Hindawi Publishing Corporation
Abstract and Applied Analysis
Volume 2014, Article ID 601092, 7 pages
http://dx.doi.org/10.1155/2014/601092

Research Article

Solvability for a Discrete Fractional Three-Point Boundary Value Problem at Resonance

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Received 18 April 2014; Accepted 24 June 2014; Published 8 July 2014

Academic Editor: Jehad Alzabut

This paper is concerned with the existence of solutions to a discrete three-point boundary value problem at resonance involving the Riemann-Liouville fractional difference of order $\alpha \in (0,1]$. Under certain suitable nonlinear growth conditions imposed on the nonlinear term, the existence result is established by using the coincidence degree continuation theorem. Additionally, a representative example is presented to illustrate the effectiveness of the main result.

1. Introduction

For any number $a \in \mathbb{R}$, we denote $\mathbb{N}_a = \{a, a+1, a+2, \ldots\}$ and $[c,d]_{\mathbb{N}_a} = \{c, c+1, \ldots, d-1, d\}$, for any $c, d \in \mathbb{N}_a$ with $c < d$, throughout this paper. It is also worth noting that, in what follows, we appeal to the conventions $\sum_{s=k_1}^{k_2} u(s) = 0$ and $\prod_{s=k_1}^{k_2} u(s) = 1$ for a given function u defined on $\mathbb{N}_a$ and $k_1, k_2 \in \mathbb{N}_a$ with $k_1 > k_2$.

In this paper, we will consider the existence of solutions for the following discrete fractional three-point boundary value problem:

$$\begin{gathered}\Delta^{\alpha} u(t) = f(t+\alpha-1, u(t+\alpha-1)), \quad t \in [0,b]_{\mathbb{N}_0}, \\ u(\alpha-1) - \beta u(\alpha+\eta) = \gamma u(\alpha+b),\end{gathered} \tag{1}$$

where $0 < \alpha \le 1$ is a real number, Δ^{α} denotes the Riemann-Liouville fractional difference of order α, $f : [\alpha-1, \alpha+b-1]_{\mathbb{N}_{\alpha-1}} \times \mathbb{R} \to \mathbb{R}$, $\eta \in [0, b-1]_{\mathbb{N}_0}$, $\beta, \gamma > 0$, and

$$\beta \prod_{i=1}^{\eta+1} \frac{\alpha+i-1}{i} + \gamma \prod_{i=1}^{b+1} \frac{\alpha+i-1}{i} = 1, \tag{2}$$

which implies that the problem (1) is at resonance. We note that the problem (1) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{gathered}\Delta^{\alpha} u(t) = 0, \quad t \in [0,b]_{\mathbb{N}_0}, \\ u(\alpha-1) - \beta u(\alpha+\eta) = \gamma u(\alpha+b),\end{gathered} \tag{3}$$

has $u(t) = ct^{\underline{\alpha-1}}, t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$, $c \in \mathbb{R}$, as a nontrivial solution.

The continuous fractional calculus has received increasing attention within the last ten years or so and the theory of fractional differential equations has been a new important mathematical branch due to its demonstrated applications in various fields of science and engineering. For more details, see [1–14] and references therein. Significantly less is known, however, about the discrete fractional calculus, but in recent several years, a lot of papers have appeared; see [15–36]. For example, in [19], Atıcı and Eloe explored a discrete fractional conjugate boundary value problem with the Riemann-Liouville fractional difference. To the best of our knowledge, this is a pioneering work on discussing boundary value problems in discrete fractional calculus. After that, Goodrich studied discrete fractional boundary value problems involving the Riemann-Liouville fractional difference intensively and obtained a series of excellent results; see [20–26]. Bastos et al. in [28, 29] considered the discrete fractional calculus of variations and established the necessary conditions for fractional difference variational problems. Abdeljawad

introduced the Caputo fractional difference and developed some useful properties of it in [30]. Ferreira [35] investigated the existence and uniqueness of solutions for some discrete fractional nonresonance boundary value problems of order less than one by using the Banach fixed point theorem.

Although the solvability of fractional boundary value problems has been studied extensively, there are few papers dealing with it under resonance conditions, besides [37–41]. Additionally, as far as we know, the existence of solutions to discrete fractional boundary value problems at resonance has not been studied.

