Research Article

Convergence Theorems for Right Bregman Strongly Nonexpansive Mappings in Reflexive Banach Spaces

H. Zegeye¹ and N. Shahzad²

¹ Department of Mathematics, University of Botswana, Private Bag 00704, Gaborone, Botswana
² Department of Mathematics, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia

Correspondence should be addressed to N. Shahzad; nshahzad@kau.edu.sa

Received 21 February 2014; Accepted 7 May 2014; Published 27 May 2014

Academic Editor: Rudong Chen

Copyright © 2014 H. Zegeye and N. Shahzad. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We prove a strong convergence theorem for a common fixed point of a finite family of right Bregman strongly nonexpansive mappings in the framework of real reflexive Banach spaces. Furthermore, we apply our method to approximate a common zero of a finite family of maximal monotone mappings and a solution of a finite family of convex feasibility problems in reflexive real Banach spaces. Our theorems complement some recent results that have been proved for this important class of nonlinear mappings.

1. Introduction

In this paper, without other specifications, let E be a real reflexive Banach space and $E^*$ as its dual, let $\mathbb{R}$ be the set of real numbers, and let $C$ be a nonempty, closed, and convex subset of E. Let $f : E \rightarrow (-\infty, \infty]$ be a proper convex and lower semicontinuous function. Denote the domain of f by $\text{dom } f$; that is, $\text{dom } f = \{x \in E : f(x) < \infty\}$. The Fenchel conjugate of f is the function $f^* : E^* \rightarrow (-\infty, +\infty]$ defined by $f^*(y) = \sup\{\langle y, x \rangle - f(x) : x \in E\}$. f is called cofinite if $\text{dom } f^* = E^*$. For any $x \in \text{int(dom } f)$ and $y \in E$, the right-hand derivative of f at x in the direction of y is defined by $f^0(x, y) := \lim_{t \to 0^+} \frac{f(x + ty) - f(x)}{t}$.

The function f is called Gâteaux differentiable at x if $\lim_{t \to 0} f^0(x, y) = \frac{f(x + ty) - f(x)}{t}$ exists for any y. In this case, $f^0(x, y)$ coincides with $\nabla f(x)$, the value of the gradient $\nabla f$ of f at x. The function f is called Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \text{int(dom } f)$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$ and f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

The function f is said to be bounded if it maps bounded subsets of E into bounded sets. We note that if $f : E \rightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded, then $\nabla f$ is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of $E^*$ (Proposition 2.1, [1]) and $f^*$ is uniformly Fréchet on bounded subsets of $E^*$ (see [2]) and hence $\nabla f^*$ is uniformly continuous on bounded subsets of $E^*$ from the strong topology of $E^*$ to the strong topology of E.

Let $f : E \rightarrow (-\infty, +\infty]$ be a Gâteaux differentiable function. The function $D_f : \text{dom } f \times \text{int(dom } f) \rightarrow [0, +\infty)$ defined by

$$D_f(x, y) = f(x) - f(y) - \langle \nabla f(y), x - y \rangle$$

is called the Bregman distance with respect to f [3].

A Bregman projection [4] of $x \in \text{int(dom } f)$ onto the nonempty closed and convex set $C \subset \text{int(dom } f)$ is the unique vector $P_C^f(x) \in C$ satisfying

$$D_f(P_C^f(x), x) = \inf \{D_f(y, x) : y \in C\}.$$  

Remark 1. If E is a smooth and strictly convex Banach space and $f(x) = \|x\|^2$ for all $x \in E$, then we have that $\nabla f(x) = 2x$ for all $x \in E$, where f is the normalized duality mapping from E into $2^{E^*}$, and hence

(i) $D_f(x, y)$ reduces to $\phi(x, y) = \|x\|^2 - 2\langle x, y \rangle + \|y\|^2$, for all $x, y \in E$, which is the Lyapunov function introduced by Alber [5] and
\( P_f C(x) \) reduces to the generalized projection \( \Pi_C(x) \) (see, e.g., [5]) which is defined by
\[
\phi(\Pi_C(x), x) = \min_{y \in C} \phi(y, x).
\]

If \( E = H \), a Hilbert space, \( J \) is the identity mapping and hence the Bregman distance becomes \( D_f(x, y) = \|x - y\|^2 \),
for \( x, y \in H \), and the Bregman projection \( P_f^C(x) \) reduces to the metric projection of \( H \) onto \( C \), \( P_C(x) \).

Let \( T : C \rightarrow C \) be a nonlinear mapping. Denote by \( F(T) = \{ x \in C : Tx = x \} \) the set of fixed points of \( T \). A mapping \( T \) is said to be nonexpansive if \( \|Tx - Ty\| \leq \|x - y\| \), for all \( x, y \in C \), and \( T \) is called quasi-Bregman nonexpansive if \( \|Tx - p\| \leq \|x - p\| \), for all \( x \in C \) and \( p \in F(T) \). A point \( p \in C \) is called an asymptotic fixed point of \( T \) (see [6]) if \( C \) contains a sequence \( \{x_n\} \) which converges weakly to \( p \) such that \( \lim_{n \to \infty} \|x_n - Tx_n\| = 0 \). We denote by \( \bar{F}(T) \) the set of asymptotic fixed points of \( T \).

