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Research Article

New Results and Generalizations for Approximate Fixed Point Property and Their Applications

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We first introduce the concept of manageable functions and then prove some new existence theorems related to approximate fixed point property for manageable functions and α-admissible multivalued maps. As applications of our results, some new fixed point theorems which generalize and improve Du's fixed point theorem, Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, and Nadler's fixed point theorem and some well-known results in the literature are given.

1. Introduction and Preliminaries

In 1922, Banach established the most famous fundamental fixed point theorem (so-called the Banach contraction principle [1]) which has played an important role in various fields of applied mathematical analysis. It is known that the Banach contraction principle has been extended and generalized in many various different directions by several authors; see [2–40] and references therein. An interesting direction of research is the extension of the Banach contraction principle to multivalued maps, known as Nadler's fixed point theorem [2], Mizoguchi-Takahashi's fixed point theorem [3], and Berinde-Berinde's fixed point theorem [5] and references therein.

Let us recall some basic notations, definitions, and well-known results needed in this paper. Throughout this paper, we denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of positive integers and real numbers, respectively. Let (X, d) be a metric space. For each $x \in X$ and $A \subseteq X$, let $d(x, A) = \inf_{y\in A} d(x, y)$. Denote by $\mathcal{N}(X)$ the class of all nonempty subsets of X, $\mathcal{C}(X)$ the family of all nonempty closed subsets of X, and $\mathcal{CB}(X)$ the family of all nonempty closed and bounded subsets of X. A function $\mathcal{H} : \mathcal{CB}(X) \times \mathcal{CB}(X) \to [0, \infty)$ defined by

$$\mathcal{H}(A, B) = \max\left\{\sup_{x\in B} d(x, A), \sup_{x\in A} d(x, B)\right\} \tag{1}$$

is said to be the Hausdorff metric on $\mathcal{CB}(X)$ induced by the metric d on X. A point v in X is a fixed point of a map T, if $v = Tv$ (when $T : X \to X$ is a single-valued map) or $v \in Tv$ (when $T : X \to \mathcal{N}(X)$ is a multivalued map). The set of fixed points of T is denoted by $\mathcal{F}(T)$. The map T is said to have the *approximate fixed point property* [29–34] on X provided $\inf_{x\in X} d(x, Tx) = 0$. It is obvious that $\mathcal{F}(T) \neq \emptyset$ implies that T has the approximate fixed point property, but the converse is not always true.

Definition 1 (see [6, 13]). A function $\varphi : [0, \infty) \to [0, 1)$ is said to be an $\mathcal{MT}$*-function* (or $\mathcal{R}$*-function*) if $\limsup_{s\to t^+} \varphi(s) < 1$ for all $t \in [0, \infty)$.

It is evident that if $\varphi : [0, \infty) \to [0, 1)$ is a nondecreasing function or a nonincreasing function, then φ is a $\mathcal{MT}$-function. So the set of $\mathcal{MT}$-functions is a rich class.

Recently, Du [6] first proved the following characterizations of $\mathcal{MT}$-functions which are quite useful for proving our main results.

Theorem 2 (see [6]). *Let $\varphi : [0,\infty) \to [0,1)$ be a function. Then the following statements are equivalent.*

(a) *φ is an $\mathcal{MT}$-function.*

(b) *For each $t \in [0,\infty)$, there exist $r_t^{(1)} \in [0,1)$ and $\varepsilon_t^{(1)} > 0$ such that $\varphi(s) \le r_t^{(1)}$ for all $s \in (t, t+\varepsilon_t^{(1)})$.*

(c) *For each $t \in [0,\infty)$, there exist $r_t^{(2)} \in [0,1)$ and $\varepsilon_t^{(2)} > 0$ such that $\varphi(s) \le r_t^{(2)}$ for all $s \in [t, t+\varepsilon_t^{(2)}]$.*

(d) *For each $t \in [0,\infty)$, there exist $r_t^{(3)} \in [0,1)$ and $\varepsilon_t^{(3)} > 0$ such that $\varphi(s) \le r_t^{(3)}$ for all $s \in (t, t+\varepsilon_t^{(3)}]$.*

(e) *For each $t \in [0,\infty)$, there exist $r_t^{(4)} \in [0,1)$ and $\varepsilon_t^{(4)} > 0$ such that $\varphi(s) \le r_t^{(4)}$ for all $s \in [t, t+\varepsilon_t^{(4)})$.*

(f) *For any nonincreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, one has $0 \le \sup_{n\in\mathbb{N}}\varphi(x_n) < 1$.*

(g) *φ is a function of contractive factor [15]; that is, for any strictly decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$ in $[0,\infty)$, one has $0 \le \sup_{n\in\mathbb{N}}\varphi(x_n) < 1$.*

In 1989, Mizoguchi and Takahashi [3] proved a famous generalization of Nadler's fixed point theorem which gives a partial answer of Problem 9 in Reich [4].

Theorem 3 (Mizoguchi and Takahashi [3]). *Let (X,d) be a complete metric space, let $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function, and let $T : X \to \mathcal{CB}(X)$ be a multivalued map. Assume that*

$$\mathcal{H}(Tx,Ty) \le \varphi(d(x,y))\, d(x,y), \tag{2}$$

for all $x, y \in X$. Then $\mathcal{F}(T) \ne \emptyset$.

In 2007, M. Berinde and V. Berinde [5] proved the following interesting fixed point theorem which generalized and extended Mizoguchi-Takahashi's fixed point theorem.

Theorem 4 (M. Berinde and V. Berinde [5]). *Let (X,d) be a complete metric space, let $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function, let $T : X \to \mathcal{CB}(X)$ be a multivalued map, and $L \ge 0$. Assume that*

$$\mathcal{H}(Tx,Ty) \le \varphi(d(x,y))\, d(x,y) + L d(y,Tx), \tag{3}$$

for all $x, y \in X$. Then $\mathcal{F}(T) \ne \emptyset$.

In 2012, Du [6] established the following fixed point theorem which is an extension of Berinde-Berinde's fixed point theorem and hence Mizoguchi-Takahashi's fixed point theorem.