Motivated by the aforementioned results, we will investigate the discrete fractional boundary value problem (1) at resonance and establish some sufficient conditions for the existence of solutions to it by using the coincidence degree theory.

For the sake of convenience, we will always assume that the following conditions hold in this paper:

(H_1) $\alpha \in (0,1]$, $\eta \in [0, b-1]_{\mathbb{N}_0}$, $\beta, \gamma > 0$, and $\beta\prod_{i=1}^{\eta+1}(\alpha + i - 1)/i + \gamma\prod_{i=1}^{b+1}(\alpha + i - 1)/i = 1$;

(H_2) $f : [\alpha - 1, \alpha + b - 1]_{\mathbb{N}_{\alpha-1}} \times \mathbb{R} \to \mathbb{R}$ is continuous.

The remainder of this paper is organized as follows. Section 2 preliminarily provides some necessary basic knowledge for the theory of discrete fractional calculus and the coincidence degree continuation theorem. In Section 3, the existence result of solutions for problem (1) will be established with the help of the coincidence degree theory. Finally in Section 4, a concrete example is provided to illustrate the possible application of the established analytical result.

2. Preliminaries

Since the theory of discrete fractional calculus is in its infancy to some extent, in order to make this paper self-contained, we begin by presenting here some necessary basic definitions and lemmas about it. For more details, see [15, 16, 19, 34].

Definition 1 (see [15]). For any t and ν, the falling factorial function is defined as

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}, \tag{4}$$

provided that the right-hand side is well defined. We appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 2 (see [42]). The νth discrete fractional sum of a function $f : \mathbb{N}_a \to \mathbb{R}$, for $\nu > 0$, is defined by

$$\Delta_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s), \quad t \in \mathbb{N}_{a+\nu}. \tag{5}$$

Also, we define the trivial sum $\Delta_a^0 f(t) = f(t)$, $t \in \mathbb{N}_a$.

Definition 3 (see [15]). The νth discrete Riemann-Liouville fractional difference of a function $f : \mathbb{N}_a \to \mathbb{R}$, for $\nu > 0$, is defined by

$$\Delta_a^{\nu} f(t) = \Delta^n \Delta_a^{-(n-\nu)} f(t), \quad t \in \mathbb{N}_{a+n-\nu}, \tag{6}$$

where n is the smallest integer greater than or equal to ν and Δ^n is the nth order forward difference operator. If $\nu = n \in \mathbb{N}_1$, then $\Delta_a^n f(t) = \Delta^n f(t)$.

Remark 4. From the Definitions 2 and 3, it is easy to see that $\Delta_a^{-\nu}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\nu}$ and Δ_a^{ν} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+n-\nu}$, where n is the smallest integer greater than or equal to ν. Also, it is worth reminding the reader that the t in $\Delta_a^{\nu} f(t)$ (or $\Delta_a^{-\nu} f(t)$) represents an input for the function $\Delta_a^{\nu} f$ (or $\Delta_a^{-\nu} f$) and not for the function f. For ease of notation, we throughout this paper omit the subscript a in $\Delta_a^{\nu} f(t)$ and $\Delta_a^{-\nu} f(t)$ when it does not to lead to domains confusion and general ambiguity.

Lemma 5 (see [19]). *Let $\nu > 0$. Then $\Delta^{-\nu}\Delta^{\nu} f(t) = f(t) + c_1 t^{\underline{\nu-1}} + c_2 t^{\underline{\nu-2}} + \cdots + c_n t^{\underline{\nu-n}}$, where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n$, and n is the smallest integer greater than or equal to ν.*

Lemma 6 (see [21]). *Let $\nu \in \mathbb{R}$ and $t, s \in \mathbb{R}$ such that $(t-s)^{\underline{\nu}}$ is well defined. Then*

$$\Delta_s (t-s)^{\underline{\nu}} = -\nu (t-s-1)^{\underline{\nu-1}}. \tag{7}$$

Next, we will briefly recall some notations in the frame of Mawhin's coincidence degree continuous theorem. For more details, see [43].