A mapping \( T : C \rightarrow \text{int}(\text{dom } f) \) is called

(i) left quasi-Bregman nonexpansive [7] if \( F(T) \neq \emptyset \) and
\[
D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, \ p \in F(T);
\]

(ii) left Bregman relatively nonexpansive [7] if \( F(T) \neq \emptyset \) and
\[
D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, \ p \in F(T), \ \bar{F}(T) = F(T);
\]

(iii) left Bregman strongly nonexpansive (see [8, 9]), with respect to nonempty \( \bar{F}(T) \), if
\[
D_f(p, Tx) \leq D_f(p, x), \quad \forall x \in C, \ p \in \bar{F}(T);
\]

and, if, whenever \( \{x_n\} \subset C \) is bounded, \( p \in \bar{F}(T) \) and
\[
\lim_{n \to \infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0;
\]

it follows that
\[
\lim_{n \to \infty} D_f(Tx_n, x_n) = 0;
\]

(iv) left Bregman firmly nonexpansive [10] if \( F(T) \neq \emptyset \) and for all \( x, y \in C \),
\[
\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty \rangle \leq \langle \nabla f(x) - \nabla f(y), Tx - Ty \rangle,
\]
or, equivalently,
\[
D_f(Tx, Ty) + D_f(Ty, Tx) + D_f(Tx, x) + D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x).
\]

If \( T \) is left Bregman firmly nonexpansive and \( f \) is Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of \( E \), then it is known in [10] that \( F(T) = \bar{F}(T) \) and \( F(T) \) is closed and convex (see [10]). It follows that every left Bregman firmly nonexpansive mapping is Bregman strongly nonexpansive with respect to a nonempty set \( F(T) = \bar{F}(T) \).

Existence and approximation of fixed points of nonexpansive and quasinonexpansive mappings have been intensively studied for almost fifty years or so by various authors (see, e.g., [11–24] and the references therein) in Hilbert spaces. But most of the methods failed to give the same conclusion in Banach spaces more general than Hilbert spaces. One of the reasons is that a nonexpansive mapping in Hilbert spaces may not be nonexpansive in Banach spaces (e.g., the resolvent \( R_\delta = (I + A)^{-1} \) of a maximal monotone mapping \( A : H \rightarrow 2^H \) and the metric projection \( P_K \) onto a nonempty, closed, and convex subset \( C \) of \( H \)).

To overcome this problem, researchers use the distance function \( D_f(\cdot, \cdot) \) introduced by Bregman [4] instead of norm which opened a growing area of research in designing and analyzing iterative techniques for solving variational inequalities, approximating equilibria, computing fixed points of nonlinear mappings, and approximating solutions of convex feasibility problems (see, e.g., [4, 25–28] and the references therein).

In [29], Reich and Sabach proposed the following algorithm for finding a common fixed point of finitely many left Bregman firmly nonexpansive self-mappings \( T_i \) \((i = 1, 2, \ldots, N)\) on \( E \) satisfying \( \cap_{i=1}^N F(T_i) \neq \emptyset \). For \( x_1 \in E \) let the sequence \( \{x_n\} \) be defined by
\[
Q_0 = E,
\]
\[
y_n = T_1(x_n + e_n),
\]
\[
Q_{m+1} = \{ z \in Q_m : \langle \nabla f(x_n + e_n) - \nabla f(y_n), z - y_n \rangle \leq 0 \},
\]
\[
Q_n = \bigcap_{i=1}^N Q_i,
\]
\[
x_{n+1} = P_f^C(x_n), \quad \forall n \geq 1.
\]
(11)

They proved that, under some suitable conditions, the sequence \( \{x_n\} \) generated by (11) converges strongly to a point in \( \cap_{i=1}^N F(T_i) \) and applied it to the solution of convex feasibility and equilibrium problems.

Very recently, by using Bregman projection, Reich and Sabach [9] proposed an algorithm for finding a common fixed point of finitely many left Bregman strongly nonexpansive mappings \( T_i : C \rightarrow C \) \((i = 1, 2, \ldots, N)\) satisfying \( \cap_{i=1}^N F(T_i) \neq \emptyset \) in a reflexive Banach space \( E \) as follows:
\[
x_0 \in E, \quad \text{chosen arbitrarily},
\]
\[
y_n = T_1(x_n + e_n),
\]
\[
C_n = \{ z \in E : D_f(z, y_n) \leq D_f(z, x_n + e_n) \},
\]

and if \( \{x_n\} \subset C \) is bounded, \( p \in \bar{F}(T) \) and
\[
\lim_{n \to \infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0;
\]

it follows that
\[
\lim_{n \to \infty} D_f(Tx_n, x_n) = 0;
\]

and, if, whenever \( \{x_n\} \subset C \) is bounded, \( p \in \bar{F}(T) \) and
\[
\lim_{n \to \infty} (D_f(p, x_n) - D_f(p, Tx_n)) = 0;
\]

it follows that
\[
\lim_{n \to \infty} D_f(Tx_n, x_n) = 0;
\]
\[ C_n = \bigcap_{i=1}^{N} C_i, \]
\[ Q_i = \{ z \in E : \langle \nabla f(x) - \nabla f(x_n), z - x_n \rangle \leq 0 \}, \]
\[ x_{n+1} = P_{C_n \cap Q_n}(x_0), \] (see [35], Corollary 5.5); if \( f \) is Legendre, then \( \nabla f \) is a bijection satisfying \( \nabla f = (\nabla f^*)^{-1} \), and \( \nabla f = \text{dom} \nabla f^* = \text{int} \text{dom} f^* \).

