Research Article
Existence of Nontrivial Solutions for Periodic Schrödinger Equations with New Nonlinearities

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We study the Schrödinger equation:

\[-\Delta u + V(x)u + f(x,u) = 0, \quad u \in H^1(\mathbb{R}^N),\]

where $V$ is 1-periodic and $f$ is 1-periodic in the $x$-variables; 0 is in a gap of the spectrum of the operator $-\Delta + V$. We prove that, under some new assumptions for $f$, this equation has a nontrivial solution. Our assumptions for the nonlinearity $f$ are very weak and greatly different from the known assumptions in the literature.

1. Introduction and Statement of Results

In this paper, we consider the following Schrödinger equation:

\[-\Delta u + V(x)u + f(x,u) = 0, \quad u \in H^1(\mathbb{R}^N),\]

where $N \geq 1$. For $V$ and $f$, we assume the following.

$\textbf{(v)}$ $V \in C(\mathbb{R}^N)$ is 1-periodic in $x_j$ for $j = 1, \ldots, N$, 0 is in a spectral gap $(-\mu_-, \mu_1)$ of $-\Delta + V$, and $-\mu_1$ and $\mu_1$ lie in the essential spectrum of $-\Delta + V$.

Denote

\[\mu_0 := \min \{\mu_-, \mu_1\}.\]

$\textbf{(f_1)}$ $f \in C(\mathbb{R}^N \times \mathbb{R})$ is 1-periodic in $x_j$ for $j = 1, \ldots, N$. And there exist constants $C > 0$ and $2 < p < 2^*$ such that

\[|f(x,t)| \leq C \left(1 + |t|^{p-1}\right), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R},\]

where

\[2^* := \begin{cases} \frac{2N}{N-2}, & N \geq 3 \\ \infty, & N = 1, 2. \end{cases}\]

$\textbf{(f_2)}$ The limit $\lim_{t \to 0} f(x,t)/t = 0$ holds uniformly for $x \in \mathbb{R}^N$. And there exists $D > 0$ such that

\[\inf_{x \in \mathbb{R}^N, |t| \geq D} \frac{f(x,t)}{t} > \max V_-,\]

where $V_+(x) = \max\{|\pm V(x)|, 0\}$, $\forall x \in \mathbb{R}^N$.

$\textbf{(f_3)}$ For any $(x,t) \in \mathbb{R}^N \times \mathbb{R}$, $\bar{F}(x,t) \geq 0$, where

\[\bar{F}(x,t) := \frac{1}{2} tf(x,t) - F(x,t), \quad F(x,t) = \int_0^t f(x,s) \, ds.\]

$\textbf{(f_4)}$ There exist $0 < \kappa < D$ and $\nu \in (0, \mu_0)$ such that, for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with $|t| < \kappa$,

\[|f(x,t)| \leq \nu |t|\]

and, for every $(x,t) \in \mathbb{R}^N \times \mathbb{R}$ with $\kappa \leq |t| \leq D$,

\[\bar{F}(x,t) > 0.\]

Remark 1. By the definitions of $F$ and $\bar{F}$, it is easy to verify that, for all $(x,t) \in \mathbb{R}^N \times (\mathbb{R} \setminus \{0\})$,

\[\frac{\partial}{\partial t} \left(\frac{F(x,t)}{t^2}\right) = \frac{2\bar{F}(x,t)}{t^3}.\]
Together with \( f(x,t) = o(t) \) as \(|t| \to 0\) and \((f_3)\), this implies that
\[
F(x,t) \geq 0 \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}.
\]

**Remark 2.** There are many functions satisfying \((f_1)-(f_4)\). We give several examples here.

**Example 1.** \( D = 1 + \mu_0/2 + e^{1+\max_{\mathbb{R}^N} V} \), \( \kappa = 1 + \mu_0/2 \), \( \gamma = \mu_0/2 \), and
\[
f(x,t) = \begin{cases} 0, & |t| \leq 1, \\ t \ln |t|, & |t| > 1. \end{cases}
\]

**Example 2.** \( D = 3 + \mu_0/2 + 2\max_{\mathbb{R}^N} V \), \( \kappa = 3/2 \), \( \gamma = \mu_0/2 \), and
\[
f(x,t) = \begin{cases} 0, & |t| \leq 1, \\ D(1-t), & t > 1, \\ D(t+1), & t < -1. \end{cases}
\]

**Example 3.** \( D = \mu_0/2 + e^{1+\max_{\mathbb{R}^N} V} \), \( \kappa = \gamma = \mu_0/2 \), and
\[
f(x,t) = t \ln(1+|t|).
\]

A solution \( u \) of (1) is called nontrivial if \( u \neq 0 \). Our main results are as follows.

**Theorem 3.** Suppose \((v)\) and \((f_1)-(f_4)\) are satisfied. Then (1) has a nontrivial solution.

Note that
\[
(f_1^\dag) \text{ the limits } \lim_{t \to 0} f(x,t)/t = 0 \text{ and } \lim_{|t| \to \infty} (f(x,t)/t) = +\infty \text{ hold uniformly for } x \in \mathbb{R}^N.
\]

Implying \((f_2)\), we have the following corollary.

**Corollary 4.** Suppose \((v)\), \((f_1)-(f_4)\), and \((f_4)\) are satisfied. Then (1) has a nontrivial solution.

It is easy to verify that the condition
\[
(f_4^\dag) \quad F(x,t) > 0, \text{ for every } (x,t) \in \mathbb{R}^N \times \mathbb{R}.
\]

And the assumption that \( f(x,t)/t \to 0 \) as \( t \to 0 \) uniformly for \( x \in \mathbb{R}^N \) imply \((f_2)\) and \((f_4)\). Therefore, we have the following corollary.