Let X and Y be two real Banach spaces. Consider an operator equation $Lu = Nu$, where $L : \operatorname{Dom} L \subset X \to Y$ is a linear operator and $N : X \to Y$ is a nonlinear operator. The operator L will be called a Fredholm operator of index zero if $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L < \infty$ and $\operatorname{Im} L$ is closed in Y. If L is a Fredholm operator of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Y \to Y$ such that $\operatorname{Ker} L = \operatorname{Im} P$, $\operatorname{Im} L = \operatorname{Ker} Q$ and $X = \operatorname{Ker} L \oplus \operatorname{Ker} P$, $Y = \operatorname{Im} L \oplus \operatorname{Im} Q$. It follows that $L|_{\operatorname{Dom} L \cap \operatorname{Ker} P} : \operatorname{Dom} L \cap \operatorname{Ker} P \to \operatorname{Im} L$ is invertible and its inverse is denoted by K_p.

If Ω is an open bounded subset of X and $\operatorname{Dom} L \cap \overline{\Omega} \neq \emptyset$, the operator $N : X \to Y$ will be called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact.

Now, we present the coincidence degree continuation theorem as follows, which will be used in the sequel to establish the existence of solutions for problem (1).

Theorem 7 (see [43]). *Let $L : \operatorname{Dom} L \subset X \to Y$ be a Fredholm operator of index zero and let $N : X \to Y$ be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x, \lambda) \in [(\operatorname{Dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0, 1)$;*

(ii) *$Nx \notin \operatorname{Im} L$ for every $x \in \operatorname{Ker} L \cap \partial\Omega$;*

(iii) *$\deg(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0) \neq 0$, where Q is a projection such that $\operatorname{Im} L = \operatorname{Ker} Q$.*

Then the operator equation $Lx = Nx$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$.

Finally, we wish to fix our framework for the study of problem (1). First of all, we denote $X = \{u : [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}} \to \mathbb{R}\}$ and $Y = \{y : [0, b]_{\mathbb{N}_0} \to \mathbb{R}\}$, and it is clear that X and Y are two Banach spaces when equipped with the usual maximum norm; that is, for any $u \in X$ and $y \in Y$, $\|u\| = \max\{|u(t)| : t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}\}$ and $\|y\| = \max\{|y(t)| : t \in [0, b]_{\mathbb{N}_0}\}$. Next, we define the linear operator $L : \operatorname{Dom} L \subset X \to Y$ by

$$Lu = \Delta^{\alpha} u, \tag{8}$$

with

$$\operatorname{Dom} L = \{u : u \in X, u(\alpha-1) - \beta u(\alpha+\eta) = \gamma u(\alpha+b)\}, \tag{9}$$

and the nonlinear operator $N : X \to Y$ as

$$Nu(t) = f(t+\alpha-1, u(t+\alpha-1)), \quad \text{for } t \in [0, b]_{\mathbb{N}_0}. \tag{10}$$

Then the problem (1) is equivalent to an operator equation

$$Lu = Nu, \quad u \in \operatorname{Dom} L. \tag{11}$$

3. Main Results

In this section, we will establish the existence of at least one solution for the problem (1). To accomplish this, we firstly present here several lemmas which will be used in the sequel.

For convenience, we define the operator $Q_1 : Y \to Y$ by

$$\begin{aligned}(Q_1 y)(t) = {} & \beta \sum_{s=0}^{\eta} (\alpha+\eta-s-1)^{\underline{\alpha-1}} y(s) \\ & + \gamma \sum_{s=0}^{b} (\alpha+b-s-1)^{\underline{\alpha-1}} y(s), \\ & t \in [0, b]_{\mathbb{N}_0},\end{aligned} \tag{12}$$

and by Lemma 6, we can find that $Q_1(1) = (\beta(\alpha+\eta)^{\underline{\alpha}} + \gamma(\alpha+b)^{\underline{\alpha}})/\alpha > 0$.

Lemma 8. *If* (H_1) *holds, then*

$$\operatorname{Ker} L = \left\{u \in X : u = ct^{\underline{\alpha-1}}, c \in \mathbb{R}\right\}, \tag{13}$$

$$\operatorname{Im} L = \{y \in Y : Q_1 y = 0\}, \tag{14}$$

where L and Q_1 are defined by (8) *and* (12)*, respectively.*

Proof. At first, in view of Lemma 5 and (H_1), we can easily verify that (13) holds. Next, we prove that (14) also holds.

For any $y \in \operatorname{Im} L$, then there exists a function $u \in \operatorname{Dom} L$ such that $y = \Delta^{\alpha} u$. Based on Lemma 5, we have

$$u(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} y(s) + ct^{\underline{\alpha-1}}, \tag{15}$$

where $t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$, $c \in \mathbb{R}$. From conditions $u(\alpha-1) - \beta u(\alpha+\eta) = \gamma u(\alpha+b)$ and (H_1), we can easily obtain that $Q_1 y = 0$.