Remark 2. It is shown in [10] that if \( T \) is right Bregman strongly nonexpansive, then \( \tilde{F}(T) = F(T) \) and hence it is right Bregman relatively nonexpansive mapping provided that the Legendre function \( f \) is uniformly Fréchet differentiable and bounded on bounded sets of \( E \).

The class of right Bregman firmly nonexpansive mappings associated with the Bregman distance induced by a convex function was introduced and studied by Martin-Marques et al. [30]. Examples of right Bregman firmly nonexpansive mappings are given in [30]. If \( C \) is a nonempty and closed subset of \( \text{int}(\text{dom} f) \), where \( f \) is a Legendre and Fréchet differentiable function, and \( T : C \to \text{int}(\text{dom} f) \) is a right Bregman strongly nonexpansive mapping, it is proved that \( F(T) \) is closed (see [30]). In addition, they have shown that this class of mappings is closed under composition and convex combination and proved weak convergence of the Picard iterative method to a fixed point of a mapping under suitable conditions (see [31]). However, Picard iteration process has only weak convergence.

In this paper, it is our purpose to introduce an iterative scheme which converges strongly to a common fixed point of a finite family of right Bregman strongly nonexpansive mappings. As a consequence, we use our results to approximate a common zero of a finite family of right Bregman strongly nonexpansive mappings. Our results complements the recent results due to Reich and Sabach [9], Suantai et al. [32], and Zhang and Cheng [33] in the sense that our scheme is applicable for right Bregman strongly nonexpansive self-mappings on \( C \subseteq E \).

2. Preliminaries

Let \( f : E \to (-\infty, +\infty] \) be a convex and Gâteaux differentiable function. The modulun of total convexity of \( f \) at \( x \in \text{dom} f \) is the function \( \nu_f(x, \cdot) : [0, +\infty) \to [0, +\infty) \) defined by
\[ \nu_f(x, t) := \inf \{ D_f(y,x) : y \in \text{dom} f, \|y-x\| = t \}. \] (20)

The function \( f \) is called totally convex at \( x \) if \( \nu_f(x, t) > 0 \), whenever \( t > 0 \). The function \( f \) is called totally convex if it is totally convex at any point \( x \in \text{int}(\text{dom} f) \) and is said to be totally convex on bounded sets if \( \nu_f(B, t) > 0 \) for any nonempty bounded subset \( B \) of \( E \) and \( t > 0 \), where the modulus of total convexity of the function \( f \) on the set \( B \) is the function \( \nu_f : \text{int}(\text{dom} f) \times [0, +\infty) \to [0, +\infty) \) defined by
\[ \nu_f(B, t) := \inf \{ \nu_f(x, t) : x \in B \cap \text{dom} f \}. \] (21)

We know that \( f \) is totally convex on bounded sets if and only if \( f \) is uniformly convex on bounded sets (see [27], Theorem 2.10).

The function \( f \) is called essentially smooth, if \( \partial f \) is both locally bounded and single-valued on its domain and it is called essentially strictly convex, if \( (\partial f)^{-1} \) is locally bounded on its domain and \( f \) is strictly convex on every convex subset of \( \text{dom} \partial f \). \( f \) is said to be Legendre, if it is both essentially smooth and essentially strictly convex. Since \( E \) is reflexive, we know that \( (\partial f)^{-1} = \partial f^* \) (see [34]), \( f \) is essentially smooth if and only if \( f^* \) is essentially strictly convex (see [35], Theorem 5.4), and \( f \) is Legendre if and only if \( f^* \) is Legendre (see [35], Corollary 5.3); if \( f \) is Legendre, then \( \nabla f \) is a bijection satisfying \( \nabla f = (\nabla f^*)^{-1} \), ran \( \nabla f = \text{dom} \nabla f^* = \text{int} \text{dom} f^* \),
and ran $\nabla f^* = \text{dom } f = \text{int } \text{dom } f$ (see [35], Theorem 5.10). From now on, we assume that the convex function $f : E \to (-\infty, +\infty]$ is Legendre.

If $E$ is a smooth and strictly convex Banach space, then an important and interesting Legendre function is $f(x) := (1/p)\|x\|^p$ $(1 < p < \infty)$. In this case, the gradient $\nabla f$ of $f$ coincides with the generalized duality mapping of $E$; that is, $\nabla f = J_p$ $(1 < p < \infty)$. In particular, $\nabla f = I$, the identity mapping in Hilbert spaces.

In the sequel, we shall use the following lemmas.