**Corollary 5.** Suppose \((v)\), \((f_1)-(f_2)\), and \((f_4)\) are satisfied. Then (1) has a nontrivial solution.

Semilinear Schrödinger equations with periodic coefficients have attracted much attention in recent years due to its numerous applications. One can see [1–24] and the references therein. In [2], the authors used the dual variational method to obtain a nontrivial solution of (1) with \( f(x,t) = \pm W(x)|t|^{\beta-2}t \), where \( W \) is an asymptotically periodic function. In [20], Troestler and Willem firstly obtained nontrivial solutions for (1) with \( f \) being a \( C^1 \) function satisfying the Ambrosetti-Rabinowitz condition:

**(AR)** there exists \( \alpha > 2 \) such that, for every \( u \neq 0 \), \( 0 < \alpha G(x,u) \leq g(x,u)u \), where \( g(x,u) = -f(x,u), G(x,u) = F(x,u), \) and
\[
\left| \frac{\partial f(x,u)}{\partial u} \right| \leq C \left( |u|^{p-2} + |u|^{q-2} \right)
\]
with \( 2 < p < q < 2^\ast \). Then, in [9], Kryszewski and Szulkin developed some infinite-dimensional linking theorems. Using these theorems, they improved Troestler and Willem’s results and obtained nontrivial solutions for (1) with \( f \) only satisfying \((f_1)\) and the (AR) condition. These generalized linking theorems were also used by Li and Szulkin to obtain nontrivial solution for (1) under some asymptotically linear assumptions for \( f \) (see [11]). In [13] (see also [14]), existence of nontrivial solutions for (1) under \((f_1)\) and the (AR) condition was also obtained by Pankov and Pflüger through approximating (1) by a sequence of equations defined in bounded domains. In the celebrated paper [17], Schechter and Zou combined a generalized linking theorem with the monotonicity methods of Jeanjean (see [8]). They obtained a nontrivial solution of (1) when \( f \) exhibits the critical growth. A similar approach was applied by Szulkin and Zou to obtain homoclinic orbits of asymptotically linear Hamiltonian systems (see [19]). Moreover, in [5] (see also [6]), Ding and Lee obtained nontrivial solutions for (1) under some new superlinear assumptions on \( f \) different from the classical (AR) conditions.

Our assumptions on \( f \) are very weak and greatly different from the assumptions mentioned above. In fact, our assumptions \((f_1)-(f_4)\) do not involve the properties of \( f \) at infinity. It may be asymptotically linear growth at infinity, that is, \( \lim \sup_{|t| \to \infty} (f(x,t)/t) < +\infty \), or superlinear growth at infinity as well, that is, \( \lim \inf_{|t| \to \infty} (f(x,t)/t) = +\infty \). Moreover, the assumptions \((f_1)-(f_4)\) allow \( f(x,t) \equiv 0 \) in a neighborhood of \( t = 0 \) (see Remark 2).

In this paper, we use the generalized linking theorem for a class of parameter-dependent functions (see [17, Theorem 2.1] or Proposition 8 in the present paper) to obtain a sequence of approximate solutions for (1). Then, we prove that these approximate solutions are bounded in \( L^p(\mathbb{R}^N) \) and \( H^1(\mathbb{R}^N) \) (see Lemmas 13 and 14). Finally, using the concentration-compactness principle, we obtain a nontrivial solution of (1).

**Notation.** \( B_r(a) \) denotes the open ball of radius \( r \) and center \( a \). For a Banach space \( E \), we denote the dual space of \( E \) by \( E' \) and denote strong and weak convergence in \( E \) by \( \rightarrow \) and \( \rightharpoonup \), respectively. For \( \varphi \in C^1(E; \mathbb{R}) \), we denote the Fréchet derivative of \( \varphi \) at \( u \) by \( \varphi'(u) \). The Gateaux derivative of \( \varphi \) is denoted by \( (\varphi'(u), v) \), \( \forall u, v \in E \). \( L^p(\mathbb{R}^N) \) denotes the standard \( L^p \) space \( (1 \leq p \leq \infty) \), and \( H^1(\mathbb{R}^N) \) denotes
the standard Sobolev space with norm \( \|u\|_{H^1} = (\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)dx)^{1/2} \). We use \( O(h) \), \( o(h) \) to mean \( |O(h)| \leq C|h| \), \( o(h)/|h| \to 0 \).

2. Existence of Approximate Solutions for (1)

Under the assumptions (v), (f₁), and (f₂), the functional

\[
\Phi(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx
\]

\[
+ \int_{\mathbb{R}^N} F(x,u) dx
\]

(14)

is of class \( C^1 \) on \( X := H^1(\mathbb{R}^N) \), and the critical points of \( \Phi \) are weak solutions of (1).

There is a standard variational setting for the quadratic form \( \int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx \). For the reader's convenience, we state it here. One can consult [5] or [6] for more details.

Assume that (v) holds, and let \( S = -\Delta + V \) be the self-adjoint operator acting on \( L^2(\mathbb{R}^N) \) with domain \( D(S) = H^2(\mathbb{R}^N) \). By virtue of (v), we have the orthogonal decomposition

\[
L^2 = L^2(\mathbb{R}^N) = L^+ + L^-
\]

(15)

such that \( S \) is negative (resp., positive) in \( L^- \) (resp., in \( L^+ \)). As in [5, Section 2] (see also [6, Chapter 6.2]), let \( X = D(|S|^{1/2}) \) be equipped with the inner product

\[
(u,v) = (|S|^{1/2}u,|S|^{1/2}v)_{L^2}
\]

(16)

and norm \( \|u\| = \|S|^{1/2}u\|_{L^2} \), where \((\cdot,\cdot)_{L^2}\) denotes the inner product of \( L^2 \). From (v),

\[
X = H^1(\mathbb{R}^N)
\]

(17)

with equivalent norms. Therefore, \( X \) continuously embeds in \( L^q(\mathbb{R}^N) \) for all \( 2 \leq q \leq 2N/(N - 2) \) if \( N \geq 3 \) and for all \( q \geq 2 \) if \( N = 1, 2 \). In addition, we have the decomposition

\[
X = X^+ + X^-,
\]

(18)

where \( X^k = X \cap L^k \) is orthogonal with respect to both \((\cdot,\cdot)_{L^2}\) and \((\cdot,\cdot)\). Therefore, for every \( u \in X \), there is a unique decomposition

\[
u = u^+ + u^-\quad \text{with } (u^+, u^-) = 0 \text{ and}
\]

\[
\int_{\mathbb{R}^N} |\nabla u|^2 dx + \int_{\mathbb{R}^N} V(x) u^2 dx = \|u^+\|^2 - \|u^-\|^2, \quad u \in X.
\]

(20)

Moreover,

\[
\mu_- \|u^-\|^2 \leq \|u\|^2, \quad \forall u \in X,
\]

(21)

\[
\mu_+ \|u^+\|^2 \leq \|u\|^2, \quad \forall u \in X.
\]

(22)

The functional \( \Phi \) defined by (14) can be rewritten as

\[
\Phi(u) = \frac{1}{2} \left(\|u^+\|^2 - \|u^-\|^2\right) + \psi(u),
\]

(23)

where

\[
\psi(u) = \int_{\mathbb{R}^N} F(x,u) dx.
\]


Let \( \{e_k\} \) be the orthonormal sequence in \( X^+ \). Let \( P : X \to X^+ \), \( Q : X \to X^- \) be the orthogonal projections. We define

\[
\|\|u\|\| = \max \left\{ \|P u\| \sum_{k=1}^{\infty} \frac{1}{2^k} \|Q u, e_k \| \right\}
\]

(25)

on \( X \). The topology generated by \( \|\cdot\|\) is denoted by \( \tau \), and all topological notation related to it will include this symbol.

