Research Article

Third-Order Newton-Type Methods Combined with Vector Extrapolation for Solving Nonlinear Systems

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We present a third-order method for solving the systems of nonlinear equations. This method is a Newton-type scheme with the vector extrapolation. We establish the local and semilocal convergence of this method. Numerical results show that the composite method is more robust and efficient than a number of Newton-type methods with the other vector extrapolations.

1. Introduction

Finding the solution of nonlinear equations is important in scientific and engineering computing areas. In this paper, we focus on the following nonlinear system of equations:

$$F(x) = 0, \tag{1}$$

where $F : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable. Here, $F(x) = (f_1(x), f_2(x), \ldots, f_n(x))^T$ and $x \in \mathbb{R}^n$.

Some efficient methods for solving the system of (1) have been brought forward [1]. The Newton method for (1) is a second-order method. Its iterative formula is given by

$$x_{k+1} = x_k - F'(x_k)^{-1} F(x_k), \tag{2}$$

where x_k is the current approximate solution and $F'(x_k)$ is the Jacobian matrix of $F(x)$ at x_k. Potra and Pták [2] propose the modified Newton method (PPM) given by

$$
\begin{aligned}
y_k &= x_k - F'(x_k)^{-1} F(x_k), \\
x_{k+1} &= y_k - F'(x_k)^{-1} F(y_k).
\end{aligned}
\tag{3}
$$

In each iteration, PPM needs two evaluations of the vector function and one evaluation of the Jacobian matrix and the order is three.

Though the PPM can reduce the computational cost of Jacobian matrix, in some cases, the sequences produced by PPM converge slowly and even cannot converge because of the accumulation of the computational error. This problem limits its practical application.

In order to solve this problem, we will introduce the vector extrapolation technique to improve the convergence of PPM. Many vector extrapolation methods have been developed, such as the minimal polynomial extrapolation (MPE) method [3], the reduced rank extrapolation (RRE) method [4, 5], the modified minimal polynomial extrapolation (MMPE) method [6–8], the topological ε-algorithm (TEA) [6], and vector ε-algorithms (VEA) [9, 10]; also see [11, 12] and the references therein. These methods could be applied to the solvers of linear and nonlinear systems and accelerate their convergence.

In this paper, we construct a new extrapolation method and combine it with PPM, thus obtaining a Newton-type method. We will show by numerical results that the composite method can be of practical interest. The local and semilocal convergence are also established for the method.

2. The Method

We introduce the following Newton-type method:

$$y_k = x_k - F'(x_k)^{-1} F(x_k),$$

$$z_k = y_k - F'(x_k)^{-1} F(y_k),$$

$$x_{k+1} = z_k - 2 \frac{(x_k - y_k - \omega(y_k - z_k)) \cdot (z_k - y_k)}{\|x_k - y_k - \omega(y_k - z_k)\|^2} (z_k - y_k),$$

(4)

where $\| \cdot \|$ is Euclidean norm and $0 < \omega \le 2$.

This iteration scheme consists of a PPM iterate to get z_k from x_k, followed by a modified iterate to calculate x_{k+1} from x_k, y_k, and z_k.

We now derive the last substep. Let $f(x) = 0$ be a scalar real equation; then King's method [13] is described as

$$y_k = x_k - \frac{f(x_k)}{f'(x_k)},$$

$$z_k = y_k - \frac{f(y_k)}{f'(x_k)},$$

(5)

$$x_{k+1} = z_k - \frac{2(z_k - y_k)^2}{x_k - y_k - \omega(y_k - z_k)}.$$

In order to extend the method (5) to the case of vector functions, we define the vector inverse as

$$v^{-1} = \frac{v^T}{\|v\|^2}, \quad v \in \mathbb{R}^n.$$

(6)

The last substep is obtained by applying the above vector inverse to the scalar King method.

The following theorem will give the order of convergence of the method with $0 < \omega \le 2$ given by (4).