Conversely, for any $y \in Y$ with $Q_1 y = 0$, if we set $u(t) = \Delta^{-\alpha} y(t)$, $t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$, then $u(\alpha-1) = 0$, and it is easy to verify that $u \in \operatorname{Dom} L$. Moreover, by the relation $\Delta^{\alpha}\Delta^{-\alpha} y = y$, we have $Lu = y$, which lead to $y \in \operatorname{Im} L$. So we get that (14) holds. The proof is complete. □

Lemma 9. *If* (H_1) *holds, then L defined by* (8) *is a Fredholm operator of index zero.*

Proof. In order to show that L is a Fredholm operator of index zero, firstly, we consider the following operator $Q : Y \to Y$ defined by

$$Qy = \frac{\alpha(Q_1 y)}{\beta(\alpha+\eta)^{\underline{\alpha}} + \gamma(\alpha+b)^{\underline{\alpha}}} = \frac{Q_1 y}{Q_1(1)}, \tag{16}$$

where Q_1 is defined by (12). Evidently, $\operatorname{Im} Q = \mathbb{R}$ and Q is a continuous linear projector. In fact, for any $y \in Y$, we have

$$\begin{aligned} Q_1(Qy) &= Q_1\left(\frac{Q_1 y}{Q_1(1)}\right) = \frac{Q_1 y}{Q_1(1)} Q_1(1) = Q_1 y, \\ Q^2 y &= Q(Qy) = \frac{Q_1(Qy)}{Q_1(1)} = \frac{Q_1 y}{Q_1(1)} = Qy; \end{aligned} \tag{17}$$

that is to say, $Q : Y \to Y$ is idempotent. Hence, Q is a projector.

From the definition of Q and (14), it is easy to see that $y \in \operatorname{Im} L$ leads to $Qy = 0$, and if $y \in \operatorname{Ker} Q$, we can get $Q_1 y = 0$, which implies that $y \in \operatorname{Im} L$. So, we derive $\operatorname{Ker} Q = \operatorname{Im} L$.

For any $y \in Y$, set $y = (y - Qy) + Qy$. Since $Qy \in \operatorname{Im} Q$ and $(I-Q)y \in \operatorname{Ker} Q$, we have $Y = \operatorname{Im} Q + \operatorname{Ker} Q$. Moreover, take $y_0 \in \operatorname{Ker} Q \cap \operatorname{Im} Q$. Then y_0 can be written as $y_0(t) = c$, $t \in [0, b]_{\mathbb{N}_0}$, $c \in \mathbb{R}$, for $y_0 \in \operatorname{Im} Q$. On the other hand, since $y_0 \in \operatorname{Ker} Q = \operatorname{Im} L$, by (14), we can get $Q_1 y_0 = Q_1(c) = cQ_1(1) = 0$, which implies that $y_0 = c = 0$. So, we have $\operatorname{Im} Q \cap \operatorname{Ker} Q = \{0\}$ and $Y = \operatorname{Im} Q \oplus \operatorname{Ker} Q = \operatorname{Im} Q \oplus \operatorname{Im} L$.

Now, since $\dim \operatorname{Ker} L = \operatorname{codim} \operatorname{Im} L = \dim \operatorname{Im} Q = 1$ and $\operatorname{Im} L$ is closed in Y, L is a Fredholm operator of index zero. The proof is complete. □

Let $P : X \to X$ be defined by

$$Pu(t) = \frac{1}{\Gamma(\alpha)} u(\alpha-1) t^{\underline{\alpha-1}}, \quad t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}. \tag{18}$$

It is clear that $P : X \to X$ is a linear continuous projector and

$$\operatorname{Im} P = \left\{u \in X : u = ct^{\underline{\alpha-1}}, c \in \mathbb{R}\right\} = \operatorname{Ker} L. \tag{19}$$

Also, proceeding as the proof of Lemma 9, we can show that

$$X = \operatorname{Ker} P \oplus \operatorname{Im} P = \operatorname{Ker} P \oplus \operatorname{Ker} L. \tag{20}$$

Define operator $K_p : \operatorname{Im} L \to \operatorname{Dom} L \cap \operatorname{Ker} P$ by

$$\begin{aligned} K_p y(t) &= \Delta^{-\alpha} y(t) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} y(s), \\ & \qquad t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}, \end{aligned} \tag{21}$$

where $y \in \operatorname{Im} L$.