Theorem 5 (Du [6]). *Let (X,d) be a complete metric space, let $T : X \to \mathcal{CB}(X)$ be a multivalued map, let $\varphi : [0,\infty) \to [0,1)$ be a $\mathcal{MT}$-function, and let $h : X \to [0,\infty)$ be a function. Assume that*

$$\begin{aligned}\mathcal{H}(Tx,Ty) \le{}& \varphi(d(x,y))\, d(x,y) \\ &+ h(y)\, d(y,Tx) \quad \forall x, y \in X.\end{aligned} \tag{4}$$

Then T has a fixed point in X.

The paper is organized as follows. In Section 2, we first introduce the concept of manageable function and give some examples of it. Section 3 is dedicated to the study of some new existence theorems related to approximate fixed point property for manageable functions and α-admissible multivalued maps. As applications of our results, some new fixed point theorems which generalize and improve Du's fixed point theorem, Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, and Nadler's fixed point theorem and some well-known results in the literature are given in Section 4. Consequently, some of our results in this paper are original in the literature, and we obtain many results in the literature as special cases.

2. Manageable Functions

In this paper, we first introduce the concept of manageable functions.

Definition 6. A function $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is called *manageable* if the following conditions hold:

(η1) $\eta(t,s) < s - t$ for all $s, t > 0$;

(η2) for any bounded sequence $\{t_n\} \subset (0,+\infty)$ and any nonincreasing sequence $\{s_n\} \subset (0,+\infty)$, it holds that

$$\limsup_{n\to\infty} \frac{t_n + \eta(t_n, s_n)}{s_n} < 1. \tag{5}$$

We denote the set of all manageable functions by $\widehat{\mathrm{Man}(\mathbb{R})}$.

Here, we give simple examples of manageable function.

Example A. Let $\gamma \in [0,1)$. Then $\eta_\gamma : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by

$$\eta_\gamma(t,s) = \gamma s - t \tag{6}$$

is a manageable function.

Example B. Let $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be any function. Then the function $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ defined by

$$\eta(t,s) = \begin{cases} \dfrac{s}{s+9}\ln(s+10) - t, & \text{if } (t,s) \in [0,+\infty)\times[0,+\infty), \\ f(t,s), & \text{otherwise,} \end{cases} \tag{7}$$

is a manageable function. Indeed, let

$$g(x) = \frac{\ln(x+10)}{x+9} \quad \forall x > -9. \tag{8}$$

Then $g(s) < 1$ for all $s > 0$, and

$$\eta(t,s) = \begin{cases} sg(s) - t, & \text{if } (t,s) \in [0,+\infty) \times [0,+\infty), \\ f(t,s), & \text{otherwise.} \end{cases} \tag{9}$$

For any $s, t > 0$, we have

$$\eta(t,s) = sg(s) - t < s - t, \tag{10}$$

so $(\eta 1)$ holds. Let $\{t_n\} \subset (0,+\infty)$ be a bounded sequence and let $\{s_n\} \subset (0,+\infty)$ be a nonincreasing sequence. Then $\lim_{n\to\infty} s_n = \inf_{n\in\mathbb{N}} s_n = a$ for some $a \in [0,+\infty)$. Since g is continuous, we get

$$\limsup_{n\to\infty} \frac{t_n + \eta(t_n, s_n)}{s_n} = \lim_{n\to\infty} g(s_n) = g(a) < 1, \tag{11}$$

which means that $(\eta 2)$ holds. Hence, $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$.

Example C. Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be any function and let $\varphi : [0,\infty) \to [0,1)$ be an $\mathcal{MT}$-function. Define $\eta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\eta(t,s) = \begin{cases} s\varphi(s) - t, & \text{if } (t,s) \in [0,+\infty) \times [0,+\infty), \\ f(t,s), & \text{otherwise.} \end{cases} \tag{12}$$

Then η is a manageable function. Indeed, one can verify easily that $(\eta 1)$ holds. Next, we verify that η satisfies $(\eta 2)$. Let $\{t_n\} \subset (0,+\infty)$ be a bounded sequence and let $\{s_n\} \subset (0,+\infty)$ be a nonincreasing sequence. Then $\lim_{n\to\infty} s_n = \inf_{n\in\mathbb{N}} s_n = a$ for some $a \in [0,+\infty)$. Since φ is an $\mathcal{MT}$-function, by Theorem 2, there exist $r_a \in [0,1)$ and $\varepsilon_a > 0$ such that $\varphi(s) \le r_a$ for all $s \in [a, a+\varepsilon_a)$. Since $\lim_{n\to\infty} s_n = \inf_{n\in\mathbb{N}} s_n = a$, there exists $n_a \in \mathbb{N}$, such that

$$a \le s_n < a + \varepsilon_a \quad \forall n \in \mathbb{N} \text{ with } n \ge n_a. \tag{13}$$

Hence, we have

$$\limsup_{n\to\infty} \frac{t_n + \eta(t_n, s_n)}{s_n} = \limsup_{n\to\infty} \varphi(s_n) \le r_a < 1, \tag{14}$$

which means that $(\eta 2)$ holds. So we prove $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$.

The following result is quite obvious.

Proposition 7. *Let $\zeta : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a function. If there exists $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$ such that $\zeta(t,s) \le \eta(t,s)$ for all $s, t > 0$, then $\zeta \in \widehat{\mathrm{Man}(\mathbb{R})}$.*

Proposition 8. *Let $\{\eta_i\}_{i\in\mathbb{N}} \subset \widehat{\mathrm{Man}(\mathbb{R})}$. Then the following statements hold.*

(a) *For each $k \in \mathbb{N}$, the function $\eta_{(k)}^{\min} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by*

$$\eta_{(k)}^{\min}(t,s) = \min\{\eta_1(t,s), \eta_2(t,s), \ldots, \eta_k(t,s)\}, \tag{15}$$

is a manageable function (i.e., $\eta_{(k)}^{\min} \in \widehat{\mathrm{Man}(\mathbb{R})}$ for any $k \in \mathbb{N}$).

(b) *For each $k \in \mathbb{N}$, the function $\overline{\eta_{(k)}} : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, defined by*

$$\overline{\eta_{(k)}}(t,s) = \frac{1}{k}\sum_{i=1}^{k} \eta_i(t,s), \tag{16}$$

is a manageable function (i.e., $\overline{\eta_{(k)}} \in \widehat{\mathrm{Man}(\mathbb{R})}$ for any $k \in \mathbb{N}$).