Theorem 1. *Suppose that the function $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $F'(x^*)$ is nonsingular, where D is an open set and $x^* \in D$ is the solution of $F(x) = 0$. Define $\lambda = \|F'(x^*)^{-1}\|$. Further, assume that there exists a positive number γ such that for any $x \in D$,*

$$\|F'(x) - F'(x^*)\| \le \gamma \|x - x^*\|;$$

(7)

then there exists a set S such that for any $x_0 \in S$, the sequence $\{x_k\}$ generated by (4) with $0 < \omega \le 2$ converges to x^* and the order of convergence is three.

Proof. We can write (4) as $x_{k+1} = G(x_k)$ where

$$y = x - F'(x)^{-1} F(x), \qquad z = y - F'(x)^{-1} F(y),$$

$$G(x) = z - 2 \frac{(x - y - \omega(y - z)) \cdot (z - y)}{\|x - y - \omega(y - z)\|^2} (z - y).$$

(8)

Without loss of generality, we use the Euclidean norms as $\| \cdot \|$ in the following. Let $\delta = 1/20\lambda\gamma$ and $S = \{x \mid \|x - x^*\| \le \delta\} \cap D$. Let $x \in S$ and $x \ne x^*$.

It is obtained from (7) that

$$\|F'(x) - F'(x^*)\| \le \gamma\delta \le \frac{1}{20\lambda}.$$

(9)

By Banach lemma, we obtain that $F'(x)$ is nonsingular and $\|F'(x)^{-1}\| \le (20/19)\lambda$. So y and z are well defined.

By making use of Taylor expansion and (7), we have

$$\|F(x^*) - F(x) - F'(x)(x^* - x)\|$$

$$= \left\| \int_0^1 F'(x + t(x^* - x))(x^* - x) dt - F'(x)(x^* - x) \right\|$$

$$= \left\| \int_0^1 [F'(x + t(x^* - x)) - F'(x)](x^* - x) dt \right\|$$

$$\le \int_0^1 \|F'(x + t(x^* - x)) - F'(x)\| \|x^* - x\| dt$$

$$\le \gamma \|x - x^*\|^2 \int_0^1 t \, dt$$

$$= \frac{1}{2} \gamma \|x - x^*\|^2.$$

(10)

So for y we obtain

$$\|y - x^*\| = \|F'(x)^{-1}\| \|F(x^*) - F(x) - F'(x)(x^* - x)\|$$

$$\le \frac{10}{19} \lambda\gamma \|x - x^*\|^2.$$

(11)

Similarly to (10), we get

$$\|F(y) - F(x^*) - F'(x^*)(y - x^*)\| \le \frac{1}{2} \gamma \|y - x^*\|^2.$$

(12)

It is obtained by (7) and (12) that

$$\|z - x^*\|$$

$$\le \|F'(x)^{-1}\| [\|F'(x) - F'(x^*)\| \|y - x^*\|$$

$$+ \|F(x^*) - F(y) - F'(x^*)(x^* - y)\|]$$

$$\le \frac{200}{361} \lambda^2 \gamma^2 \|x - x^*\|^3 + \frac{1000}{6859} \lambda^3 \gamma^3 \|x - x^*\|^4.$$

(13)

Therefore it follows that

$$\|G(x) - x^*\|$$

$$\leq \left(\|x - x^*\| \|z - x^*\| + (3 - \omega) \|z - x^*\| \|y - x^*\| \right.$$

$$\left. + (2 - \omega) \|z - x^*\|^2 + 2\|y - x^*\|^2 \right)$$

$$\times \left(\|x - x^*\| - (\omega + 1) \|y - x^*\| - \omega \|z - x^*\| \right)^{-1}$$

$$\leq \frac{292700}{480111} \lambda^2 \gamma^2 \|x - x^*\|^3$$

$$\leq \frac{2927}{1920444} \|x - x^*\| < \|x - x^*\|. \tag{14}$$

This proves that $G(x) \in S$ and G is a contraction mapping. Thus, for any $x_0 \in S$, the sequence $\{x_k\}$ produced by (4) is well defined and it converges to x^*. Finally, it is shown from (13) that the order of the method (4) is three. □

3. The Semilocal Convergence

In this section, we will establish the semilocal convergence of method (4). This convergence may be derived by using recurrence relations, which have been used in establishing the convergence of Newton's method and some third-order methods [14–29]. In what follows, an attempt is made to use recurrence relations to establish the semilocal convergence for the method (4). The recurrence relations based on one constant which depend on F are derived. Further, based on these recurrence relations, the error estimate is obtained for the present iterative method.