From the definitions of P and K_p, it is easy to see that the inverse of $L|_{\mathrm{Dom}\,L\cap\mathrm{Ker}\,P}$ is K_p. In fact, if $y \in \mathrm{Im}\,L$, then we have

$$LK_p y = \Delta^{\alpha}\Delta^{-\alpha} y = y. \tag{22}$$

Also, if $u \in \mathrm{Dom}\,L \cap \mathrm{Ker}\,P$, by Lemma 5, we have

$$K_p Lu(t) = u(t) + ct^{\underline{\alpha-1}}, \tag{23}$$

where $t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$, $c \in \mathbb{R}$. Then it follows from $K_p Lu \in \mathrm{Ker}\,P$ and $u \in \mathrm{Ker}\,P$ that

$$K_p Lu(\alpha-1) = u(\alpha-1) + c(\alpha-1)^{\underline{\alpha-1}} = \Gamma(\alpha)\,c = 0, \tag{24}$$

which implies that $c = 0$. Consequently, we have $K_p Lu = u$ for $u \in \mathrm{Dom}\,L \cap \mathrm{Ker}\,P$. So, $K_p = (L|_{\mathrm{Dom}\,L\cap\mathrm{Ker}\,P})^{-1}$.

Lemma 10. *Suppose that* (H_2) *holds. If* $\Omega \subset X$ *is an open bounded subset and* $\mathrm{Dom}\,L \cap \overline{\Omega} \neq \emptyset$, *then* N *is* L*-compact on* $\overline{\Omega}$.

Proof. By the continuity of f, we can verify that $QN(\overline{\Omega})$ and $K_p(I-Q)N(\overline{\Omega})$ are bounded. So we get that $K_p(I-Q)N(\overline{\Omega})$ is compact. Therefore N is L-compact on $\overline{\Omega}$. The proof is complete. □

To establish the main result, we need the following conditions.

(H_3) There exist two nonnegative functions $p, q \in X$ with $\{\prod_{i=1}^{b}((\alpha+i)/i) + ((b+1)/\alpha)\}\|p\| < 1$ such that

$$|f(t,u)| \le p(t)\,|u| + q(t), \quad \text{for } (t,u) \in [\alpha-1, \alpha+b-1]_{\mathbb{N}_{\alpha-1}} \times (-\infty, +\infty). \tag{25}$$

(H_4) There exists a constant $M > 0$ such that

$$Q_1 Nu \neq 0, \tag{26}$$

for each $u \in X$ satisfying $|u(t)| > M$, $t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$.

(H_5) There exists a constant $M^* > 0$ such that for any $u(t) = ct^{\underline{\alpha-1}}$, $t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$, $c \in \mathbb{R}$, if $|c| > M^*$, then either

$$cQ_1 Nu > 0 \tag{27}$$

or

$$cQ_1 Nu < 0. \tag{28}$$

Theorem 11. *If* (H_1)–(H_5) *hold, then the problem* (1) *has at least one solution in* X.

Proof. This proof will be divided into four main steps. Now let us prove the steps one by one.

Step 1. Set $\Omega_1 = \{u \in \mathrm{Dom}\,L \setminus \mathrm{Ker}\,L : Lu = \lambda Nu, \lambda \in (0,1)\}$ and prove that Ω_1 is bounded in X.

For $u \in \Omega_1$, then $u \in \mathrm{Dom}\,L \setminus \mathrm{Ker}\,L$ and $Lu = \lambda Nu$, so $Nu \in \mathrm{Im}\,L$. By (14), we have $Q_1 Nu = 0$. From (H_4), there exists a constant $t_0 \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$ such that $|u(t_0)| \le M$. Since $Lu = \lambda Nu$, by Lemma 5, we have

$$u(t) = \frac{\lambda}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}} Nu(s) + ct^{\underline{\alpha-1}}, \quad t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}},\ c \in \mathbb{R}. \tag{29}$$

Considering $|u(t_0)| \le M$, we get

$$\left|ct_0^{\underline{\alpha-1}}\right| \le M + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t_0-\alpha}(t_0-s-1)^{\underline{\alpha-1}}\,|Nu(s)|. \tag{30}$$