Proof. Since $\eta_{(k)}^{\min}(t,s) \le \eta_1(t,s)$ for all $t, s > 0$, the conclusion (a) is a direct consequence of Proposition 7. Next, we prove the conclusion (b). Let $k \in \mathbb{N}$ be given. It is obvious that $\overline{\eta_{(k)}}(t,s) < s-t$ for all $s, t > 0$. Let $\{t_n\} \subset (0,+\infty)$ be a bounded sequence and let $\{s_n\} \subset (0,+\infty)$ be a nonincreasing sequence. For any $n \in \mathbb{N}$, we have

$$\begin{aligned} \frac{t_n + \overline{\eta_{(k)}}(t_n, s_n)}{s_n} &= \frac{1}{s_n}\left(t_n + \frac{1}{k}\sum_{i=1}^{k}\eta_i(t,s)\right) \\ &= \frac{1}{k}\sum_{i=1}^{k}\frac{t_n + \eta_i(t_n, s_n)}{s_n}. \end{aligned} \tag{17}$$

Because each η_i satisfies $(\eta 2)$, we get

$$\begin{aligned} \limsup_{n\to\infty} \frac{t_n + \overline{\eta_{(k)}}(t_n, s_n)}{s_n} &= \frac{1}{k}\limsup_{n\to\infty}\left(\sum_{i=1}^{k}\frac{t_n + \eta_i(t_n, s_n)}{s_n}\right) \\ &\le \frac{1}{k}\sum_{i=1}^{k}\left(\limsup_{n\to\infty}\frac{t_n + \eta_i(t_n, s_n)}{s_n}\right) \\ &< 1. \end{aligned} \tag{18}$$

Hence, for each $k \in \mathbb{N}$, the function $\overline{\eta_{(k)}}$ is a manageable function. □

3. New Existence Results for Manageable Functions and Approximate Fixed Point Property

Recall that a multivalued map $T : X \to \mathcal{CB}(X)$ is called

(1) a Nadler's type contraction (or a multivalued k-contraction [3, 33]), if there exists a number $0 < k < 1$ such that

$$\mathcal{H}(Tx, Ty) \le kd(x,y) \quad \forall x, y \in X; \tag{19}$$

(2) a Mizoguchi-Takahashi's type contraction [33], if there exists an $\mathcal{MT}$-function $\alpha : [0,\infty) \to [0,1)$ such that

$$\mathcal{H}(Tx, Ty) \le \alpha(d(x,y))\, d(x,y) \quad \forall x, y \in X; \tag{20}$$

(3) A multivalued (θ, L)-almost contraction [28, 29, 33], if there exist two constants $\theta \in (0,1)$ and $L \ge 0$ such that

$$\mathcal{H}(Tx, Ty) \le \theta d(x,y) + Ld(y, Tx) \quad \forall x, y \in X; \tag{21}$$

(4) a Berinde-Berinde's type contraction [33] (or a generalized multivalued almost contraction [28, 29, 33]), if there exists an $\mathcal{MT}$-function $\alpha : [0,\infty) \to [0,1)$ and $L \geq 0$ such that

$$\mathcal{H}(Tx,Ty) \leq \alpha(d(x,y))\, d(x,y) + L d(y,Tx) \quad \forall x,y \in X; \tag{22}$$

(5) a Du's strong type contraction, if there exist an $\mathcal{MT}$-function $\alpha : [0,\infty) \to [0,1)$ and a function $h : X \to [0,\infty)$ such that

$$\mathcal{H}(Tx,Ty) \leq \alpha(d(x,y))\, d(x,y) + h(y)\, d(y,Tx) \quad \forall x,y \in X; \tag{23}$$

(6) a Du's weak type contraction, if there exist an $\mathcal{MT}$-function $\alpha : [0,\infty) \to [0,1)$ and a function $h : X \to [0,\infty)$ such that

$$d(y,Ty) \leq \alpha(d(x,y))\, d(x,y) \quad \forall y \in Tx. \tag{24}$$

Definition 9 (see [36–39]). Let (X,d) be a metric space and let $T : X \to \mathcal{N}(X)$ be a multivalued map. One says that T is *α-admissible,* if there exists a function $\alpha : X \times X \to [0,+\infty)$ such that for each $x \in X$ and $y \in Tx$ with $\alpha(x,y) \geq 1$, one has $\alpha(y,z) \geq 1$ for all $z \in Ty$.

The following existence theorem is one of the main results of this paper.

Theorem 10. *Let (X,d) be a metric space, let $T : X \to \mathcal{N}(X)$ be an α-admissible multivalued map, and $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$. Let*

$$\Omega = \{(\alpha(x,y)\, d(y,Ty), d(x,y)) \in [0,+\infty) \times [0,+\infty) : x \in X, y \in Tx\}. \tag{25}$$

If $\eta(t,s) \geq 0$ for all $(t,s) \in \Omega$ and there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$, then the following statements hold.

(a) *There exists a Cauchy sequence $\{w_n\}_{n\in\mathbb{N}}$ in X such that*

(i) *$w_{n+1} \in Tw_n$ for all $n \in \mathbb{N}$,*
(ii) *$\alpha(w_n,w_{n+1}) \geq 1$ for all $n \in \mathbb{N}$,*
(iii) *$\lim_{n\to\infty} d(w_n,w_{n+1}) = \inf_{n\in\mathbb{N}} d(w_n,w_{n+1}) = 0$.*

(b) *$\inf_{x\in X} d(x,Tx) = 0$; that is, T has the approximate fixed point property on X.*

Proof. By our assumption, there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0,x_1) \geq 1$. If $x_1 = x_0$, then $x_0 \in Tx_0$ and

$$\inf_{x\in X} d(x,Tx) \leq d(x_0,Tx_0) \leq d(x_0,x_0) = 0, \tag{26}$$

which implies $\inf_{x\in X} d(x,Tx) = 0$. Let $w_n = x_0$ for all $n \in \mathbb{N}$. Then $\{w_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X and

$$\lim_{n\to\infty} d(w_n,w_{n+1}) = \inf_{n\in\mathbb{N}} d(w_n,w_{n+1}) = d(x_0,x_0) = 0. \tag{27}$$