In order to establish the recurrence relations for the present iterative method, we will use the following scalar functions which are defined by

$$g_1(t) = 1 + \frac{1}{2}t + \frac{t^2}{2 - \omega t},$$

$$g_2(t) = \frac{1}{1 - tg_1(t)},$$

$$g_3(t) = \frac{1}{2}t^2 g_2(t) \left[\frac{4 + (2 - \omega) t}{2 - \omega t} + \frac{1}{4}t \left(\frac{2 + (2 - \omega)t}{2 - \omega t} \right)^2 \right],$$

$$g_4(t) = g_2(t) g_3(t), \tag{15}$$

where $0 < \omega \leq 2$.

Let $h(t) = (2 - \omega t)(tg_1(t) - 1) = (1 - (1/2)\omega)t^3 + (1 - \omega)t^2 + (2 + w)t - 2$. For any positive real number ω, it is easy to obtain $h(0)h(2/\omega) < 0$, so $h(t)$ has at least a real zero point $\hat{t} \in (0, 2/\omega)$. Furthermore, let $f(t) = g_4(t) - 1$. It can be included $f(0) < 0$ and $f(\hat{t}) \to +\infty$, so $f(t)$ has at least a real zero in $t^* \in (0, \hat{t})$. Furthermore, it can be obtained that $f(t)$ is an increasing function in $(0, 2/\omega)$. So t^* is the unique zero of $f(t)$ in $(0, 2/\omega)$. For the functions defined by (15), we have the following results.

Lemma 2. *Let t^* be the unique real root of $g_4(t) - 1 = 0$ in $(0, 2/\omega)$. Then*

(a) *$g_1(t)$ is an increasing function in $[0, t^*]$ and satisfies $1 \leq g_1(t) \leq g_1(t^*)$;*

(b) *$g_2(t)$ is an increasing function in $[0, t^*]$ and satisfies $1 \leq g_2(t) \leq g_2(t^*)$;*

(c) *$g_3(t)$ is an increasing function in $[0, t^*]$ and satisfies $0 \leq g_3(t) \leq g_3(t^*) < 1$;*

(d) *$g_4(t)$ is an increasing function in $[0, t^*]$ and satisfies $0 \leq g_4(t) \leq g_4(t^*) = 1$;*

(e) *$t^* g_1(t^*)/(1 - g_3(t^*)) = 1$.*

Proof. The results (a)–(d) can be obtained by simple derivations. We only prove the validity of (e). Noticing that

$$g_4(t) = g_2(t) g_3(t),$$

$$g_2(t) = \frac{1}{1 - tg_1(t)}, \tag{16}$$

we get

$$1 = g_4(t^*) = \frac{g_3(t^*)}{1 - t^* g_1(t^*)}, \tag{17}$$

which can be converted to (e). □

Theorem 3. *Assume that the function $F : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable where D is an open set and there exists a positive number γ such that for any $x, v \in D$*

$$\|F'(x) - F'(v)\| \leq \gamma \|x - v\|. \tag{18}$$

Let $g_1(t)$, $g_2(t)$, $g_3(t)$, and $g_4(t)$ be defined by (15). Further, define α_k, β_k, and ρ_k as

$$\alpha_k = \|F'(x_k)^{-1} F(x_k)\|, \qquad \beta_k = \|F'(x_k)^{-1}\|,$$

$$\rho_k = \alpha_k \beta_k \gamma. \tag{19}$$

Let $x_0 \in D$ satisfy $F(x_0) \neq 0$, $F'(x_0)$ be nonsingular, $\rho_0 \in (0, t^*) \subset (0, 2/\omega)$, and $S = \{x \mid \|x - x_0\| \leq \theta \alpha_0\} \subset D$ where $\theta = g_1(t^*)/(1 - g_3(t^*))$ and t^* is the root of $g_4(t) - 1 = 0$ in $(0, 2/\omega)$; then we have that