Consequently, by (29), (30), Lemma 6, (H_3), and the monotonicity of functions $t^{\underline{\alpha}}$ and $t^{\underline{\alpha-1}}$ on $[\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$, we have

$$\begin{aligned}
|u(t)| &\le \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}\,|Nu(s)| + \frac{t^{\underline{\alpha-1}}}{t_0^{\underline{\alpha-1}}}\left|ct_0^{\underline{\alpha-1}}\right| \\
&\le \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}\,|Nu(s)| + \frac{(\alpha-1)^{\underline{\alpha-1}}}{(\alpha+b)^{\underline{\alpha-1}}} \\
&\quad\times\left\{M + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t_0-\alpha}(t_0-s-1)^{\underline{\alpha-1}}\,|Nu(s)|\right\} \\
&= \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}}\left|f(s+\alpha-1, u(s+\alpha-1))\right| \\
&\quad+\prod_{i=1}^{b+1}\frac{i}{\alpha+i-1}\left\{M + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t_0-\alpha}(t_0-s-1)^{\underline{\alpha-1}}\right. \\
&\qquad\left.\times\left|f(s+\alpha-1, u(s+\alpha-1))\right|\right\} \\
&\le \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t-\alpha}(t-s-1)^{\underline{\alpha-1}} \\
&\quad\times\{p(s+\alpha-1)\,|u(s+\alpha-1)| + q(s+\alpha-1)\} \\
&\quad+\prod_{i=1}^{b+1}\frac{i}{\alpha+i-1} \\
&\quad\times\left\{M + \frac{1}{\Gamma(\alpha)}\sum_{s=0}^{t_0-\alpha}(t_0-s-1)^{\underline{\alpha-1}}\right. \\
&\qquad\left.\times\{p(s+\alpha-1)\,|u(s+\alpha-1)| + q(s+\alpha-1)\}\right\}
\end{aligned}$$

$$\leq \frac{\|p\|\,\|u\| + \|q\|}{\Gamma(\alpha)} \left\{ \sum_{s=0}^{t-\alpha} (t-s-1)^{\underline{\alpha-1}} + \left(\prod_{i=1}^{b+1} \frac{i}{\alpha+i-1} \right) \times \sum_{s=0}^{t_0-\alpha} (t_0-s-1)^{\underline{\alpha-1}} \right\}$$
$$+ M \prod_{i=1}^{b+1} \frac{i}{\alpha+i-1}$$
$$= \frac{\|p\|\,\|u\| + \|q\|}{\Gamma(\alpha+1)} \left\{ t^{\underline{\alpha}} + \left(\prod_{i=1}^{b+1} \frac{i}{\alpha+i-1} \right) t_0^{\underline{\alpha}} \right\}$$
$$+ M \prod_{i=1}^{b+1} \frac{i}{\alpha+i-1}$$
$$\leq \frac{\|p\|\,\|u\| + \|q\|}{\Gamma(\alpha+1)} \left\{ (\alpha+b)^{\underline{\alpha}} + \left(\prod_{i=1}^{b+1} \frac{i}{\alpha+i-1} \right) \times (\alpha+b)^{\underline{\alpha}} \right\} + M \prod_{i=1}^{b+1} \frac{i}{\alpha+i-1}$$
$$\leq \left\{ \prod_{i=1}^{b} \frac{\alpha+i}{i} + \frac{b+1}{\alpha} \right\} (\|p\|\,\|u\| + \|q\|) + M \prod_{i=1}^{b+1} \frac{i}{\alpha+i-1}. \tag{31}$$

So, by the fact that $\{\prod_{i=1}^{b}((\alpha+i)/i) + ((b+1)/\alpha)\}\|p\| < 1$ in (H_3) and (31), we can derive that Ω_1 is bounded.

Step 2. Set $\Omega_2 = \{u \in \operatorname{Ker} L : Nu \in \operatorname{Im} L\}$ and prove that Ω_2 is bounded in X.

For any $u \in \Omega_2$, there exists a constant $c \in \mathbb{R}$ such that $u(t) = ct^{\underline{\alpha-1}}$, $t \in [\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$, and $Nu \in \operatorname{Im} L$. So it follows from (14) that $Q_1 Nu = 0$. By virtue of (H_5) and the fact that $t^{\underline{\alpha-1}}$ is decreasing for t on $[\alpha-1, \alpha+b]_{\mathbb{N}_{\alpha-1}}$, we can derive that $|u(t)| \leq |c|(\alpha-1)^{\underline{\alpha-1}} \leq M^* \Gamma(\alpha)$, which implies that Ω_2 is bounded in X.