Clearly, $\alpha(w_n,w_{n+1}) = \alpha(x_0,x_1) \geq 1$ for all $n \in \mathbb{N}$. Hence, the conclusions (a) and (b) hold in this case. Assume $x_1 \notin x_0$ or $d(x_0,x_1) > 0$. If $x_1 \in Tx_1$, then, following a similar argument as above, we can prove the conclusions (a) and (b) by taking a Cauchy sequence $\{w_n\}_{n\in\mathbb{N}}$ with $w_1 = x_0$ and $w_n = x_1$ for all $n \geq 2$. Suppose $x_1 \notin Tx_1$. Thus $d(x_1,Tx_1) > 0$. Define $\lambda : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by

$$\lambda(t,s) = \begin{cases} \dfrac{t+\eta(t,s)}{s}, & \text{if } (t,s) \in \Omega \setminus \{(0,0)\}, \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

By (η1), we know that

$$0 < \lambda(t,s) < 1 \quad \forall (t,s) \in \Omega \setminus \{(0,0)\}. \tag{29}$$

Since $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$ and $\eta(t,s) \geq 0$ for all $(t,s) \in \Omega$, we have

$$0 < t \leq s\lambda(t,s) \quad \forall (t,s) \in \Omega \setminus \{(0,0)\}. \tag{30}$$

Clearly, $(\alpha(x_0,x_1)d(x_1,Tx_1), d(x_0,x_1)) \in \Omega \setminus \{(0,0)\}$. So, by (29), we obtain

$$0 < \lambda(\alpha(x_0,x_1)\, d(x_1,Tx_1), d(x_0,x_1)) < 1. \tag{31}$$

Let

$$\epsilon_1 = \left(\frac{\alpha(x_0,x_1)}{\sqrt{\lambda(\alpha(x_0,x_1)\, d(x_1,Tx_1), d(x_0,x_1))}} - 1 \right) \times d(x_1,Tx_1). \tag{32}$$

Taking into account $\alpha(x_0,x_1) \geq 1$, $d(x_1,Tx_1) > 0$, and the last inequality, we get $\epsilon_1 > 0$. Since

$$d(x_1,Tx_1) < d(x_1,Tx_1) + \epsilon_1 = \frac{\alpha(x_0,x_1)}{\sqrt{\lambda(\alpha(x_0,x_1)\, d(x_1,Tx_1), d(x_0,x_1))}} \times d(x_1,Tx_1), \tag{33}$$

there exists $x_2 \in Tx_1$ such that $x_2 \neq x_1$ and

$$d(x_1,x_2) < \frac{\alpha(x_0,x_1)}{\sqrt{\lambda(\alpha(x_0,x_1)\, d(x_1,Tx_1), d(x_0,x_1))}} \times d(x_1,Tx_1). \tag{34}$$

If $x_2 \in Tx_2$, then the proof can be finished by a similar argument as above. Otherwise, we have $d(x_2,Tx_2) > 0$. Since T is α -admissible, we obtain $\alpha(x_1,x_2) \geq 1$. By taking

$$\epsilon_2 = \left(\frac{\alpha(x_1,x_2)}{\sqrt{\lambda(\alpha(x_1,x_2)\, d(x_2,Tx_2), d(x_1,x_2))}} - 1 \right) \times d(x_2,Tx_2), \tag{35}$$

then there exists $x_3 \in Tx_2$ with $x_3 \neq x_2$ such that

$$d(x_2, x_3) < \frac{\alpha(x_1, x_2)}{\sqrt{\lambda(\alpha(x_1, x_2) d(x_2, Tx_2), d(x_1, x_2))}} \times d(x_2, Tx_2). \tag{36}$$

By induction, if $x_{k-1}, x_k, x_{k+1} \in X$ is known satisfying $x_{k-1} \in Tx_k$, $x_{k+1} \in Tx_{k+2}$, $d(x_k, Tx_k) > 0$, $\alpha(x_{k-1}, x_k) \geq 1$, and

$$\begin{aligned} 0 &< d(x_k, x_{k+1}) \\ &< \frac{\alpha(x_{k-1}, x_k)}{\sqrt{\lambda(\alpha(x_{k-1}, x_k) d(x_k, Tx_k), d(x_{k-1}, x_k))}} \\ &\quad \times d(x_k, Tx_k), \quad k \in \mathbb{N}, \end{aligned} \tag{37}$$

then, by taking

$$\epsilon_k = \left(\frac{\alpha(x_{k-1}, x_k)}{\sqrt{\lambda(\alpha(x_{k-1}, x_k) d(x_k, Tx_k), d(x_{k-1}, x_k))}} - 1 \right) \times d(x_k, Tx_k), \tag{38}$$

one can obtain $x_{k+2} \in Tx_{k+1}$ with $x_{k+2} \neq x_{k+1}$ such that

$$\begin{aligned} &d(x_{k+1}, x_{k+2}) \\ &\quad < \frac{\alpha(x_k, x_{k+1})}{\sqrt{\lambda(\alpha(x_k, x_{k+1}) d(x_{k+1}, Tx_{k+1}), d(x_k, x_{k+1}))}} \\ &\quad \times d(x_{k+1}, Tx_{k+1}). \end{aligned} \tag{39}$$

Hence, by induction, we can establish sequences $\{x_n\}$ in X satisfying, for each $n \in \mathbb{N}$,

$$\begin{gathered} x_n \in Tx_{n-1}, \\ d(x_{n-1}, x_n) > 0, \\ d(x_n, Tx_n) > 0, \\ \alpha(x_{n-1}, x_n) \geq 1, \\ d(x_n, x_{n+1}) < \frac{\alpha(x_{n-1}, x_n)}{\sqrt{\lambda(\alpha(x_{n-1}, x_n) d(x_n, Tx_n), d(x_{n-1}, x_n))}} \\ \times d(x_n, Tx_n). \end{gathered} \tag{40}$$

By (30), we have

$$\begin{aligned} &\alpha(x_{n-1}, x_n) d(x_n, Tx_n) \\ &\quad \leq d(x_{n-1}, x_n) \lambda(\alpha(x_{n-1}, x_n) d(x_n, Tx_n), d(x_{n-1}, x_n)) \\ &\qquad \text{for each } n \in \mathbb{N}. \end{aligned} \tag{41}$$

Hence, for each $n \in \mathbb{N}$, by combining (40) and (41), we get

$$d(x_n, x_{n+1}) < \sqrt{\lambda(\alpha(x_{n-1}, x_n) d(x_n, Tx_n), d(x_{n-1}, x_n))} \times d(x_{n-1}, x_n), \tag{42}$$

which means that the sequence $\{d(x_{n-1}, x_n)\}_{n\in\mathbb{N}}$ is strictly decreasing in $(0, +\infty)$. So

$$\gamma := \lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) \geq 0 \text{ exists.} \tag{43}$$