(i) *$\{x_k\}$ generated by the method (4) is well defined in S and satisfies*

$$\|x_{k+1} - x_k\| \leq g_1(\rho_k) \alpha_k; \tag{20}$$

(ii) *α_k, β_k, and ρ_k are well defined and satisfy*

$$\alpha_{k+1} \leq g_3(\rho_k) \alpha_k, \tag{21}$$

$$\beta_{k+1} \leq g_2(\rho_k) \beta_k, \tag{22}$$

$$\rho_{k+1} \leq g_4(\rho_k) \rho_k. \tag{23}$$

Proof. Without loss of generality, we use the Euclidean norms as $\|\cdot\|$ in the following. We firstly consider the case $k = 0$.

Since $1 < \theta$, it is obvious that $y_0 \in S$ by the definition $\alpha_0 = \|y_0 - x_0\|$. By Taylor expansion we have

$$
\begin{aligned}
\|F(y_0)\| \\
&= \left\| F(x_0) + F'(x_0)(y_0 - x_0) + \int_{x_0}^{y_0} \left[F'(x) - F'(x_0) \right] dx \right\| \\
&= \left\| \int_0^1 \left[F'(x_0 + t(y_0 - x_0)) - F'(x_0) \right] (y_0 - x_0) \, dt \right\| \\
&\le \int_0^1 \| F'(x_0 + t(y_0 - x_0)) - F'(x_0) \| \| y_0 - x_0 \| \, dt \\
&\le \gamma \| y_0 - x_0 \|^2 \int_0^1 t \, dt \\
&= \frac{1}{2} \gamma \alpha_0^2.
\end{aligned}
\tag{24}
$$

Furthermore,

$$
\| z_0 - y_0 \| \le \| F'(x_0)^{-1} \| \| F(y_0) \| \le \frac{1}{2} \beta_0 \gamma \alpha_0^2 = \frac{1}{2} \rho_0 \alpha_0. \tag{25}
$$

It then follows that

$$
\| z_0 - x_0 \| \le \| z_0 - y_0 \| + \| y_0 - x_0 \| \le \left(1 + \frac{1}{2} \rho_0 \right) \alpha_0. \tag{26}
$$

Taking account of the relation

$$
1 + \frac{1}{2} \rho_0 < g_1(\rho_0) < g_1(t^*) < \theta, \tag{27}
$$

we have $z_0 \in S$. Since

$$
\begin{aligned}
\| x_0 - y_0 - \omega(y_0 - z_0) \| &\ge \| y_0 - x_0 \| - \omega \| z_0 - y_0 \| \\
&\ge \left(1 - \frac{1}{2} \omega \rho_0 \right) \alpha_0 > 0,
\end{aligned}
\tag{28}
$$

we obtain that x_1 is well defined and

$$
\begin{aligned}
\| x_1 - z_0 \| &\le \frac{2 \| z_0 - y_0 \|^2}{\| x_0 - y_0 - \omega(y_0 - z_0) \|} \\
&\le \frac{\rho_0^2 \alpha_0}{2 - \omega \rho_0},
\end{aligned}
\tag{29}
$$

$$
\| x_1 - y_0 \| \le \| x_1 - z_0 \| + \| z_0 - y_0 \| \le \frac{\rho_0 \alpha_0}{2} \frac{2 + (2 - \omega) \rho_0}{2 - \omega \rho_0}, \tag{30}
$$

$$
\begin{aligned}
\| x_1 - x_0 \| &\le \| x_1 - y_0 \| + \| y_0 - x_0 \| \\
&\le \alpha_0 \left[1 + \frac{1}{2} \rho_0 + \frac{\rho_0^2}{2 - \omega \rho_0} \right] \\
&= g_1(\rho_0) \alpha_0 < \theta \alpha_0.
\end{aligned}
\tag{31}
$$

This shows $x_1 \in S$ and the validity of (20).