**Lemma 3** (see [31]). Let $f : E \to \mathbb{R}$ be a bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of $E$. For each $i = 1, \ldots, N$, let $T_i : K \subseteq E \to K$ be a right Bregman strongly nonexpansive mapping with respect to $\mathcal{F}(T_i) = F(T_i)$, and let $T := T_N \circ T_{N-1} \circ \cdots \circ T_1$. If $\mathcal{F} := \cap_{i=1}^N F(T_i)$ is nonempty, then $T$ is also right Bregman strongly nonexpansive and $F(T) = \cap_{i=1}^N F(T_i)$.

**Lemma 4** (see [30]). Let $f : E \to \mathbb{R}$ be a Fréchet differentiable function. Let $C$ be a nonempty closed convex subset of $\text{int} \text{dom } f$ and let $T : C \to \text{int} \text{dom } f$ be a right quasi-Bregman nonexpansive mapping. Then $F(T)$ is closed.

**Lemma 5** (see [36]). The function $f : E \to (-\infty, +\infty)$ is totally convex on bounded subsets of $E$ if and only if for any two sequences $\{x_n\}$ and $\{y_n\}$ in $\text{int} \text{dom } f$ and $\text{dom } f$, respectively, such that the first one is bounded and

$$
\lim_{n \to \infty} D_f(y_n, x_n) = 0 \implies \lim_{n \to \infty} \|y_n - x_n\| = 0.
$$

**Lemma 6** (see [27]). Let $C$ be a nonempty, closed, and convex subset of $E$. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$. Then

(i) $z = P_C^f(x)$ if and only if $(\nabla f(x) - \nabla f(z), y - z) \leq 0$, $\forall y \in C$;

(ii) $D_f(y, P_C^f(x)) + D_f(P_C^f(x), y) \geq D_f(y, x)$, $\forall y \in C$.

**Lemma 7** (see [37]). If $f : E \to (-\infty, +\infty]$ is proper, lower semicontinuous, and convex function, then $f^* : E^* \to (-\infty, +\infty]$ is proper, weak* lower semicontinuous and convex function. Thus, for all $z, x \in E$, we have

$$
D_f\left(z, \nabla f^*\left(\sum_{i=1}^N l_i \nabla f^* (x_i)\right)\right) \leq \sum_{i=1}^N l_i D_f(z, x_i).
$$

**Lemma 8** (see [31]). Let $f : E \to \mathbb{R}$ be admissible and totally bounded at a point $x \in \text{int} \text{dom } f$. Let $\{x_n\} \subseteq \text{dom } f$. If $\{D_f(x_n, x)\}$ is bounded, then so is the sequence $\{x_n\}$.

Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable function. Following [3, 5], we make use of the function $V_f : E \times E^* \to [0, +\infty)$ associated with $f$, which is defined by

$$
V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad \forall x \in E, \; x^* \in E^*.
$$

Then, $V_f$ is nonnegative and

$$
V_f(x, x^*) = D_f(x, \nabla f^*(x^*)) \quad \forall x \in E, \; x^* \in E^*.
$$

Moreover, by the subdifferential inequality,

$$
V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x \rangle \leq V_f(x, x^* + y^*), \quad (26)
$$

for all $x \in E$ and $x^*, y^* \in E^*$ (see [38]).

**Lemma 9** (see [39]). Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:

$$
a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n \delta_n, \quad n \geq n_0, \quad (27)
$$

where $\{\alpha_n\} \subseteq (0, 1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfying the following conditions: $\lim_{n \to \infty} \alpha_n = 0$, $\sum_{n=1}^\infty \alpha_n = \infty$, and $\limsup_{n \to \infty} \delta_n \leq 0$.

Then, $\lim_{n \to \infty} \alpha_n = 0$.

**Lemma 10** (see [40]). Let $\{a_n\}$ be sequences of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_n < a_{n+1}$ for all $i \in \mathbb{N}$. Then there exists an increasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:

$$
a_{m_k} \leq a_{m_k+1}, \quad a_k \leq a_{m_k+1}. \quad (28)
$$

In fact, $m_k$ is the largest number $n$ in the set $\{1, 2, \ldots, k\}$ such that the condition $a_n \leq a_{m_k+1}$ holds.

### 3. Main Results

**Theorem 11.** Let $f : E \to \mathbb{R}$ be a cofinite function which is bounded, uniformly Fréchet differentiable and totally convex on bounded subsets of $E$. Let $C$ be a nonempty, closed, and convex subset of $\text{int} \text{dom } f$ and let $T_i : C \to C$, for $i = 1, 2, \ldots, N$, be a finite family of right Bregman strongly nonexpansive mappings such that $F(T_i) = F(T_i)$, for each $i \in \{1, 2, \ldots, N\}$. Assume that $\mathcal{F} := \cap_{i=1}^N F(T_i)$ is nonempty. For $u, x_1 \in C$, let $\{x_n\}$ be a sequence generated by

$$
x_{n+1} = \alpha_n u + (1 - \alpha_n) T(x_n), \quad n = 1, 2, \ldots, (29)
$$

where $T = T_N \circ T_{N-1} \circ \cdots \circ T_1$, $\{\alpha_n\} \subset (0, 1)$ satisfy $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=1}^\infty \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to a point $\tilde{x} \in \mathcal{F}$.