By (41), we have

$$\alpha(x_{n-1}, x_n) d(x_n, Tx_n) \leq d(x_{n-1}, x_n) \quad \forall n \in \mathbb{N}, \tag{44}$$

which means that $\{\alpha(x_{n-1}, x_n) d(x_n, Tx_n)\}_{n\in\mathbb{N}}$ is a bounded sequence. By $(\eta 2)$, we have

$$\limsup_{n\to\infty} \lambda(\alpha(x_{n-1}, x_n) d(x_n, Tx_n), d(x_{n-1}, x_n)) < 1. \tag{45}$$

Now, we claim $\gamma = 0$. Suppose $\gamma > 0$. Then, by (45) and taking lim sup in (42), we get

$$\gamma \leq \sqrt{\limsup_{n\to\infty} \lambda(\alpha(x_{n-1}, x_n) d(x_n, Tx_n), d(x_{n-1}, x_n))}\gamma < \gamma, \tag{46}$$

a contradiction. Hence we prove

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} d(x_n, x_{n+1}) = 0. \tag{47}$$

To complete the proof of (a), it suffices to show that $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in X. For each $n \in \mathbb{N}$, let

$$\rho_n := \sqrt{\lambda(\alpha(x_{n-1}, x_n) d(x_n, Tx_n), d(x_{n-1}, x_n))}. \tag{48}$$

Then $\rho_n \in (0, 1)$ for all $n \in \mathbb{N}$. By (42), we obtain

$$d(x_n, x_{n+1}) < \rho_n d(x_{n-1}, x_n) \quad \forall n \in \mathbb{N}. \tag{49}$$

From (45), we have $\limsup_{n\to\infty} \rho_n < 1$, so there exist $c \in [0, 1)$ and $n_0 \in \mathbb{N}$, such that

$$\rho_n \leq c \quad \forall n \in \mathbb{N} \text{ with } n \geq n_0. \tag{50}$$

For any $n \geq n_0$, since $\rho_n \in (0,1)$ for all $n \in \mathbb{N}$ and $c \in [0,1)$, taking into account (49) and (50) concludes that

$$\begin{aligned} d(x_n, x_{n+1}) &< \rho_n d(x_{n-1}, x_n) \\ &< \cdots < \rho_n \rho_{n-1} \rho_{n-2} \cdots \rho_{n_0} d(x_0, x_1) \\ &\leq c^{n-n_0+1} d(x_0, x_1). \end{aligned} \tag{51}$$

Put $\alpha_n = (c^{n-n_0+1}/(1-c))\, d(x_0, x_1)$, $n \in \mathbb{N}$. For $m, n \in \mathbb{N}$ with $m > n \geq n_0$, we have from the last inequality that

$$d(x_n, x_m) \leq \sum_{j=n}^{m-1} d(x_j, x_{j+1}) < \alpha_n. \tag{52}$$

Since $c \in [0,1)$, $\lim_{n\to\infty} \alpha_n = 0$. Hence

$$\lim_{n\to\infty} \sup\{d(x_n, x_m) : m > n\} = 0. \tag{53}$$

So $\{x_n\}$ is a Cauchy sequence in X. Let $w_n = x_{n-1}$ for all $n \in \mathbb{N}$. Then $\{w_n\}_{n\in\mathbb{N}}$ is the desired Cauchy sequence in (a).

To see (b), since $x_n \in Tx_{n-1}$ for each $n \in \mathbb{N}$, we have

$$\inf_{x\in X} d(x, Tx) \leq d(x_n, Tx_n) \leq d(x_n, x_{n+1}) \quad \forall n \in \mathbb{N}. \tag{54}$$

Combining (47) and (54) yields

$$\inf_{x\in X} d(x, Tx) = 0. \tag{55}$$

The proof is completed. □

Applying Theorem 10, we can establish the following new existence theorem related to approximate fixed point property for α-admissible multivalued maps.

Theorem 11. *Let (X,d) be a metric space and let $T : X \to \mathcal{N}(X)$ be an α-admissible multivalued map. Suppose that there exists an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that*

$$\alpha(x,y)\, d(y, Ty) \leq \varphi(d(x,y))\, d(x,y) \quad \forall y \in Tx. \tag{56}$$

If there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$, then the following statements hold.

(a) *There exists $\eta \in \widehat{\mathrm{Man}}(\mathbb{R})$ such that $\eta(t,s) \geq 0$ for all $(t,s) \in \Omega$, where*

$$\begin{aligned} \Omega = \{&(\alpha(x,y)\, d(y,Ty), d(x,y)) \in [0,+\infty) \\ &\times [0,+\infty) : x \in X, y \in Tx\}. \end{aligned} \tag{57}$$

(b) *There exists a Cauchy sequence $\{w_n\}_{n\in\mathbb{N}}$ in X such that*

(i) *$w_{n+1} \in Tw_n$ for all $n \in \mathbb{N}$,*
(ii) *$\alpha(w_n, w_{n+1}) \geq 1$ for all $n \in \mathbb{N}$,*
(iii) *$\lim_{n\to\infty} d(w_n, w_{n+1}) = \inf_{n\in\mathbb{N}} d(w_n, w_{n+1}) = 0$.*

(c) *$\inf_{x\in X} d(x, Tx) = 0$; that is, T has the approximate fixed point property on X.*

Proof. Define $\eta : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by

$$\eta(t,s) = \begin{cases} s\varphi(s) - t, & \text{if } (t,s) \in [0,+\infty)\times[0,+\infty), \\ 0, & \text{otherwise.} \end{cases} \tag{58}$$

By Example C, we know $\eta \in \widehat{\mathrm{Man}}(\mathbb{R})$. By (56), we obtain $\eta(t,s) \geq 0$ for all $(t,s) \in \Omega$. Therefore (a) is proved. It is obvious that the desired conclusions (b) and (c) follow from Theorem 10 immediately. □

The following interesting results are immediate from Theorem 11.