Lemma 6. Suppose that (v) holds. Then

(a) \( \max_{\mathbb{R}^N} V_+ \geq \mu_- \), where \( \mu_- \) is defined in (v);

(b) for any \( C > \mu_- \), there exists \( u_0 \in X^- \) with \( \|u_0\| = 1 \) such that \( C\|u_0\|_{L^2} > 1 \).

Proof. (a) We apply an indirect argument, and assume by contradiction that

\[
\max_{\mathbb{R}^N} V_+ < \mu_-.
\]

(26)

From assumption (v), \( -\mu_- \) is in the essential spectrum of the operator (with domain \( D(L) = H^2(\mathbb{R}^N) \)):

\[
L = -\Delta + V : L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N).
\]

(27)

Then, by Weyl's criterion (see, e.g., [25, Theorem VII.12] or [26, Theorem 7.2]), there exists a sequence \( \{u_n\} \subset H^2(\mathbb{R}^N) \) with the properties that \( \|u_n\|_2 = 1, \forall n \) and \( \|\Delta u_n + V u_n + \mu_- u_n\|_2 \to 0 \).

Since \( \mu_- > \max_{\mathbb{R}^N} V_+ \), we deduce that \( -V_+(x) + \mu_- > 0 \) for all \( x \in \mathbb{R}^N \). Together with the facts that \( V \) is a continuous periodic function and \( V = V_+ - V_- \), this implies

\[
\inf_{x \in \mathbb{R}^N} (V(x) + \mu_-) > 0.
\]

(28)

It follows that there exists a constant \( C' > 0 \) such that

\[
\int_{\mathbb{R}^N} (|\nabla u|^2 + (V(x) + \mu_-) u^2) dx \geq C' \|u\|^2, \quad \forall u \in X.
\]

(29)

Note that

\[
\int_{\mathbb{R}^N} (-\Delta u_n + V(x) u_n + \mu_- u_n) u_n dx
\]

\[
= \int_{\mathbb{R}^N} (|\nabla u_n|^2 + (V(x) + \mu_-) u_n^2) dx.
\]

(30)

Together with (29) and the fact that \( \|\Delta u_n + V u_n + \mu_- u_n\|_2 \to 0 \) and \( \|u_n\|_2 = 1 \), this implies \( \|u_n\| \to 0 \). It contradicts \( \|u_n\|_2 = 1, \forall n \). Therefore, \( \max_{\mathbb{R}^N} V_+ \geq \mu_- \).


It suffices to prove that
\[ \mu_{-1} = C_- := \inf \{ \|u\|^2 \mid u \in X^-, \|u\|_{L^2} = 1 \} \cdot \quad (31) \]
From (21), we deduce that \( \mu_{-1} \leq C_- \). From assumption (v), \(-\mu_{-1}\) is in the essential spectrum of \( L \). By Weyl’s criterion, there exists \( \{u_n\} \subset H^2(\mathbb{R}^N) \) such that \( \|u_n\|_{L^2} = 1 \) and
\[ \| -\Delta u_n + V u_n + \mu_{-1} u_n \|_{L^2} \to 0. \]
Multiplying \(-\Delta u_n + V u_n + \mu_{-1} u_n \) by \( u_n^{-} \) and then integrating it into \( \mathbb{R}^N \), by (20) and (22), we get that
\[
(\mu_1 + \mu_{-1}) \|u_n^+\|_{L^2}^2 \\
\leq \int_{\mathbb{R}^N} \left( |\nabla u_n^+|^2 + V(x) (u_n^+)^2 + \mu_{-1} (u_n^+)^2 \right) dx \\
= \int_{\mathbb{R}^N} (-\Delta u_n + V(x) u_n + \mu_{-1} u_n) u_n^- dx \to 0.
\]
It follows that \( \|u_n^-\|_{L^2} \to 1 \). Multiplying \(-\Delta u_n + V u_n + \mu_{-1} u_n \) by \( u_n^- \) and then integrating it into \( \mathbb{R}^N \), we get that
\[
- \|u_n^-\|_{L^2}^2 + \mu_{-1} \|u_n^-\|_{L^2}^2 \\
= \int_{\mathbb{R}^N} \left( |\nabla u_n^-|^2 + V(x) (u_n^-)^2 + \mu_{-1} (u_n^-)^2 \right) dx \\
= \int_{\mathbb{R}^N} (-\Delta u_n + V u_n + \mu_{-1} u_n) u_n^- dx \to 0.
\]
It implies that \( \mu_{-1} \geq C_- \). This together with \( \mu_{-1} \leq C_- \) implies \( \mu_{-1} = C_- \).

Let \( R > r > 0 \) and
\[
A := \inf_{x \in \mathbb{R}^N, |t| < 2} \frac{f(x, t)}{t}. \quad (34)
\]
From assumption (5), we have \( A > \max_{\mathbb{R}^N} V_- \). Together with the result (a) of Lemma 6, this implies that \( (1/2)(A + \mu_{-1}) > \mu_{-1} \). Choose
\[
\gamma \in \left( \mu_{-1}, \frac{A + \mu_{-1}}{2} \right). \quad (35)
\]
Then by the result (b) of Lemma 6, there exists \( u_0 \in X^- \) with \( \|u_0\| = 1 \) such that
\[
\gamma \|u_0\|_{L^2}^2 > 1. \quad (36)
\]
Set
\[
N = \{ u \in X^- \mid \|u\| = r \}, \quad (37)
\]
\[
M = \{ u \in X^+ \oplus \mathbb{R}^+ u_0 \mid \|u\| \leq R \}. \]
Then, \( M \) is a submanifold of \( X^+ \oplus \mathbb{R}^+ u_0 \) with boundary
\[
\partial M = \{ u \in X^- \mid \|u\| \leq R \} \\
\cup \{ u^- + tu_0 \mid u^- \in X^-, t > 0, \|u^- + tu_0\| = R \}. \quad (38)
\]

**Definition 7.** Let \( \phi \in C^1(X; \mathbb{R}) \). A sequence \( \{u_n\} \subset X \) is called a Palais-Smale sequence at level \( c \) (\( (PS)_c \)-sequence for short) for \( \phi \), if \( \phi(u_n) \to c \) and \( \|\phi'(u_n)\|_{X'} \to 0 \) as \( n \to \infty \).