By condition (18) we have

$$
\| F'(x_1) - F'(x_0) \| \le \gamma \| x_1 - x_0 \| \le \gamma g_1(\rho_0) \alpha_0. \tag{32}
$$

Because

$$
\gamma g_1(\rho_0) \alpha_0 \beta_0 = g_1(\rho_0) \rho_0 < 1, \tag{33}
$$

by Banach lemma we obtain that $F'(x_1)$ is nonsingular and

$$
\beta_1 = \| F'(x_1)^{-1} \| \le \frac{\beta_0}{1 - g_1(\rho_0) \rho_0} = g_2(\rho_0) \beta_0. \tag{34}
$$

This is to say that (22) holds.

Now we consider $F(x_1)$. By making use of (24), (25), and (30), we obtain

$$
\begin{aligned}
\|F(x_1)\| \\
&= \left\| F(x_0) + F'(x_0)(x_1 - x_0) \right. \\
&\qquad \left. + \int_{x_0}^{x_1} \left[F'(x) - F'(x_0) \right] dx \right\| \\
&\le \frac{2 \| z_0 - y_0 \|}{\| x_0 - y_0 - \omega(y_0 - z_0) \|} \| F(y_0) \| \\
&\qquad + \left\| -F(y_0) + \int_{x_0}^{x_1} \left[F'(x) - F'(x_0) \right] dx \right\| \\
&\le \frac{2 \| z_0 - y_0 \| \| F(y_0) \|}{\| y_0 - x_0 \| - \omega \| z_0 - y_0 \|} \\
&\qquad + \left\| - \int_{x_0}^{y_0} \left[F'(x) - F'(x_0) \right] dx \right. \\
&\qquad \left. + \int_{x_0}^{x_1} \left[F'(x) - F'(x_0) \right] dx \right\| \\
&= \frac{\gamma \rho_0 \alpha_0^2}{2 - \omega \rho_0} + \left\| \int_{y_0}^{x_1} \left[F'(x) - F'(x_0) \right] dx \right\| \\
&\le \frac{\gamma \rho_0 \alpha_0^2}{2 - \omega \rho_0} + \int_0^1 \| F'(y_0 + t(x_1 - y_0)) - F'(x_0) \| \\
&\qquad \times \| x_1 - y_0 \| \, dt
\end{aligned}
$$

$$\leq \frac{\gamma \rho_0 \alpha_0^2}{2 - \omega \rho_0}$$

$$+ \gamma \int_0^1 \left[\|y_0 - x_0\| + t \|x_1 - y_0\| \right] \|x_1 - y_0\| \, dt$$

$$\leq \frac{\gamma \rho_0 \alpha_0^2}{2 - \omega \rho_0}$$

$$+ \gamma \left[\|y_0 - x_0\| + \frac{1}{2} \|x_1 - y_0\| \right] \|x_1 - y_0\|$$

$$\leq \frac{\gamma \rho_0 \alpha_0^2}{2 - \omega \rho_0}$$

$$+ \frac{\gamma}{2} \rho_0 \alpha_0^2 \frac{2 + (2 - \omega) \rho_0}{2 - \omega \rho_0} \left[1 + \frac{1}{4} \rho_0 \frac{2 + (2 - \omega) \rho_0}{2 - \omega \rho_0} \right]$$

$$= \frac{\gamma}{2} \rho_0 \alpha_0^2 \left[\frac{4 + (2 - \omega) \rho_0}{2 - \omega \rho_0} + \frac{1}{4} \rho_0 \left(\frac{2 + (2 - \omega)\rho_0}{2 - \omega \rho_0} \right)^2 \right]. \tag{35}$$