Corollary 12. *Let (X,d) be a metric space and let $T : X \to \mathcal{CB}(X)$ be an α-admissible multivalued map. Assume that one of the following conditions holds.*

(L1) *there exist an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ and a function $h : X \to [0,\infty)$ such that*

$$\begin{aligned} \alpha(x,y)\, \mathcal{H}(Tx, Ty) \leq{}& \varphi(d(x,y))\, d(x,y) \\ &+ h(y)\, d(y, Tx) \quad \forall x, y \in X; \end{aligned} \tag{59}$$

(L2) *there exist an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ and $L \geq 0$ such that*

$$\begin{aligned} \alpha(x,y)\, \mathcal{H}(Tx, Ty) \leq{}& \varphi(d(x,y))\, d(x,y) \\ &+ L d(y, Tx) \quad \forall x, y \in X; \end{aligned} \tag{60}$$

(L3) *there exist two constants $\theta \in (0,1)$ and $L \geq 0$ such that*

$$\alpha(x,y)\, \mathcal{H}(Tx, Ty) \leq \theta d(x,y) + L d(y, Tx) \quad \forall x, y \in X; \tag{61}$$

(L4) *there exists an $\mathcal{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that*

$$\alpha(x,y)\, \mathcal{H}(Tx, Ty) \leq \varphi(d(x,y))\, d(x,y) \quad \forall x, y \in X; \tag{62}$$

(L5) *there exists a number $0 < k < 1$ such that*

$$\alpha(x,y)\, \mathcal{H}(Tx, Ty) \leq k d(x,y) \quad \forall x, y \in X. \tag{63}$$

If there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \geq 1$, then the following statements hold.

(a) *There exists $\eta \in \widehat{\mathrm{Man}}(\mathbb{R})$ such that $\eta(t,s) \geq 0$ for all $(t,s) \in \Omega$, where*

$$\begin{aligned} \Omega = \{&(\alpha(x,y)\, d(y,Ty), d(x,y)) \in [0,+\infty) \\ &\times [0,+\infty) : x \in X, y \in Tx\}. \end{aligned} \tag{64}$$

(b) *There exists a Cauchy sequence $\{w_n\}_{n\in\mathbb{N}}$ in X such that*

(i) *$w_{n+1} \in Tw_n$ for all $n \in \mathbb{N}$,*
(ii) *$\alpha(w_n, w_{n+1}) \geq 1$ for all $n \in \mathbb{N}$,*
(iii) *$\lim_{n\to\infty} d(w_n, w_{n+1}) = \inf_{n\in\mathbb{N}} d(w_n, w_{n+1}) = 0$.*

(c) $\inf_{x\in X} d(x,Tx) = 0$; *that is, T has the approximate fixed point property on X.*

Proof. It suffices to verify the conclusion under (L1). Note first that, for each $x \in X$, $d(y,Tx) = 0$ for all $y \in Tx$. So, for each $x \in X$, by (L1), we obtain

$$\alpha(x,y)\, d(y,Ty) \le \varphi(d(x,y))\, d(x,y) \quad \forall y \in Tx, \tag{65}$$

which means (56) holds. Therefore, the conclusion follows from Theorem 11. □

In Corollary 12, if we take $\alpha : X \times X \to [0,+\infty)$ by $\alpha(x,y) = 1$ for all $x, y \in X$, then we obtain the following existence theorem.

Corollary 13. *Let (X,d) be a metric space and let $T : X \to \mathscr{CB}(X)$ be a multivalued map. Assume that one of the following conditions holds.*

(1) *T is a Du's weak type contraction;*

(2) *T is a Du's strong type contraction;*

(3) *T is a Berinde-Berinde's type contraction;*

(4) *T is a multivalued (θ, L)-almost contraction;*

(5) *T is a Mizoguchi-Takahashi's type contraction;*

(6) *T is a Nadler's type contraction.*

Then the following statements hold.

(a) *There exists $\eta \in \widehat{\mathrm{Man}(\mathbb{R})}$ such that $\eta(t,s) \ge 0$ for all $(t,s) \in \mathscr{D}$, where*

$$\mathscr{D} = \{(d(y,Ty), d(x,y)) \in [0,+\infty) \times [0,+\infty) : x \in X, y \in Tx\}. \tag{66}$$

(b) *There exists a Cauchy sequence $\{w_n\}_{n\in\mathbb{N}}$ in X such that*

(i) *$w_{n+1} \in Tw_n$ for all $n \in \mathbb{N}$,*

(ii) $\lim_{n\to\infty} d(w_n, w_{n+1}) = \inf_{n\in\mathbb{N}} d(w_n, w_{n+1}) = 0$.

(c) $\inf_{x\in X} d(x,Tx) = 0$; *that is, T has the approximate fixed point property on X.*

4. Some Applications to Fixed Point Theory

Definition 14 (see [36–39]). Let (X,d) be a metric space and let $\alpha : X \times X \to [0,+\infty)$ be a function. α is said to *have* the property (B) if any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \ge 1$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} x_n = v$, we have $\alpha(x_n, v) \ge 1$ for all $n \in \mathbb{N}$.

Theorem 15. *Let (X,d) be a complete metric space and let $T : X \to \mathscr{CB}(X)$ be an α-admissible multivalued map. Suppose that there exists an $\mathscr{MT}$-function $\varphi : [0,\infty) \to [0,1)$ such that*

$$\alpha(x,y)\, d(y,Ty) \le \varphi(d(x,y))\, d(x,y) \quad \forall y \in Tx. \tag{67}$$

If there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$, and one of the following conditions is satisfied:

(H1) *T is $\mathscr{H}$-continuous (i.e., $x_n \to v$ implies $\mathscr{H}(Tx_n, Tv) \to 0$ as $n \to \infty$);*

(H2) *T is closed (i.e., $GrT := \{(x,y) \in X \times X : y \in Tx\}$; the graph of T is a closed subset of $X \times X$);*

(H3) *the map $g : X \to [0,\infty)$ defined by $g(x) = d(x,Tx)$ is l.s.c.;*

(H4) *for any sequence $\{z_n\}$ in X with $\alpha(z_n, z_{n+1}) \ge 1$, $z_{n+1} \in Tz_n$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} z_n = c$, one has $\lim_{n\to\infty} d(z_n, Tc) = 0$,*

then T admits a fixed point in X.

Proof. Applying Theorem 11, there exists a Cauchy sequence $\{w_n\}_{n\in\mathbb{N}}$ in X such that

$$\begin{gathered} w_{n+1} \in Tw_n, \\ \alpha(w_n, w_{n+1}) \ge 1 \quad \forall n \in \mathbb{N}. \end{gathered} \tag{68}$$

By the completeness of X, there exists $v \in X$ such that $w_n \to v$ as $n \to \infty$.