Step 3. Set $\Omega_3 = \{u \in \operatorname{Ker} L : \lambda Ju + (1-\lambda)\theta QNu = 0, \lambda \in [0,1]\}$ and prove that Ω_3 is bounded in X, where $J : \operatorname{Ker} L \to \operatorname{Im} Q$ is a linear isomorphism defined by

$$J\left(ct^{\underline{\alpha-1}}\right) = c,$$
$$\theta = \begin{cases} 1, & \text{if } (H_5)\ (27) \text{ holds}, \\ -1, & \text{if } (H_5)\ (28) \text{ holds}. \end{cases} \tag{32}$$

For any $ct^{\underline{\alpha-1}} \in \Omega_3$, there exists $\lambda \in [0,1]$ such that

$$\lambda c = -(1-\lambda)\theta QN\left(ct^{\underline{\alpha-1}}\right). \tag{33}$$

If $\lambda = 0$, then $QN(ct^{\underline{\alpha-1}}) = 0$. Hence $Q_1 N(ct^{\underline{\alpha-1}}) = 0$. By (H_5), we get $|c| \leq M^*$. If $\lambda = 1$, then $c = 0$. For $\lambda \in (0,1)$, if $|c| > M^*$, then, from (H_5), we can obtain that

$$\theta c Q_1 N\left(ct^{\underline{\alpha-1}}\right) > 0. \tag{34}$$

Therefore, we have

$$\lambda c^2 = -(1-\lambda)\theta c QN\left(ct^{\underline{\alpha-1}}\right) < 0, \tag{35}$$

which is a contradiction. So, $\Omega_3 \subset \{u \in \operatorname{Ker} L : u = ct^{\underline{\alpha-1}}, |c| \leq M^*\}$ is bounded in X.

Step 4. Let Ω be a bounded open set such that $\Omega \supset \cup_{i=1}^{3} \overline{\Omega}_i$ and prove that

$$\deg\left(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\right) \neq 0. \tag{36}$$

It follows from Lemma 10 that N is L-compact on $\overline{\Omega}$. Then by Steps 1 and 2, we have

(i) $Lu \neq \lambda Nu$ for every $(u, \lambda) \in [(\operatorname{Dom} L \setminus \operatorname{Ker} L) \cap \partial\Omega] \times (0,1)$;

(ii) $Nu \notin \operatorname{Im} L$ for every $u \in \operatorname{Ker} L \cap \partial\Omega$.

At last, we prove that condition (iii) of Theorem 7 is satisfied. Let

$$H(u, \lambda) = \lambda Ju + (1-\lambda)\theta QNu. \tag{37}$$

According to the arguments in Step 3, we have

$$H(u, \lambda) \neq 0, \quad \forall u \in \operatorname{Ker} L \cap \partial\Omega, \tag{38}$$

and therefore, via the homotopy property of degree, we get that

$$\begin{aligned} &\deg\left(QN|_{\operatorname{Ker} L}, \Omega \cap \operatorname{Ker} L, 0\right) \\ &\quad = \deg\left(\theta H(\cdot, 0), \Omega \cap \operatorname{Ker} L, 0\right) \\ &\quad = \deg\left(\theta H(\cdot, 1), \Omega \cap \operatorname{Ker} L, 0\right) \\ &\quad = \deg\left(\theta J, \Omega \cap \operatorname{Ker} L, 0\right) \\ &\quad \neq 0, \end{aligned} \tag{39}$$

which implies that condition (iii) of Theorem 7 is satisfied. Then by Theorem 7, we can conclude that $Lu = Nu$ has at least one solution in $\operatorname{Dom} L \cap \overline{\Omega}$; that is, (1) has at least one solution in X. The proof is completed.

□

4. An Illustrative Example

In this section, we will illustrate the possible application of the above established analytical result with a concrete example.