**Proof.** Note that from Lemma 3 we have $\mathcal{F} = F(T) = \cap_{i=1}^N F(T_i)$ and $T$ is right Bregman strongly nonexpansive mapping. Let $p \in \mathcal{F}$. Then, using (29), the convexity of $f$, and property of $T$ we get that

$$
D_f(x_{n+1}, p) = D_f(\alpha_n u + (1 - \alpha_n) T x_n, p) \leq \alpha_n D_f(u, p) + (1 - \alpha_n) D_f(T x_n, p) \quad (30)
$$

$$
\leq \alpha_n D_f(u, p) + (1 - \alpha_n) D_f(x_n, p).$$

Thus, by induction we obtain that

$$
D_f(x_{n+1}, p) \leq \max \{D_f(u, p), D_f(x_1, p)\} \quad \forall n \geq 1, \quad (31)
$$

which implies that $\{D_f(x_n, p)\}$ and hence $D_f(T x_n, p)$ are bounded. Thus, from Lemma 8 we get that $\{x_n\}$ and $\{T x_n\}$ are
bounded. Now, let \( y_n = \nabla f(x_n) \). Then, iteration process (29) becomes
\[
y_{n+1} = \nabla f(\alpha_n \nabla f^* (\nabla f (u)) + (1 - \alpha_n) \nabla f^* (T^* y_n)), \tag{32}
\]
where \( T^* := \nabla f T \nabla f^* \), a conjugate of \( T \). Since \( \nabla f \) and \( \nabla f^* \) are uniformly continuous on bounded subsets of \( \text{int}(\text{dom} f) \) and \( \text{int}(\text{dom} f^*) \), respectively, we get that \( \{y_n\} \) and \( \{T^* y_n\} \) are bounded and by Section 6 of Martin-Marquez et al. [31] we have that \( T^* \) is left Bergman strongly nonexpansive with respect to \( \nabla f(F(T)) \). In addition, by Proposition 3.3 of [30] we have that \( \nabla f(F(T)) = F(T^*) = F(\mathcal{T}) \) is closed and convex. Let \( p' = p'_{\nabla f(F(u))} \). Now, from (32), (25), (26), and Lemma 6 we obtain that
\[
D_f^* \left( p', y_{n+1} \right) = D_f^* \left( p', \nabla f (\alpha_n \nabla f^* (\nabla f (u)) + (1 - \alpha_n) \nabla f^* (T^* y_n)) \right)
= V_f \left( p', \alpha_n \nabla f^* (\nabla f (u)) + (1 - \alpha_n) \nabla f^* (T^* y_n) \right)
\leq V_f \left( p', \alpha_n \nabla f^* (\nabla f (u)) + (1 - \alpha_n) \nabla f^* (T^* y_n) \right)
\leq \alpha_n \left( \nabla f^* (\nabla f (u)) - \nabla f^* (p') \right), \tag{33}
\]
Now, we consider two cases.

Case 1. Suppose that there exists \( n_0 \in \mathbb{N} \) such that \( \{D_f^* (p', y_n)\} \) is decreasing for all \( n \geq n_0 \). Then, we get that \( \{D_f^* (p', y_n)\} \) is convergent and hence
\[
D_f^* (p', y_n) - D_f^* (p', y_{n+1}) \to 0 \quad \text{as} \quad n \to \infty. \tag{34}
\]
In addition, from (32) and Lemma 7 we have that
\[
D_f^* (p', y_{n+1})
= D_f^* \left( p', \nabla f (\alpha_n \nabla f^* (\nabla f (u)) + (1 - \alpha_n) \nabla f^* (T^* y_n)) \right)
\leq \alpha_n D_f^* (p', y_{n+1}) + (1 - \alpha_n) D_f^* (p', T^* y_n). \tag{35}
\]
Following from (35), (34), and the fact that \( \alpha_n \to 0 \), as \( n \to \infty \), we get that
\[
D_f^* (p', y_n) - D_f^* (p', T^* y_n)
= D_f^* \left( p', y_n \right) - D_f^* \left( p', y_{n+1} \right)
+ D_f^* \left( p', y_{n+1} \right) - D_f^* (p', T^* y_n)
\leq D_f^* (p', y_n) - D_f^* (p', y_{n+1})
+ \alpha_n \left( D_f^* (p', y_{n+1}) - D_f^* (p', T^* y_n) \right) \to 0 \quad \text{as} \quad n \to \infty.
\]
This with the fact that \( T^* \) is left Bergman strongly nonexpansive implies that
\[
\lim_{n \to \infty} D_f^* (T^* y_n, y_n) = 0. \tag{37}
\]
Then, by Lemma 5 we obtain that
\[
\lim_{n \to \infty} \|T^* y_n - y_n\| = 0. \tag{38}
\]
Now, since \( E^* \) is reflexive and \( \{y_{n+1}\} \) is bounded, there exists a subsequence \( \{y_{n_{k+1}}\} \) of \( \{y_{n+1}\} \) such that
\[
y_{n_{k+1}} \to y \in E^*, \tag{39}
\]
\[
\limsup_{k \to \infty} \langle \nabla f^* (\nabla f (u)) - \nabla f^* (p'), y_{n_{k+1}} - p' \rangle
= \limsup_{k \to \infty} \langle \nabla f^* (\nabla f (u)) - \nabla f^* (p'), y_{n_{k+1}} - p' \rangle. \tag{40}
\]
Thus, from (39), (38), the fact that \( T^* \) is left Bergman strongly nonexpansive mapping with \( F(T^*) = F(T^*) \), and Lemma 6 we get that \( y \in F(T^*) = \mathcal{F} \) and
\[
\limsup_{n \to \infty} \langle \nabla f^* (\nabla f (u)) - \nabla f^* (p'), y_{n+1} - p' \rangle
= \limsup_{k \to \infty} \langle \nabla f^* (\nabla f (u)) - \nabla f^* (p'), y_{n_{k+1}} - p' \rangle. \tag{41}
\]
Therefore, it follows from (33), (41), and Lemma 9 that \( D_f^* (p', y_n) \to 0 \) as \( n \to \infty \). Consequently, by Lemma 5 we obtain that \( y_n \to p' = p'_{\nabla f(F(u))} \) and hence \( x_n = \nabla f^* (y_n) \to \nabla f^* (p') = p \in \mathcal{F} \).