Now, we verify $v \in \mathscr{F}(T)$. If (H1) holds, since T is $\mathscr{H}$-continuous on X, $w_{n+1} \in Tw_n$ for each $n \in \mathbb{N}$, and $w_n \to v$ as $n \to \infty$, we get

$$d(v,Tv) = \lim_{n\to\infty} d(w_{n+1}, Tv) \le \lim_{n\to\infty} \mathscr{H}(Tw_n, Tv) = 0, \tag{69}$$

which implies $d(v,Tv) = 0$. By the closeness of Tv, we have $v \in Tv$. If (H2) holds, since T is closed, $w_{n+1} \in Tw_n$ for each $n \in \mathbb{N}$, and $w_n \to v$ as $n \to \infty$, we have $v \in \mathscr{F}(T)$. Suppose that (H3) holds. Since $\{w_n\}_{n\in\mathbb{N}}$ is convergent in X, we have

$$\lim_{n\to\infty} d(w_n, w_{n+1}) = 0. \tag{70}$$

Since

$$d(v,Tv) = g(v) \le \liminf_{n\to\infty} g(w_n) \le \lim_{n\to\infty} d(w_n, w_{n+1}) = 0, \tag{71}$$

we obtain $d(v,Tv) = 0$, and hence $v \in \mathscr{F}(T)$. Finally, assume (H4) holds. Then we obtain

$$d(v,Tv) = \lim_{n\to\infty} d(w_n, Tv) = 0. \tag{72}$$

Hence $v \in Tv$. Therefore, in any case, we prove $v \in \mathscr{F}(T)$. This completes the proof. □

Theorem 16. *Let (X,d) be a complete metric space and let $T : X \to \mathscr{CB}(X)$ be an α-admissible multivalued map. Suppose that there exist an $\mathscr{MT}$-function $\varphi : [0,\infty) \to [0,1)$ and a function $h : X \to [0,\infty)$ such that*

$$\begin{aligned} \alpha(x,y)\, \mathscr{H}(Tx,Ty) \le{}& \varphi(d(x,y))\, d(x,y) \\ &+ h(y)\, d(y,Tx) \quad \forall x, y \in X. \end{aligned} \tag{73}$$

If there exist $x_0 \in X$ and $x_1 \in Tx_0$ such that $\alpha(x_0, x_1) \ge 1$, and one of the following conditions is satisfied:

(S1) *T is $\mathscr{H}$-continuous;*

(S2) *T is closed;*

(S3) *the map $g : X \to [0, \infty)$ defined by $g(x) = d(x, Tx)$ is l.s.c.;*

(S4) *the function α has the property (B),*

then T admits a fixed point in X.

Proof. It is obvious that (73) implies (67). If one of the conditions (S1), (S2), and (S3) is satisfied, then the desired conclusion follows from Theorem 15 immediately. Suppose that (S4) holds. We claim that (H4) as in Theorem 15 is satisfied. Let $\{z_n\}$ be in X with $\alpha(z_n, z_{n+1}) \geq 1$, $z_{n+1} \in Tz_n$, $n \in \mathbb{N}$, and $\lim_{n\to\infty} z_n = c$. Since α has the property (B), $\alpha(z_n, c) \geq 1$ for all $n \in \mathbb{N}$. So, it follows from (73) that

$$\begin{aligned} \lim_{n\to\infty} d(z_{n+1}, Tc) &\leq \lim_{n\to\infty} \mathscr{H}(Tz_n, Tc) \\ &\leq \lim_{n\to\infty} \alpha(z_n, c)\, \mathscr{H}(Tz_n, Tc) \\ &\leq \lim_{n\to\infty} \{\varphi(d(z_n, c))\, d(z_n, c) \\ &\quad + h(c)\, d(c, z_{n+1})\} = 0, \end{aligned} \tag{74}$$

which implies $\lim_{n\to\infty} d(z_n, Tc) = 0$. Hence (H4) holds. By Theorem 15, we also prove $\mathscr{F}(T) \neq \emptyset$. The proof is completed. □

Applying Theorem 16, we can give a short proof of Du's fixed point theorem.

Corollary 17 (Du [[6]]). *Let (X, d) be a complete metric space, let $T : X \to \mathscr{CB}(X)$ be a multivalued map, let $\varphi : [0, \infty) \to [0, 1)$ be a $\mathscr{MT}$-function, let and $h : X \to [0, \infty)$ be a function. Assume that*

$$\begin{aligned} \mathscr{H}(Tx, Ty) &\leq \varphi(d(x, y))\, d(x, y) \\ &\quad + h(y)\, d(y, Tx) \quad \forall x, y \in X. \end{aligned} \tag{75}$$

Then $\mathscr{F}(T) \neq \emptyset$.

Proof. Take $\alpha : X \times X \to [0, +\infty)$ by $\alpha(x, y) = 1$ for all $x, y \in X$. Then (75) implies (73). Moreover, T is an α-admissible multivalued map and the function α has the property (B). Therefore the conclusion follows from Theorem 16. □

Remark 18. Theorems 15 and 16 and Corollary 17 all generalize and improve Berinde-Berinde's fixed point theorem, Mizoguchi-Takahashi's fixed point theorem, Nadler's fixed point theorem, and Banach contraction principle.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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References

[1] S. Banach, "Sur les opérations dans les ensembles abstraits et leurs applications aux équations integrales," *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

[2] S. B. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, pp. 475–488, 1969.

[3] N. Mizoguchi and W. Takahashi, "Fixed point theorems for multivalued mappings on complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 141, no. 1, pp. 177–188, 1989.

[4] S. Reich, "Some problems and results in fixed point theory," *Contemporary Mathematics*, vol. 21, pp. 179–187, 1983.

[5] M. Berinde and V. Berinde, "On a general class of multi-valued weakly Picard mappings," *Journal of Mathematical Analysis and Applications*, vol. 326, no. 2, pp. 772–782, 2007.

[6] W.-S. Du, "On coincidence point and fixed point theorems for nonlinear multivalued maps," *Topology and its Applications*, vol. 159, no. 1, pp. 49–56, 2012.

[7] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, Japan, 2000.

[8] A. Uderzo, "Fixed points for directional multi-valued $k(\cdot)$-contractions," *Journal of Global Optimization*, vol. 31, no. 3, pp. 455–469, 2005.