Example 1. Consider the following discrete fractional boundary value problem:

$$\Delta^{1/2} u(t) = f\left(t - \frac{1}{2}, u\left(t - \frac{1}{2}\right)\right), \quad t \in [0,3]_{\mathbb{N}_0},$$
$$u\left(-\frac{1}{2}\right) - \frac{16}{9} u\left(\frac{3}{2}\right) = \frac{128}{105} u\left(\frac{7}{2}\right), \tag{40}$$

where

$$f(t,u)=\begin{cases}2-\dfrac{t^2}{100000}\left[\sin\left(t+u^2\right)\right]^2,\\ (t,u)\in\left[-\dfrac{1}{2},\dfrac{5}{2}\right]_{\mathbb{N}_{-1/2}}\times\left(\dfrac{62}{3},+\infty\right),\\ \dfrac{3}{31}u-\dfrac{t^2}{100000}\left[\sin\left(t+u^2\right)\right]^2,\\ (t,u)\in\left[-\dfrac{1}{2},\dfrac{5}{2}\right]_{\mathbb{N}_{-1/2}}\times\left(-\dfrac{62}{3},+\dfrac{62}{3}\right),\\ -2-\dfrac{t^2}{100000}\left[\sin\left(t+u^2\right)\right]^2,\\ (t,u)\in\left[-\dfrac{1}{2},\dfrac{5}{2}\right]_{\mathbb{N}_{-1/2}}\times\left(-\infty,-\dfrac{62}{3}\right).\end{cases}\tag{41}$$

It is obvious that f is continuous. Corresponding to problem (1), there exist $\alpha=1/2$, $\beta=16/9$, $\gamma=128/105$, $\eta=1$, $b=3$, and

$$\begin{aligned}&\beta\prod_{i=1}^{\eta+1}\frac{\alpha+i-1}{i}+\gamma\prod_{i=1}^{b+1}\frac{\alpha+i-1}{i}\\ &\quad=\frac{16}{9}\prod_{i=1}^{2}\frac{1/2+i-1}{i}+\frac{128}{105}\prod_{i=1}^{4}\frac{1/2+i-1}{i}\\ &\quad=1.\end{aligned}\tag{42}$$

Therefore, the problem (40) is at resonance.

Choosing $p(t)=3/31$, $q(t)=t^2/100000$, $t\in[-1/2,7/2]_{\mathbb{N}_{-1/2}}$, then we have

$$\begin{gathered}\left|f(t,u)\right|\le p(t)|u|+q(t),\quad \text{for } (t,u)\in\left[-\frac{1}{2},\frac{5}{2}\right]_{\mathbb{N}_{-1/2}}\times\mathbb{R},\\ \left\{\prod_{i=1}^{3}\frac{1/2+i}{i}+\frac{3+1}{1/2}\right\}\|p\|=\frac{163}{16}\times\frac{3}{31}=\frac{489}{496}<1.\end{gathered}\tag{43}$$

Hence, the condition (H_3) in Theorem 11 holds.

Let $M=21$. If $|u(t)|>M$ holds for any $t\in[-1/2,7/2]_{\mathbb{N}_{-1/2}}$, then we can easily verify that

$$Q_1Nu\neq 0,\tag{44}$$

which implies that condition (H_4) of Theorem 11 holds.

Furthermore, we can choose $M^*=50$, to show that the condition (H_5) of Theorem 11 holds. In fact, for any $u(t)=ct^{\underline{-1/2}}$, $t\in[-1/2,7/2]_{\mathbb{N}_{-1/2}}$ satisfying $|c|>M^*$, we can get that

$$cNu(t)>0,\quad \text{for } t\in[0,3]_{\mathbb{N}_0}.\tag{45}$$

So, by the fact that $(t-s-1/2)^{\underline{-1/2}}>0$ for $(t,s)\in[0,3]_{\mathbb{N}_0}\times[0,t]_{\mathbb{N}_0}$ and (45), we can derive that, for any $u(t)=ct^{\underline{\alpha-1}}$, $t\in[-1/2,7/2]_{\mathbb{N}_{-1/2}}$ satisfying $|c|>M^*$,

$$cQ_1Nu=Q_1(cNu)>0,\tag{46}$$

which implies that (27) in (H_5) of Theorem 11 holds. Therefore, all conditions of Theorem 11 hold. Hence, we can conclude that problem (40) has at least one solution.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The author sincerely thanks the reviewers for their valuable suggestions and useful comments that have led to the present improved version of the original paper. The research is supported by the Longdong University Grant XYZK-1402.

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