Case 2. Suppose that there exists a subsequence \( \{n_k\} \) of \( \{n\} \) such that
\[
D_f^* (p', y_n) < D_f^* (p', y_{n+1}), \tag{42}
\]
for all \( i \in \mathbb{N} \). Then, by Lemma 10, there exist a nondecreasing sequence \( \{m_k\} \subset \mathbb{N} \) such that \( m_k \to \infty \) and
\[
\begin{align*}
D_{f}^* \left( p', y_{m_k} \right) & \leq D_{f}^* \left( p', y_{m_{k+1}} \right), \\
D_{f} \left( p', y_k \right) & \leq D_{f} \left( p', y_{m_{k+1}} \right),
\end{align*}
\] (43)
for all \( k \in \mathbb{N} \). Thus, we get that
\[
\begin{align*}
D_{f}^* \left( p', y_{m_k} \right) & \leq D_{f}^* \left( p', T^* y_{m_{k+1}} \right) \\
& \leq D_{f} \left( p', y_k \right) - D_{f} \left( p', y_{m_{k+1}} \right) \\
& \leq D_{f} \left( p', y_{m_k} \right) - D_{f} \left( p', y_{m_{k+1}} \right) \\
& \leq D_{f} \left( p', y_{m_k} \right) - D_{f} \left( p', y_{m_{k+1}} \right) \\
& \leq D_{f} \left( p', T^* y_{m_{k+1}} \right) + \alpha_{m_k} \left( D_{f} \left( p', \nabla f \left( u \right) \right) - D_{f} \left( p', T^* y_{m_{k+1}} \right) \right) \to 0.
\end{align*}
\] (44)

This implies that \( D_{f} \left( T^* y_{m_k}, y_{m_{k+1}} \right) \to 0 \) as \( k \to \infty \). Now, following the method in Case 1 we obtain that
\[
\limsup_{k \to \infty} \left\langle \nabla f^* \left( \nabla f \left( u \right) \right) - \nabla f^* \left( p' \right), y_{m_{k+1}} - p' \right\rangle \leq 0.
\] (45)

Now, from (33) we have that
\[
\begin{align*}
D_{f}^* \left( p', y_{m_{k+1}} \right) & \leq \left( 1 - \alpha_{m_k} \right) \left[ D_{f}^* \left( p', y_{m_k} \right) \\
& \quad + \alpha_{m_k} \left\langle \nabla f^* \left( \nabla f \left( u \right) \right), y_{m_{k+1}} - p' \right\rangle \right] \\
& \leq \left( 1 - \alpha_{m_k} \right) \left[ D_{f}^* \left( p', y_{m_k} \right) + \alpha_{m_k} \left\langle \nabla f^* \left( \nabla f \left( u \right) \right), y_{m_{k+1}} - p' \right\rangle \right].
\end{align*}
\] (46)

But (43) and (46) imply that
\[
\begin{align*}
\alpha_{m_k} D_{f}^* \left( p', y_{m_k} \right) & \leq D_{f}^* \left( p', y_{m_k} \right) - D_{f}^* \left( p', y_{m_{k+1}} \right) \\
& \quad + \alpha_{m_k} \left\langle \nabla f^* \left( \nabla f \left( u \right) \right), y_{m_{k+1}} - p' \right\rangle \\
& \leq \left( 1 - \alpha_{m_k} \right) \left[ D_{f}^* \left( p', y_{m_k} \right) + \alpha_{m_k} \left\langle \nabla f^* \left( \nabla f \left( u \right) \right), y_{m_{k+1}} - p' \right\rangle \right].
\end{align*}
\] (47)

and noting that \( \alpha_{m_k} > 0 \), we get that
\[
D_{f}^* \left( p', y_{m_k} \right) \leq \left\langle \nabla f^* \left( \nabla f \left( u \right) \right), y_{m_{k+1}} - p' \right\rangle.
\] (48)