[9] A. Petruşel and A. Sîntămărian, "Single-valued and multi-valued Caristi type operators," *Publicationes Mathematicae Debrecen*, vol. 60, no. 1-2, pp. 167–177, 2002.

[10] Y. Feng and S. Liu, "Fixed point theorems for multi-valued contractive mappings and multi-valued Caristi type mappings," *Journal of Mathematical Analysis and Applications*, vol. 317, no. 1, pp. 103–112, 2006.

[11] L.-G. Huang and X. Zhang, "Cone metric spaces and fixed point theorems of contractive mappings," *Journal of Mathematical Analysis and Applications*, vol. 332, no. 2, pp. 1468–1476, 2007.

[12] W.-S. Du, "A note on cone metric fixed point theory and its equivalence," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 72, no. 5, pp. 2259–2261, 2010.

[13] W.-S. Du, "Some new results and generalizations in metric fixed point theory," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 73, no. 5, pp. 1439–1446, 2010.

[14] W.-S. Du, "Coupled fixed point theorems for nonlinear contractions satisfied Mizoguchi-Takahashi's condition in quasiordered metric spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 876372, 2010.

[15] W.-S. Du, "Nonlinear contractive conditions for coupled cone fixed point theorems," *Fixed Point Theory and Applications*, vol. 2010, Article ID 190606, 2010.

[16] W.-S. Du, "New cone fixed point theorems for nonlinear multivalued maps with their applications," *Applied Mathematics Letters*, vol. 24, no. 2, pp. 172–178, 2011.

[17] W.-S. Du and S.-X. Zheng, "Nonlinear conditions for coincidence point and fixed point theorems," *Taiwanese Journal of Mathematics*, vol. 16, no. 3, pp. 857–868, 2012.

[18] W.-S. Du, "On Caristi type maps and generalized distances with applications," *Abstract and Applied Analysis*, vol. 2013, Article ID 407219, 8 pages, 2013.

[19] W.-S. Du, E. Karapınar, and N. Shahzad, "The study of fixed point theory for various multivalued non-self-maps," *Abstract and Applied Analysis*, vol. 2013, Article ID 938724, 9 pages, 2013.

[20] I. J. Lin and T. H. Chen, "New existence theorems of coincidence points approach to generalizations of Mizoguchi-Takahashi's fixed point theorem," *Fixed Point Theory and Applications*, vol. 2012, article 156, 2012.

[21] F. Khojasteh, S. Shukla, and S. Radenovic, "A new approach to the study of fixed point theory for simulation function," *Filomat*. In press.

[22] A. A. Eldred and P. Veeramani, "Existence and convergence of best proximity points," *Journal of Mathematical Analysis and Applications*, vol. 323, no. 2, pp. 1001–1006, 2006.

[23] E. Karapinar, "Generalizations of Caristi Kirk's theorem on partial metric spaces," *Fixed Point Theory and Applications*, vol. 2011, article 4, 2011.

[24] M. Jleli, E. Karapinar, and B. Samet, "On best proximity points under the P-property on partially ordered metric spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 150970, 6 pages, 2013.

[25] E. Karapınar, "On best proximity point of ψ-Geraghty contractions," *Fixed Point Theory and Applications*, vol. 2013, article 200, 2013.

[26] M. Jleli, E. Karapınar, and B. Samet, "Best proximity points for generalized α-ψ-proximal contractive type mappings," *Journal of Applied Mathematics*, vol. 2013, Article ID 534127, 10 pages, 2013.

[27] W.-S. Du and H. Lakzian, "Nonlinear conditions for the existence of best proximity points," *Journal of Inequalities and Applications*, vol. 2012, article 206, 2012.

[28] W.-S. Du and E. Karapinar, "A note on Caristi-type cyclic maps: related results and applications," *Fixed Point Theory and Applications*, vol. 2013, article 344, 2013.

[29] N. Hussain, A. Amini-Harandi, and Y. J. Cho, "Approximate endpoints for set-valued contractions in metric spaces," *Fixed Point Theory and Applications*, vol. 2010, Article ID 614867, 13 pages, 2010.

[30] M. A. Khamsi, "On asymptotically nonexpansive mappings in hyperconvex metric spaces," *Proceedings of the American Mathematical Society*, vol. 132, no. 2, pp. 365–373, 2004.

[31] W.-S. Du, "On approximate coincidence point properties and their applications to fixed point theory," *Journal of Applied Mathematics*, vol. 2012, Article ID 302830, 17 pages, 2012.

[32] W.-S. Du, Z. He, and Y. L. Chen, "New existence theorems for approximate coincidence point property and approximate fixed point property with applications to metric fixed point theory," *Journal of Nonlinear and Convex Analysis*, vol. 13, no. 3, pp. 459–474, 2012.

[33] W.-S. Du, "On generalized weakly directional contractions and approximate fixed point property with applications," *Fixed Point Theory and Applications*, vol. 2012, article 6, 2012.

[34] W.-S. Du, "New existence results and generalizations for coincidence points and fixed points without global completeness," *Abstract and Applied Analysis*, vol. 2013, Article ID 214230, 12 pages, 2013.

[35] B. Samet, C. Vetro, and P. Vetro, "Fixed point theorems for α-ψ-contractive type mappings," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 75, no. 4, pp. 2154–2165, 2012.

[36] J. H. Asl, S. Rezapour, and N. Shahzad, "On fixed points of α-ψ-contractive multifunctions," *Fixed Point Theory and Applications*, vol. 2012, article 212, 2012.

[37] B. Mohammadi, S. Rezapour, and N. Shahzad, "Some results on fixed points of α-ψ-Ciric generalized multifunctions," *Fixed Point Theory and Applications*, vol. 2013, article 24, 2013.

[38] H. Alikhani, Sh. Rezapour, and N. Shahzad, "Fixed points of a new type of contractive mappings and multifunctions," *Filomat*, vol. 27, no. 7, pp. 1319–1315, 2013.

[39] H. Alikhani, V. Rakocevic, Sh. Rezapour, and N. Shahzad, "Fixed points of proximinal valued β-ψ-contractive multifunctions," *Journal of Nonlinear and Convex Analysis*. In press.

[40] H.-K. Xu, "Metric fixed point theory for multivalued mappings," *Dissertationes Mathematicae*, vol. 389, 39 pages, 2000.