The following proposition is proved in [17] (see [17, Theorem 2.11]).

**Proposition 8.** Let \( 0 < K < 1 \). The family of \( C^1 \)-functional \( \{H_\lambda\} \) has the form
\[
H_\lambda(u) = \lambda I(u) - J(u), \quad u \in X, \lambda \in [K, 1]. \quad (39)
\]
Assume
(a) \( J(u) \geq 0 \), \( \forall u \in X \);
(b) \( |J(u)| + J(u) \to +\infty \) as \( \|u\| \to +\infty \);
(c) for all \( \lambda \in [K, 1] \), \( H_\lambda \) is \( \tau \)-sequentially upper semicontinuous; that is, if \( \|u_n - u\| \to 0 \), then
\[
\limsup_{n \to \infty} H_\lambda(u_n) \leq H_\lambda(u), \quad (40)
\]
and \( H_\lambda' \) is weakly sequentially continuous. Moreover, \( H_\lambda \) maps bounded sets to bounded sets;
(d) there exist \( u_0 \in X^- \setminus \{0\} \) with \( \|u_0\| = 1 \) and \( R > r > 0 \) such that, for all \( \lambda \in [K, 1] \),
\[
\inf_{\lambda \in \Delta M} H_\lambda > \sup_{N \in \Delta M} H_\lambda. \quad (41)
\]
Then there exists \( E \subset [K, 1] \) such that the Lebesgue measure of \([K, 1] \setminus E\) is zero and, for every \( \lambda \in E \), there exist \( c_\lambda \) and a bounded \((PS)_{c_\lambda}\)-sequence for \( H_\lambda \), where \( c_\lambda \) satisfies
\[
\sup_{M \in \Delta M} H_\lambda \geq c_\lambda \geq \inf_{N \in \Delta M} H_\lambda. \quad (42)
\]
For \( 0 < K < 1 \) and \( \lambda \in [K, 1] \), let
\[
\Psi_\lambda(u) = \frac{1}{2} \int_{\mathbb{R}^N} V_-(x) u^2 dx \\
- \left( \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_+(x) u^2) dx + \psi(u) \right), \quad u \in X. \quad (43)
\]
Then
\[
\Psi_1 = -\Phi \quad (44)
\]
and it is easy to verify that a critical point \( u \) of \( \Psi_\lambda \) is a weak solution of
\[
-\Delta u + V_\lambda(x) u + f(x, u) = 0, \quad u \in X, \quad (45)
\]
where
\[
V_\lambda = V_+ - \lambda V_- \quad (46)
\]
**Lemma 9.** Suppose that (v) and (f_1)–(f_3) hold. Then, there exist \( 0 < K_* < 1 \) and \( E \subset [K_*, 1] \) such that the Lebesgue measure of \([K_*], 1 \] \setminus E \) is zero and, for every \( \lambda \in E \), there exist \( c_\lambda \) and a bounded \((PS)_{c_\lambda}\)-sequence for \( \Psi_\lambda \), where \( c_\lambda \) satisfies
\[
\sup_{\lambda \in E} \inf_{c_\lambda} c_\lambda > 0. \quad (47)
\]
Proof. For $u \in X$, let
\[ I(u) = \frac{1}{2} \int_{\mathbb{R}^N} V_-(x) u^2 \, dx \]
\[ J(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + V_+(x) u^2) \, dx + \psi(u). \]
Then, $I$ and $J$ satisfy assumptions (a) and (b) in Proposition 8, and, by (43), $\Psi(u) = \lambda I(u) - J(u)$.

From (43) and (20), for any $u \in X$ and $\lambda \in [K, 1]$, we have
\[ \Psi_\lambda(u) = \frac{\lambda - 1}{2} \int_{\mathbb{R}^N} V_-(x) u^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \frac{1 - \lambda}{2} \int_{\mathbb{R}^N} V_+(x) u^2 \, dx - \int_{\mathbb{R}^N} F(x, u) \, dx. \]

Let $u_n \in X$ and $\{u_n\} \subset X$ be such that $\|u_n - u_*\| \to 0$. It follows that $u_n^* \to u_*, u_n^* - u_n \to u_*^*$. In addition, up to a subsequence, we can assume that $u_n \to u_*$ a.e. in $\mathbb{R}^N$. Then, we have
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} V_-(x) u_n^2 \, dx \to \int_{\mathbb{R}^N} V_-(x) u_*^2 \, dx, \]
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} V_+(x) u_n^2 \, dx \geq \int_{\mathbb{R}^N} V_+(x) u_*^2 \, dx \quad \text{(by Fatou's lemma)}, \]
\[ \lim_{n \to \infty} \|u_n^*\|^2 \geq \|u_*^*\|^2. \]

By Remark 1, $F(x, t) \geq 0$ for all $x$ and $t$. This together with Fatou's lemma implies
\[ \lim_{n \to \infty} \int_{\mathbb{R}^N} F(x, u_n) \, dx \geq \int_{\mathbb{R}^N} F(x, u_*) \, dx. \]

Then, by (49), we obtain
\[ \limsup_{n \to \infty} \Psi_\lambda(u_n) \leq \Psi_\lambda(u_*). \]

This implies that $\Psi_\lambda$ is $r$-sequentially upper semicontinuous. If $u_n \to u_*$ in $X$, then, for any fixed $\varphi \in X$, as $n \to \infty$,
\[ \langle -\Psi_\lambda'(u_n), \varphi \rangle = \int_{\mathbb{R}^N} (V_\varphi u_n + V_+ u_n \varphi) \, dx + \int_{\mathbb{R}^N} f(x, u_n) \varphi \, dx \to \int_{\mathbb{R}^N} (V_\varphi u_* + V_+ u_\varphi) \, dx + \int_{\mathbb{R}^N} f(x, u_*) \varphi \, dx = \langle -\Psi_\lambda'(u_*), \varphi \rangle. \]

This implies that $\Psi_\lambda'$ is weakly sequentially continuous. Moreover, it is easy to see that $\Psi_\lambda$ maps bounded sets to bounded sets. Therefore, $\Psi_\lambda$ satisfies assumption (c) in Proposition 8.

Finally, we will verify assumption (d) in Proposition 8 for $\Psi_\lambda$.