Finally, we prove (21) and (23). By making use of (34) and (35), we have

$$\alpha_1 \leq \|F'(x_1)^{-1}\| \|F(x_1)\|$$

$$\leq \frac{1}{2} \alpha_0 \rho_0^2 g_2(\rho_0)$$

$$\times \left[\frac{4 + (2 - \omega) \rho_0}{2 - \omega \rho_0} + \frac{1}{4} \rho_0 \left(\frac{2 + (2 - \omega) \rho_0}{2 - \omega \rho_0} \right)^2 \right] \tag{36}$$

$$= g_3(\rho_0) \alpha_0.$$

It then holds that

$$\rho_1 = \alpha_1 \beta_1 \gamma$$

$$\leq \rho_0 g_2(\rho_0) g_3(\rho_0) \tag{37}$$

$$= g_4(\rho_0) \rho_0$$

$$< g_4(t^*) \rho_0 = \rho_0 < t^*.$$

Now we consider the cases $k \geq 1$. By induction we can obtain the following facts.

(P1) By Lemma 2, we obtain that

$$\rho_k \leq g_4(\rho_{k-1}) \rho_{k-1} < g_4(t^*) \rho_{k-1} = \rho_{k-1} < t^*, \tag{38}$$

which leads to

$$g_3(\rho_k) < g_3(\rho_{k-1}) < \cdots < g_3(\rho_0) < g_3(t^*) < 1,$$

$$g_1(\rho_k) < g_1(t^*). \tag{39}$$

It follows that

$$\alpha_k \leq g_3(\rho_{k-1}) \alpha_{k-1} \leq g_3(\rho_{k-1}) \cdots g_3(\rho_0) \alpha_0 < g_3(t^*)^k \alpha_0. \tag{40}$$

This further yields

$$\|x_k - x_{k-1}\| \leq g_1(\rho_{k-1}) \alpha_{k-1} < g_1(t^*) g_3(t^*)^{k-1} \alpha_0. \tag{41}$$

Thus it is obtained that

$$\|x_k - x_0\| \leq \sum_{i=1}^{k} \|x_i - x_{i-1}\| \tag{42}$$

$$< \sum_{i=1}^{k} g_1(t^*) g_3(t^*)^{i-1} \alpha_0.$$

Next we show that y_k, z_k are well defined in S. By Lemma 2 and (42), we have

$$\|y_k - x_0\| \leq \alpha_k + \|x_k - x_0\|$$

$$\leq g_3(t^*)^k \alpha_0 + \sum_{i=1}^{k} g_1(t^*) g_3(t^*)^{i-1} \alpha_0$$

$$< \sum_{i=0}^{k} g_1(t^*) g_3(t^*)^i \alpha_0 \tag{43}$$

$$< \frac{g_1(t^*)}{1 - g_3(t^*)} \alpha_0.$$

This means that $y_k \in S$. Furthermore, by analogous procedures to (24), (25), and (26), we obtain that

$$\|z_k - y_k\| \leq \frac{1}{2} \rho_k \alpha_k,$$

$$\|z_k - x_k\| \leq \|z_k - y_k\| + \|y_k - x_k\| \leq \left(1 + \frac{1}{2} \rho_k \right) \alpha_k. \tag{44}$$

Since

$$1 + \frac{1}{2} \rho_k < g_1(\rho_k) < g_1(t^*), \tag{45}$$

we get

$$\|z_k - x_k\| < g_1(\rho_k) \alpha_k < g_1(t^*) g_3(t^*)^k \alpha_0. \tag{46}$$

Hence it follows that

$$\|z_k - x_0\| \leq \|z_k - x_k\| + \|x_k - x_0\|$$

$$\leq \sum_{i=0}^{k} g_1(t^*) g_3(t^*)^i \alpha_0 \tag{47}$$

$$< \frac{g_1(t^*)}{1 - g_3(t^*)} \alpha_0.$$

This shows that $z_k \in S$. Similarly to the case $k = 0$, we obtain that x_{k+1} is well defined and have

$$\|x_{k+1} - z_k\| \leq \frac{\rho_k^2 \alpha_k}{2 - \omega \rho_k},$$

$$\|x_{k+1} - y_k\| \leq \|x_{k+1} - z_k\| + \|z_k - y_k\| \leq \frac{\rho_k \alpha_k}{2} \frac{2 + (2 - \omega) \rho_k}{2 - \omega \rho_k},$$

$$\|x_{k+1} - x_k\| \leq g_1(\rho_k) \alpha_k. \tag{48}$$

By (42) we obtain

$$\|x_{k+1} - x_0\| < \frac{g_1(t^*)}{1 - g_3(t^*)} \alpha_0, \tag{49}$$

which shows $x_{k+1} \in S$.