Thus, using (45) we get that \( D_{f}^* \left( p', y_{m_k} \right) \to 0 \) and hence from (46) we have that \( D_{f} \left( p', y_{m_{k+1}} \right) \to 0 \) as \( k \to \infty \). But \( D_{f} \left( p', y_k \right) \leq D_{f} \left( p', y_{m_{k+1}} \right) \), for all \( k \in \mathbb{N} \), implies that \( D_{f} \left( p', y_k \right) \to 0 \) and hence by Lemma 5 we obtain that \( y_k \to p' \) and \( x_k = \nabla f^* \left( y_k \right) \to p' = \nabla f^* \left( p' \right) \in \mathcal{F} \). Therefore, from the above two cases, we can conclude that \( \{x_n\} \) converges strongly to \( p \in \mathcal{F} \) and the proof is complete.

Remark 12. We note that the sequence \( \{x_n\} \) in Theorem II converges strongly to a point \( p \in \mathcal{F} \) such that \( p = \nabla f^* \left( p' \right) \), where \( p' = P_{\mathcal{F}} \left( \nabla f \left( u \right) \right) \).

If, in Theorem II, we consider a single right Bregman strongly nonexpansive mapping, we get the following corollary.

Corollary 13. Let \( E \) be a reflexive Banach space and let \( f : E \to \mathbb{R} \) be a cofinite function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of \( E \). Let \( C \) be a nonempty, closed, and convex subset of \( \text{int}(\text{dom } f) \) and let \( T : C \to C \) be a right Bregman strongly nonexpansive mapping such that \( F(T) = F(T) \neq \emptyset \). For \( u, x_1 \in C \) let \( \{x_n\} \) be a sequence generated by
\[
x_{n+1} = \alpha_n u + \left( 1 - \alpha_n \right) T x_n, \quad n = 1, 2, \ldots,
\] (49)

where \( \alpha_n \subset (0,1) \) satisfy \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Then, \( \{x_n\} \) converges strongly to some \( p \in \mathcal{F}(T) \).

If, in Theorem II, we assume that each \( T_i, (i = 1, 2, \ldots, N) \) is right Bregman firmly nonexpansive, then we have that \( T = T_N \circ T_{N-1} \circ \cdots \circ T_1 \) is right Bregman firmly nonexpansive with \( F(T) = F(T) = \cap_{i=1}^{N} F(T_i) \) (see [10]) and hence it is right Bregman strongly nonexpansive mapping. Thus, we have the following.

Corollary 14. Let \( f : E \to \mathbb{R} \) be a cofinite function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of \( E \). Let \( C \) be a nonempty, closed, and convex subset of \( \text{int}(\text{dom } f) \) and let \( T_j : C \to C, \) for \( i = 1, 2, \ldots, N \), be a finite family of right Bregman firmly nonexpansive mappings with \( \mathcal{F} := \cap_{i=1}^{N} F(T_i) \). For \( u, x_1 \in C \) let \( \{x_n\} \) be a sequence generated by
\[
x_{n+1} = \alpha_n u + \left( 1 - \alpha_n \right) T x_n, \quad n = 1, 2, \ldots,
\] (50)

where \( T = T_N \circ T_{N-1} \circ \cdots \circ T_1 \), \( \alpha_n \subset (0,1) \) satisfying \( \lim_{n \to \infty} \alpha_n = 0 \) and \( \sum_{n=1}^{\infty} \alpha_n = \infty \). Then, \( \{x_n\} \) converges strongly to \( p \in \mathcal{F} \).

4. Applications

4.1. Zeroes of Maximal Mappings. Let \( A : E \to 2^E \) be a maximal monotone mapping. Recently, many authors studied zero points of monotone mappings using different methods (see e.g., [13, 25, 28, 30, 31, 38]). In this section we use Halpern’s type scheme to find common zeroes of a finite family of maximal monotone set-valued mappings.

Definition 15 (see [31]). Let \( f : E \to (-\infty, +\infty] \) be an admissible function and let \( A : E \to 2^E \) be a set-valued mapping such that \( \text{int}(\text{dom } f) \cap \text{dom } A \neq \emptyset \). The conjugate resolvent of \( A \) with respect to \( f \), or the conjugate \( \nabla f \)-resolvent, is the operator \( \text{CRRes}_A^f : E^* \to 2^{E^*} \) defined by
\[
\text{CRRes}_A^f := \left( I + A \circ \nabla f \right)^{-1}.
\] (51)
Remark 16. If, in addition, $A$ is monotone and $f_{\text{int}(\text{dom} f)}$ is strictly convex, then it is shown in [31] that $\text{CRes}_{A_i}^f$ is right Bregman firmly nonexpansive and $\nabla f^*(F(\text{CRes}_{A_i}^f)) = \text{int}(\text{dom} f) \cap A^{-1}(0^*)$. Moreover, we know that if $f$ is Legendre, bounded, and uniformly continuous on bounded subsets of $E$, then, for every right Bregman firmly nonexpansive operator $T$, $\hat{F}(T) = F(T)$ (see [10]). Thus, under these assumptions on $A$ and $f$, the operator $\text{CRes}_{A_i}^f$ is right Bregman strongly nonexpansive mapping.