From assumption (f) and $f(x, t)/t \to 0$ as $t \to 0$ uniformly for $x \in \mathbb{R}^N$, we deduce that, for any $\epsilon > 0$, there exists $C_\epsilon > 0$ such that
\[ F(x, t) \leq \epsilon t^2 + C_\epsilon |t|^p, \quad \forall (x, t) \in \mathbb{R}^N \times \mathbb{R}. \]

From (49) and (55), we have, for $u \in N$,
\[ \Psi_\lambda(u) \geq \frac{1}{2} \|u\|^2 - \frac{1 - \lambda}{2} \int_{\mathbb{R}^N} V_-(x) u^2 \, dx \]
\[ - \epsilon \int_{\mathbb{R}^N} u^2 \, dx - C \int_{\mathbb{R}^N} |u|^p \, dx. \]

Then by the Sobolev inequality $\|u\|_{L^p(\mathbb{R}^N)} \leq C \|u\|$ and $\|u\|_{L^2} \leq C \|u\|$ (by (21) and (22)), we deduce that there exists a constant $C > 0$ such that
\[ \Psi_\lambda(u) \geq \frac{1}{2} \|u\|^2 - C (1 - \lambda) \max_{\mathbb{R}^N} V_-(x) \|u\|^2 \]
\[ - \epsilon C \|u\|^2 - C_{\epsilon} \|u\|^p. \]

Choose $0 < K_* < 1$ and $\epsilon > 0$ such that $C(1 - K_*) \max_{\mathbb{R}^N} V_-(x) < 1/4$ and $C_\epsilon = 1/8$. Then, for every $\lambda \in [K_*, 1]$, we have
\[ \Psi_\lambda(u) \geq \frac{1}{8} \|u\|^2 - C_{\epsilon} \|u\|^p. \]

Let $r > 0$ be such that $r^{p - 2} C_{\epsilon} = 1/16$ and $\beta = r^2/16$. Then, from (58), we deduce that, for $N = \{u \in X^- | \|u\| = r\}$,
\[ \inf_{N} \Psi_\lambda \geq \beta, \quad \forall \lambda \in [K_*, 1]. \]

We will prove that $\sup_{K_* \leq \lambda \leq 1} \Psi_\lambda(u) \to -\infty$ as $\|u\| \to \infty$ and $u \in X^+ \oplus \mathbb{R}^+ \theta_0$. Arguing indirectly, assume that, for some sequences $\lambda_n \in [K_*, 1]$ and $u_n \in X^+ \oplus \mathbb{R}^+ \theta_0$ with $\|u_n\| \to +\infty$, there is $\lambda \not\in \mathcal{L}$ such that $\Psi_{\lambda_n}(u_n) \geq \lambda \not\in \mathcal{L}$ for all $n$. Then, setting $w_n = u_n/\|u_n\|$, we have $\|w_n\| = 1$, and, up to a subsequence, $w_n \rightharpoonup w, w_n \to w^* \in X^-$ and $w_n^* \to w^* \in X^*$. First, we consider the case $w \neq 0$. Dividing both sides of (49) by $\|u_n\|^2$, we get that
\[ - \frac{\mathcal{L}}{\|u_n\|^2} \leq \frac{\Psi_{\lambda_n}(u_n)}{\|u_n\|^2}. \]
\[ = \frac{1}{2} \|w_n\|^2 - \frac{1}{2} \|w_n\|^2 \]
\[ - \frac{1 - \lambda_n}{2} \int_{\mathbb{R}^N} V_-(x) w_n^2 \, dx - \int_{\mathbb{R}^N} F(x, u_n) \, dx. \]

From (5) and the result (a) of Lemma 6, we deduce that
\[ \liminf_{\|u\| \to \infty} \frac{F(x, t)}{t^2} \geq \frac{A}{2} \geq \frac{1}{2} \max_{\mathbb{R}^N} V_- \geq \frac{1}{2} \mu_1. \]
where A is defined by (34). Note that, for \( x \in \{ x \in \mathbb{R}^N \mid w \neq 0 \} \), we have \( |u_n(x)| \to +\infty \). This implies that, when \( n \) is large enough,

\[
\int_{\{x \in \mathbb{R}^N \mid w \neq 0 \}} \frac{F(x, u_n)}{u_n^2} w_n^2 dx \geq \frac{A + \mu_1}{4} \int_{\{x \in \mathbb{R}^N \mid w \neq 0 \}} u_n^2 dx.
\]

(62)

By (10), we have, when \( n \) is large enough,

\[
\int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} \|u_n\|^2 dx = \int_{\mathbb{R}^N} \frac{F(x, u_n)}{u_n^2} w_n^2 dx \\
\geq \int_{\{x \in \mathbb{R}^N \mid w \neq 0 \}} \frac{F(x, u_n)}{u_n^2} w_n^2 dx.
\]

Combining the above two inequalities yields

\[
\liminf_{n \to \infty} \left( \frac{1}{2} \|w_n\|^2 - \frac{1}{2} \|w_n^*\|^2 \right) \\
- \frac{1}{2} \int_{\mathbb{R}^N} V_-(x) w_n^2 dx - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} dx \\
\leq \liminf_{n \to \infty} \left( \frac{1}{2} \|w_n\|^2 - \frac{1}{2} \|w_n^*\|^2 \right) \\
- \frac{A + \mu_1}{4} \int_{\{x \in \mathbb{R}^N \mid w \neq 0 \}} w_n^2 dx \\
\leq \frac{1}{2} \|w^-\|^2 - \frac{1}{2} \|w^+\|^2 - \frac{A + \mu_1}{4} \|w\|^2_{L^2}.
\]

(63)

We used the inequalities

\[
\lim_{n \to \infty} \|w_n\|^2 = \|w\|^2.
\]

\[
\liminf_{n \to \infty} \|w_n^*\|^2 \geq \|w^*\|^2.
\]

\[
\liminf_{n \to \infty} \int_{\{x \in \mathbb{R}^N \mid w \neq 0 \}} w_n^2 dx \geq \int_{\mathbb{R}^N} w^2 dx
\]

in the second inequality of (64).

Since \( w = tu_0 \) for some \( t \in \mathbb{R} \), by (36), we get that

\[
\frac{A + \mu_1}{4} \|w\|^2_{L^2} \geq \frac{A + \mu_1}{4\gamma} \|w\|^2.
\]

(66)

Note that, by the choice of \( \gamma \) (see (35)), we have \( (A + \mu_1)/4\gamma > 1/2 \). Then by (64) and the fact that \( w \neq 0 \), we have that

\[
\liminf_{n \to \infty} \left( \frac{1}{2} \|w_n\|^2 - \frac{1}{2} \|w_n^*\|^2 \right) \\
- \frac{1}{2} \int_{\mathbb{R}^N} V_-(x) w_n^2 dx - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{\|u_n\|^2} dx
\]

\[
\leq - \frac{A + \mu_1}{4\gamma} \|w^+\|^2 < 0.
\]

(67)

It contradicts (60), since \(-\mathcal{L}/\|u_n\|^2 \to 0 \) as \( n \to \infty \).

