{S} \times \overline{S}, \tag{78}$$

then one has

$$\lambda^+_{0,\omega_1} \leq \lambda^+_{0,\omega_2}. \tag{79}$$

Moreover, $\lambda^+_{0,\omega_1} = \lambda^+_{0,\omega_2}$ if and only if one has

(i) *$\phi_1 = \phi_2$ on $\overline{S}$,*

(ii) *$\omega_1(x, y) = \omega_2(x, y)$ whenever $\phi_1(x) \neq \phi_1(y)$ or $\phi_2(x) \neq \phi_2(y)$,*

where ϕ_i is the eigenfunction corresponding to λ^+_{0,ω_i}, $i = 1, 2$.

Proof. Let ϕ_i^+ be an eigenfunction corresponding to λ^+_{0,ω_i} for $i = 1, 2$. By the definition of the smallest eigenvalue, we have

$$\lambda^+_{0,\omega_1} \leq \frac{(1/2)\int_{\overline{S}} \nabla_{p,w_1}\phi_2 \cdot \nabla_{w_1}\phi_2 + \int_S V|\phi_2|^p}{\int_S h|\phi_2|^p}. \tag{80}$$

It follows from $\omega_1 \leq \omega_2$ that

$$\begin{aligned} &\frac{(1/2)\int_{\overline{S}} \nabla_{p,w_1}\phi_2 \cdot \nabla_{w_1}\phi_2 + \int_S V|\phi_2|^p}{\int_S h|\phi_2|^p} \\ &\quad \leq \frac{(1/2)\int_{\overline{S}} \nabla_{p,w_2}\phi_2 \cdot \nabla_{w_2}\phi_2 + \int_S V|\phi_2|^p}{\int_S h|\phi_2|^p} = \lambda^+_{0,\omega_2}. \end{aligned} \tag{81}$$

Thus, we obtain $\lambda^+_{0,\omega_1} \leq \lambda^+_{0,\omega_2}$. Now, we suppose that $\lambda^+_{0,\omega_1} = \lambda^+_{0,\omega_2}$. Then

$$\begin{aligned} \lambda^+_{0,\omega_2} &= \frac{(1/2)\int_{\overline{S}} \nabla_{p,w_2}\phi_2 \cdot \nabla_{w_2}\phi_2 + \int_S V|\phi_2|^p}{\int_S h|\phi_2|^p} \\ &\geq \frac{(1/2)\sum_{x,y\in\overline{S}} |\phi_2(y) - \phi_2(x)|^p (\omega_2(x,y) - \omega_1(x,y))}{\int_S h|\phi_2|^p} \\ &\quad + \lambda^+_{0,\omega_1}. \end{aligned} \tag{82}$$

Since $\sum_{x\in S} h|\phi_2|^p > 0$, we have $|\phi_2(y) - \phi_2(x)|^p(\omega_2(x, y) - \omega_1(x, y)) = 0$, $x, y \in \overline{S}$. Then $\omega_1(x, y) = \omega_2(x, y)$ whenever $\phi_2(x) \neq \phi_2(y)$. This implies that

$$\begin{aligned} \lambda^+_{0,\omega_1} = \lambda^+_{0,\omega_2} &= \frac{(1/2)\int_{\overline{S}} \nabla_{p,w_2}\phi_2 \cdot \nabla_{w_2}\phi_2 + \int_S V|\phi_2|^p}{\int_S h|\phi_2|^p} \\ &= \frac{(1/2)\int_{\overline{S}} \nabla_{p,w_2}\phi_2 \cdot \nabla_{w_1}\phi_2 + \int_S V|\phi_2|^p}{\int_S h|\phi_2|^p}. \end{aligned} \tag{83}$$

Hence, ϕ_2 is an eigenfunction of λ^+_{0,ω_1}. Since λ^+_{0,ω_1} is simple, we have $\phi_2 \equiv \phi_1$. Therefore, $\omega_1(x, y) = \omega_2(x, y)$ whenever $\phi_i(x) \neq \phi_i(y)$, $i = 1, 2$.

If $\omega_1(x, y) = \omega_2(x, y)$ whenever $\phi_i(x) \neq \phi_i(y)$, $i = 1, 2$, then

$$\begin{aligned} \lambda^+_{0,\omega_1} &= \frac{(1/2)\sum_{x,y\in\overline{S}} |\phi_1(y) - \phi_1(x)|^p \omega_1(x,y) + \int_S V|\phi_1|^p}{\int_S h|\phi_1|^p} \\ &= \frac{(1/2)\sum_{x,y\in\overline{S}} |\phi_1(y) - \phi_1(x)|^p \omega_2(x,y) + \int_S V|\phi_1|^p}{\int_S h|\phi_1|^p} \\ &\geq \lambda^+_{0,\omega_2}. \end{aligned} \tag{84}$$

Thus, we have $\lambda^+_{0,\omega_1} = \lambda^+_{0,\omega_2}$. □

Theorem 19 (inverse conductivity problem). *For networks $G(\overline{S}, E_i, \omega_i)$, $i = 1, 2$, let λ^-_{0,ω_i} be the largest negative indefinite eigenvalue for $-\mathscr{L}^V_{p,\omega_i}$. One can suppose that the given functions V and h satisfy either $V \geq 0$ or $V \leq h$. If the weight functions satisfy*

$$\omega_1 \leq \omega_2 \quad \text{in } \overline{S} \times \overline{S}, \tag{85}$$

then one has

$$\lambda^-_{0,\omega_1} \geq \lambda^-_{0,\omega_2}. \tag{86}$$

Moreover, $\lambda^-_{0,\omega_1} = \lambda^-_{0,\omega_2}$ if and only if one has

(i) $\phi_1 = \phi_2$ *on* $\overline{S}$,

(ii) $\omega_1(x, y) = \omega_2(x, y)$ *whenever* $\phi_1(x) \neq \phi_1(y)$ *or* $\phi_2(x) \neq \phi_2(y)$,

where ϕ_i is the eigenfunction corresponding to λ^-_{0,ω_i}, $i = 1, 2$.

Proof. The proof is similar to that in Theorem 18 and we thus omit it. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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