We shall need the following lemma.

Lemma 17 (see [30]). Let $f: E \to \mathbb{R}$ be a strictly convex, cofinite, and admissible function, and let $A: E \to 2^{E^*}$ be a set-valued monotone mapping. Then $A$ is maximal monotone if and only if $\text{dom}(\text{CRes}_{A_i}^f) = E^*$.

Theorem 18. Let $f: E \to \mathbb{R}$ be a cofinite function such that $f^*$ is uniformly Fréchet differentiable and totally convex on bounded subsets of $E^*$. Let $A_i: E \to 2^{E^*}, i = 1, 2, \ldots, N$, be maximal monotone mappings such that $\mathcal{F} := \cap_{i=1}^N A_i^{-1}(0^*) \neq \emptyset$. For each $w, z_1 \in E^*$, consider the sequence $\{z_n\}$ generated iteratively by

$$z_{n+1} = \alpha_n w + (1 - \alpha_n) T(z_n), \quad n = 1, 2, \ldots , \tag{52}$$

where $T = \text{CRes}_{A_1}^f \circ \text{CRes}_{A_{N-1}}^f \circ \cdots \circ \text{CRes}_{A_1}^f, \{\alpha_n\} \subset (0, 1)$ satisfy $\lim_{n \to \infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{z_n\}$ converges strongly to $p^f$ in $F(T) = \cap_{i=1}^N F(\text{CRes}_{A_i}^f)$, and $\nabla f^*(p^f) = p \in \mathcal{F}$.

Proof. From Lemma 17 we know that each $T_i = \text{CRes}_{A_i}^f, i = 1, 2, \ldots, N$ is a mapping from $E^*$ into itself, since

$$\bigcap_{i=1}^N F(\text{CRes}_{A_i}^f) = \bigcap_{i=1}^N \nabla f \left( A_i^{-1}(0^*) \right) = \nabla f(\mathcal{F}) \neq \emptyset. \tag{53}$$

Remark 16 guarantees that each $T_i, i = 1, 2, \ldots, N$ is right Bregman strongly nonexpansive mapping with respect to $F(T_i) = \hat{F}(T_i)$. Now the result follows immediately from Theorem 11 applied to $E^*$.

4.2. Convex Feasibility Problems. The convex feasibility problem (CFP) is finding an element $x^* \in \cap_{i=1}^N K_i$, where $K_i$ for $i = 1, 2, \ldots, N$, are nonempty, closed, and convex subsets of $E$. Let $K \subset \text{int}(\text{dom} f)$. The right Bregman projection [30] onto $K$ is the operator $P_{K_i}^f: \text{int}(\text{dom} f) \to K$ defined by

$$P_{K_i}^f(x) := \arg \lim_{y \to K} \{D_f(x, y)\} \quad \{z \in K: D_f(x, z) \leq D_f(x, y) \forall y \in K\}. \tag{54}$$

If $f: E \to \mathbb{R}$ is Legendre and uniformly continuous on bounded subsets of $E$ and $f$ is weakly sequentially continuous, then the right Bregman projection $P_{K_i}^f$ is right Bregman strongly nonexpansive mapping with $F(P_{K_i}^f) = \hat{F}(P_{K_i}^f)$ (see [30]). Therefore, if we take $T_i = P_{K_i}^f$ for each $i \in \{1, 2, \ldots, N\}$, then we get an algorithm for solving convex feasibility problems. More precisely, we have the following result.

Theorem 19. Let $f: E \to \mathbb{R}$ be a cofinite function which is bounded, uniformly continuous, and totally convex on bounded subsets of $E$. Assume that $\nabla f$ is weakly sequentially continuous. Let $K_i, i = 1, 2, \ldots, N$ be $N$ nonempty, closed, and convex subsets of $E$ such that $\mathcal{F} := \cap_{i=1}^N K_i \neq \emptyset$. For each $x_1 \in K$, let the sequence $\{x_n\}$ be generated iteratively by

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) T(z_n), \quad n = 1, 2, \ldots , \tag{55}$$

where $T = P_{K_1}^f \circ P_{K_{N-1}}^f \circ \cdots \circ P_{K_1}^f, \{\alpha_n\} \subset (0, 1)$ satisfy $\lim_{n \to \infty} \alpha_n = 0$, and $\sum_{n=1}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges strongly to $p \in \mathcal{F}$.

Remark 20. Theorem 11 complements the results due to Reich and Sabach [9], Suantai et al. [32], and Zhang and Cheng [33] in the sense that our scheme is applicable for right Bregman strongly nonexpansive self-mappings on C, where $C$ is nonempty, closed, and convex subset of $E$.

Conflict of Interests

The authors declare that they have no conflict of interests regarding the publication of this paper.

Acknowledgments

This paper was funded by the Deanship of Scientific Research (DSR), King Abdulaziz University, Jeddah. The second author acknowledges with thanks DSR for financial support.

References


Submit your manuscripts at http://www.hindawi.com