Second, we consider the case \( w = 0 \). In this case, \( \lim_{n \to \infty} \|w_n\|^2 = 0 \). It follows that

\[
\liminf_{n \to \infty} \|w_n^*\|^2 \geq 1,
\]

since \( \|w_n\| = 1 \) and \( u_n = w_n^* + w_n^- \). Therefore, the right hand side of (60) is less than \(-1/4 \) when \( n \) is large enough. However, as \( n \to \infty \), the left hand side of (60) converges to zero. It induces a contradiction.

Therefore, there exists \( R > r \) such that

\[
\sup_{\lambda \in [K, 1]} \sup_{\delta \in M} \Psi_\lambda \leq 0.
\]

(69)

This implies that \( \Psi_\lambda \) satisfies assumption (d) in Proposition 8 if \( \lambda \in [K, 1] \). Finally, it is easy to see that

\[
\sup_{\lambda \in [K, 1]} \sup_{\delta \in M} \Psi_\lambda < +\infty.
\]

(70)

Then, the results of this lemma follow immediately from Proposition 8. 

\[ \square \]

**Lemma 10.** Suppose that (v) and \((f_1)-(f_2)\) are satisfied. Let \( \lambda \in [K, 1] \) be fixed, where \( K \) is the constant in Lemma 9. If \( |v_\lambda| \) is a bounded (PS)\_\_sequence for \( \Psi_\lambda \) with \( \epsilon > 0 \), then, for every \( n \in \mathbb{N} \), there exists \( a_n \in \mathbb{Z}^N \) such that, up to a subsequence, \( u_n := v_n( \cdot + a_n) \) satisfies

\[
u_n \to u_\lambda \neq 0, \quad \Psi_\lambda(u_\lambda) \leq \epsilon, \quad \Psi_\lambda'(u_\lambda) = 0.
\]

(71)

**Proof.** The proof of this lemma is inspired by the proof of Lemma 3.7 in [19]. Because \( |v_\lambda| \) is a bounded sequence in \( X \), up to a subsequence, either

(a) \( \lim_{n \to \infty} \sup_{y \in \mathcal{R}} \int_{B_t(y)} |v_n|^2 dx = 0 \) or

(b) there exist \( c > 0 \) and \( a_n \in \mathbb{Z}^N \) such that \( \int_{B_t(a_n)} |v_n|^2 dx \geq c \).

If (a) occurs, using the Lions lemma (see, e.g., [21, Lemma 1.21]), a similar argument as for the proof of [19, Lemma 3.6] shows that

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} F(x, v_n) dx = 0, \quad \lim_{n \to \infty} \int_{\mathbb{R}^N} f(x, v_n) \psi dx = 0.
\]

(72)
It follows that
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} (2F(x, v_n) - f(x, v_n)v_n) \, dx = 0. \tag{73}
\]
On the other hand, as \(\{v_n\}\) is a \((PS)_c\)-sequence of \(\Psi_\lambda\), we have
\[
\langle \Psi'_\lambda(v_n), v_n \rangle = c \neq 0. \tag{74}
\]
It follows that
\[
\int_{\mathbb{R}^N} f(x, v_n) v_n = 2\Psi_\lambda(v_n) - \langle \Psi'_\lambda(v_n), v_n \rangle \rightarrow 2c \neq 0, \quad n \to \infty.
\]
This contradicts (73). Therefore, case (a) cannot occur.

If case (b) occurs, let \(u_n = v_n(\cdot + a_n)\). For every \(n\),
\[
\int_{\mathbb{R}^N} |u_n|^2 \, dx \geq 0. \tag{75}
\]
Because \(V\) and \(F(x, t)\) are 1-periodic in every \(x_j\), \(\{u_n\}\) is still bounded in \(X\),
\[
\lim_{n \to \infty} \Psi_\lambda(u_n) \leq c, \quad \Psi'_\lambda(u_n) \to 0, \quad n \to \infty. \tag{76}
\]
Up to a subsequence, we assume that \(u_n \rightharpoonup u_1\) in \(X\) as \(n \to \infty\). Since \(u_n \to u_1\) in \(L^p_{\text{loc}}(\mathbb{R}^N)\), it follows from (75) that \(u_1 \neq 0\). Recall that \(\Psi'_\lambda(u_n)\) is weakly sequentially continuous. Therefore, \(\Psi'_\lambda(u_n) \to \Psi'_\lambda(u_1)\) and, by (76), \(\Psi'_\lambda(u_n) = 0\).

Finally, by (f) and Fatou's lemma
\[
c = \lim_{n \to \infty} \left( \Psi_\lambda(u_n) - \frac{1}{2} \langle \Psi'_\lambda(u_n), u_n \rangle \right)
= \lim_{n \to \infty} \int_{\mathbb{R}^N} F(x, u_n) \geq \int_{\mathbb{R}^N} F(x, u_1) = \Psi_\lambda(u_1). \tag{77}
\]

\[\square\]

**Lemma 11.** There exist \(0 < K_* < 1\) and \(\eta > 0\) such that, for \(\lambda \in (K_*, 1)\), \(u\neq 0\) satisfies \(\Psi'_\lambda(u) = 0\), then \(\|u\| \geq \eta\).

**Proof.** We adapt the arguments of Yang [23, p. 2626] and Liu [12, Lemma 2.2]. Note that, by (f1) and (f2), for any \(\epsilon > 0\), there exists \(C_\epsilon > 0\) such that
\[
|f(x, t)| \leq \epsilon |t| + C_\epsilon |t|^{p-1}. \tag{78}
\]
Let \(u \neq 0\) be a critical point of \(\Psi_\lambda\). Then \(u\) is a solution of
\[
-\Delta u + V_\lambda(x)u + f(x, u) = 0, \quad u \in X. \tag{79}
\]
Multiplying both sides of this equation by \(u^p\), respectively, and then integrating into \(\mathbb{R}^N\), we get that
\[
0 = \pm \|u^p\|^2 + (1 - \lambda) \int_{\mathbb{R}^N} V_\lambda(x)u_n u^p \, dx + \int_{\mathbb{R}^N} f(x, u)u^p \, dx. \tag{80}
\]
It follows that
\[
\|u^p\|^2 = \|u^p\|^2 + (1 - \lambda) \int_{\mathbb{R}^N} V_\lambda(x)u_n u^p \, dx \leq (1 - \lambda) \sup_{\mathbb{R}^N} V_\lambda |u| \cdot |u^p| \, dx
+ \epsilon \int_{\mathbb{R}^N} |u| \cdot |u^p| \, dx + C_\epsilon \int_{\mathbb{R}^N} |u|^{p-1} |u^p| \, dx \leq C_1 ((1 - \lambda) + \epsilon) \|u^p\|^2 + C_2 \|u^p\|^p. \tag{81}
\]
where \(C_1\) and \(C_2\) are positive constants related to the Sobolev inequalities and \(\sup_{\mathbb{R}^N} V_\lambda\). From the above two inequalities, we obtain
\[
\|u^p\|^2 = \|u^p\|^2 + \|u^p\|^2 \leq 2C_1 ((1 - \lambda) + \epsilon) \|u\|^2 + 2C_2 \|u\|^p. \tag{82}
\]
Because \(p > 2\), this implies that \(\|u\| \geq \eta\) for some \(\eta > 0\) if \(\epsilon > 0\) and \(1 - K_* > 0\) are small enough and \(\lambda \in (K_*, 1)\). The desired result follows.

Let \(K = \max\{K_*, K_*\}\), where \(K_*\) and \(K_*\) are the constants that appeared in Lemmas 9 and 11, respectively. Combining Lemmas 9–11, we obtain the following lemma.

**Lemma 12.** Suppose (v) and (f1)–(f3) are satisfied. Then, there exist \(\eta > 0\), \(\{\lambda_n\} \subset [K_*, 1]\), and \(\{u_n\} \subset X\) such that \(\lambda_n \to 1\),
\[
\sup_n \Psi_{\lambda_n}(u_n) < +\infty, \quad \|u_n\| \geq \eta, \quad \Psi'_{\lambda_n}(u_n) = 0. \tag{83}
\]

3. A Priori Bound of Approximate Solutions and Proof of the Main Theorem

In this section, we give a priori bound for the sequence of approximate solutions \(\{u_n\}\) obtained in Lemma 12. We then give the proofs of Theorem 3.

**Lemma 13.** Suppose (v) and (f1)–(f3) are satisfied. Let \(\{u_n\}\) be the sequence obtained in Lemma 12. Then, \(\{u_n\} \subset L^\infty(\mathbb{R}^N)\) and
\[
\sup_n \|u_n\|_{L^\infty(\mathbb{R}^N)} \leq D. \tag{84}
\]

**Proof.** From \(\Psi'_{\lambda_n}(u_n) = 0\), we deduce that \(u_n\) is a weak solution of (45) with \(\lambda = \lambda_n\); that is,
\[
-\Delta u_n + V_{\lambda_n}(x)u_n + f(x, u_n) = 0 \quad \text{in } \mathbb{R}^N. \tag{85}
\]
By assumption (f1) and the bootstrap argument of elliptic equations, we deduce that \(u_n \in L^\infty(\mathbb{R}^N)\).

Multiplying both sides of (85) by \(v_n = (u_n - D)^+ := \max(u_n - D, 0)\) and integrating into \(\mathbb{R}^N\), we get that
\[
\int_{\mathbb{R}^N} |V_{\lambda_n}|^2 \, dx + \int_{u_n > D} \left( V_{\lambda_n}(x)u_n + f(x, u_n) \right) v_n \, dx = 0. \tag{86}
\]
Recall that $V_{\lambda_n} = V_+ - \lambda_n V_-$ and $\lambda_n \leq 1$. Then by (5), we get that
\[ \int_{u_n \geq D} (V_{\lambda_n}(x) u_n + f(x, u_n)) v_n dx = \int_{u_n \geq D} (V_{\lambda_n}(x) + \frac{f(x, u_n)}{u_n}) u_n v_n dx \geq 0. \] (87)
This together with (86) yields $v_n = 0$. It follows that $u_n(x) \leq D$ on $\mathbb{R}^N$.

Similarly, multiplying both sides of (85) by $w_n = (u_n + D)^{-} := \max\{-(u_n + D), 0\}$ and integrating into $\mathbb{R}^N$, we can get that $u_n \geq -D$ on $\mathbb{R}^N$. Therefore, for all $n$, $\|u_n\|_{L^\infty(\mathbb{R}^N)} \leq D$. □

**Lemma 14.** Suppose that $(v)$, $(f_1)$, $(f_2)$, $(f_3)$, and $(f_4)$ are satisfied. Let $\{u_n\}$ be the sequence obtained in Lemma 12. Then
\[ 0 < \inf_n \|u_n\| \leq \sup_n \|u_n\| < +\infty. \] (88)

**Proof.** As $\Psi'_{\lambda_n}(u_n) = 0$ and $u_n \neq 0$, Lemma 11 implies that $\inf_n \|u_n\| > 0$.

To prove $\sup_n \|u_n\| < +\infty$, we apply an indirect argument and assume by contradiction that $\|u_n\| \to +\infty$.

Since $\Psi'_{\lambda_n}(u_n) = 0$, by (81), we get that
\[ \|u_n\|^2 + \int_{\mathbb{R}^N} f(x, u_n) u_n^2 dx = \int_{\mathbb{R}^N} f(x, u_n) u_n^2 dx \geq \int_{\mathbb{R}^N} f(x, u_n) u_n^2 dx \geq (1 - \lambda_n) O(\|u_n\|^2). \] (89)

It follows that
\[ \|u_n\|^2 + \int_{\mathbb{R}^N} f(x, u_n) (u_n^+ - u_n^-) dx = \|u_n^+\|^2 + \|u_n^-\|^2 \]
\[ + \int_{\mathbb{R}^N} f(x, u_n) (u_n^+ - u_n^-) dx = (1 - \lambda_n) O(\|u_n\|^2). \] (90)

Set $w_n = u_n/\|u_n\|$. Then, by (90),
\[ \|w_n\|^2 \left(1 + \int_{\mathbb{R}^N} \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n dx\right) \]
\[ = (1 - \lambda_n) O(\|w_n\|^2). \] (91)

Then, by $\lambda_n \to 1$ as $n \to \infty$, we have that
\[ \int_{\mathbb{R}^N} \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n dx \to -1, \quad n \to \infty. \] (92)

From Lemma 12,
\[ C_0 := \sup_n \Psi'_{\lambda_n}(u_n) < +\infty. \] (93)

Then, by $\Psi'_{\lambda_n}(u_n) = 0$, we obtain
\[ 2C_0 \geq 2\Psi_{\lambda_n}(u_n) - \langle \Psi'_{\lambda_n}(u_n), u_n \rangle = 2 \int_{\mathbb{R}^N} \mathcal{F}(x, u_n) dx. \] (94)

From $(f_3)$, we have
\[ 2C_0 \geq 2 \int_{\mathbb{R}^N} \mathcal{F}(x, u_n) dx \geq 2 \int_{|x|D \leq |u_n(x)| \leq \kappa} \mathcal{F}(x, u_n) dx, \] (95)
where $\kappa$ is the constant in $(f_3)$. As the continuous function $\mathcal{F}$ is 1-periodic in every $x_j$ variable, we deduce from (8) that there exists a constant $C' > 0$ such that
\[ \mathcal{F}(x, t) \geq C' t^2, \] (96)
for every $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ with $|t| \leq D$.

Combining (95) and (96) leads to
\[ \lim_{n \to \infty} \int_{|x|D \leq |u_n(x)| \leq \kappa} w_n^2 dx = 0. \] (97)

Dividing both sides of this inequality by $\|u_n\|^2$ and sending $n \to \infty$, we obtain
\[ \int_{|x|D < |u_n(x)| \leq \kappa} \mathcal{F}(x, u_n) dx \leq 0. \] (98)

From (7), (21), and (22), we have that
\[ \int_{|x|< \mu_0} \left|\frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n dx\right| \]
\[ \leq C \int_{|x|< \mu_0} \left|\frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n dx\right| \]
\[ \leq \mu_0 \|w_n\|^2 \leq \frac{\mu_0}{\mu_0} \|w_n\|^2 \leq \frac{\mu_0}{\mu_0} < 1, \] (99)
where $\mu_0$ is the constant defined in $(v)$.

Since $f \in C(\mathbb{R}^N \times \mathbb{R})$ and $\lim_{t \to 0} f(x, t)/t = 0$, we deduce that there exists $C > 0$ such that, for every $(x, t) \in \mathbb{R}^N \times \mathbb{R}$ with $|t| \leq D$,
\[ f(x, t) \leq C |t|. \] (100)

This together with (98) gives
\[ \int_{|x|D \leq |u_n(x)| \leq \kappa} \left|\frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n dx\right| \]
\[ \leq C \int_{|x|D \leq |u_n(x)| \leq \kappa} \left|\frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n dx\right| \]
\[ \leq C \int_{|x|D \leq |u_n(x)| \leq \kappa} \left|\frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n dx\right| \]
\[ \leq 2C \|w_n\|_{L^\infty} \left(\int_{|x|D \leq |u_n(x)| \leq \kappa} w_n^2 dx\right)^{1/2} \to 0, \quad n \to \infty. \] (101)
Combining (99) and (101) yields
\[
\limsup_{n \to \infty} \int_{\mathbb{R}^N} \left| \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n \right| dx
\leq \limsup_{n \to \infty} \int_{\{x \in \mathbb{R}^N \mid |u(x)| > \alpha \}} \left| \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n \right| dx
\]
\[
+ \limsup_{n \to \infty} \int_{\{x \in \mathbb{R}^N \mid |u(x)| \leq \alpha \}} \left| \frac{f(x, u_n)}{u_n} (w_n^+ - w_n^-) w_n \right| dx < 1.
\]  
(102)

This contradicts (92). Therefore, \( |u_n| \) is bounded in \( X \).

\[\square\]

Proof of Theorem 3. Let \( \{u_n\} \) be the sequence obtained in Lemma 12. From Lemma 14, \( |u_n| \) is bounded in \( X \). Therefore, up to a subsequence, either

(a) \( \lim_{n \to \infty} \sup_{y \in \mathbb{R}^N} \int_{B_1(y)} |u_n|^2 dx = 0 \) or

(b) there exist \( \alpha > 0 \) and \( y_n \in \mathbb{Z}^N \) such that

\( \int_{B_1(y_n)} |u_n|^2 dx \geq \alpha. \)

According to (72), if case (a) occurs,

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} f(x, u_n) u_n^* dx = 0.
\]  
(103)

Then, by (81) and \( \lambda_n \to 1 \), we have

\[
\|u_n^*\|_2^2 = \frac{1}{1 - \lambda_n} \int_{\mathbb{R}^N} V_-(x) u_n^* dx
\]
\[
\geq \int_{\mathbb{R}^N} f(x, u_n) u_n^* dx
\]
\[
\leq C (1 - \lambda_n) \|u_n\|_2^2 + \int_{\mathbb{R}^N} f(x, u_n) u_n^* dx \to 0.
\]  
(104)

This contradicts \( \inf_n \|u_n\| > 0 \) (see (88)). Therefore, case (a) cannot occur. As case (b) therefore occurs, \( w_n = u_n(\cdot + y_n) \) satisfies \( w_n \to w_0 \neq 0 \). From (14) and (43), we have that

\[
\Psi_{\lambda_n}(u) = -\Phi(u) + \frac{\lambda - 1}{2} \int_{\mathbb{R}^N} V(x) u^2 dx, \quad \forall u \in X.
\]  
(105)

It follows that

\[
\left\langle \Psi_{\lambda_n}'(u), \varphi \right\rangle = -\left\langle \Phi'(u), \varphi \right\rangle + (\lambda - 1) \int_{\mathbb{R}^N} V(x) u \varphi dx, \quad \forall u, \varphi \in X.
\]  
(106)

By \( \Psi_{\lambda_n}'(u_n) = 0 \) (see Lemma 12), we have \( \Psi_{\lambda_n}'(w_n) = 0. \) From (106), we have that, for any \( \varphi \in X, \)

\[
\left\langle \Psi_{\lambda_n}'(w_n), \varphi \right\rangle = -\left\langle \Phi'(w_n), \varphi \right\rangle + (\lambda_n - 1) \int_{\mathbb{R}^N} V(x) w_n \varphi dx.
\]  
(107)

Together with \( \Phi_{\lambda_n}'(w_n) = 0 \) and \( \lambda_n \to 1 \), this yields

\[
\left\langle \Phi'(w_n), \varphi \right\rangle \to 0, \quad \forall \varphi \in X.
\]  
(108)

Finally, by \( w_n \to w_0 \neq 0 \) and the weakly sequential continuity of \( \Phi' \), we have that \( \Phi'(w_0) = 0. \) Therefore, \( u_0 \) is a nontrivial solution of (1). This completes the proof.

\[\square\]

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References


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