(P2) We can prove analogously to (35) that

$$\|F(x_{k+1})\|$$
$$\leq \frac{\gamma}{2} \rho_k \alpha_k^2 \left[\frac{4 + (2 - \omega)\rho_k}{2 - \omega \rho_k} + \frac{1}{4} \rho_k \left(\frac{2 + (2 - \omega)\rho_k}{2 - \omega \rho_k} \right)^2 \right]. \tag{50}$$

(P3) Because

$$\|F'(x_{k+1}) - F'(x_k)\| \leq \gamma g_1(\rho_k) \alpha_k,$$
$$g_1(\rho_k) \rho_k < g_1(\rho_0) \rho_0 < 1, \tag{51}$$

we obtain that $F'(x_{k+1})$ is nonsingular and

$$\beta_{k+1} = \|F'(x_{k+1})^{-1}\| \leq g_2(\rho_k) \beta_k. \tag{52}$$

(P4) From (50) and (52), we have

$$\alpha_{k+1} \leq g_3(\rho_k) \alpha_k. \tag{53}$$

It then follows that

$$\rho_{k+1} = \alpha_{k+1} \beta_{k+1} \gamma$$
$$\leq g_4(\rho_k) \rho_k \tag{54}$$
$$< g_4(t^*) \rho_k = \rho_k.$$

Thus far, we have proved all conclusions of this theorem. □

The theorem given below will establish the convergence of the sequence $\{x_k\}$ and give the error estimate for it.

Theorem 4. *Let the conditions of Theorem 3 be satisfied. Denote $p = g_1(t^*)$ and $q = g_3(t^*)$. Then the sequence $\{x_k\}$ generated by (4) converges to a unique solution $x^* \in S$ of $F(x)$, and it holds that*

$$\|x^* - x_k\| < \frac{pq^k}{1 - q} \alpha_0. \tag{55}$$

Proof. Since $0 < q < 1$, it follows from (41) that

$$\|x_{k+m} - x_k\| \leq \sum_{i=0}^{m-1} \|x_{k+i+1} - x_{k+i}\|$$
$$< \sum_{i=0}^{m-1} pq^{k+i} \alpha_0 \tag{56}$$
$$< \frac{pq^k}{1 - q} \alpha_0.$$

This means that $\{x_k\}$ is a Cauchy sequence and thus there exists a x^* such that $\lim_{k \to \infty} x_k = x^*$. By letting $m \to \infty$ in (56), we obtain (55). From (56) and (42), we can get

$$\|x^* - x_0\| \leq \|x^* - x_k\| + \|x_k - x_0\|$$
$$< \sum_{i=0}^{\infty} pq^i \alpha_0 = \frac{p}{1 - q} \alpha_0. \tag{57}$$

This shows $x^* \in S$.

From (50) and (40), we obtain that

$$\|F(x_k)\| < \frac{\gamma}{2} \rho_0 \alpha_0^2 q^{2(k-1)}$$
$$\times \left[\frac{4 + (2 - \omega)\rho_0}{2 - \omega \rho_0} + \frac{1}{4} \rho_0 \left(\frac{2 + (2 - \omega)\rho_0}{2 - \omega \rho_0} \right)^2 \right]. \tag{58}$$

By letting $k \to \infty$ in (58), we obtain $F(x^*) = 0$; namely, x^* is a solution of $F(x)$.

Now, we prove the uniqueness of x^* in S. Let x^{**} be another zero of $F(x)$ in S. By mean value theorem, we have

$$0 = F(x^{**}) - F(x^*) = F'(\xi)(x^{**} - x^*), \tag{59}$$

where ξ is between x^* and x^{**}. Since

$$\beta_0 \|F'(\xi) - F'(x_0)\| \leq \beta_0 \gamma \|\xi - x_0\|$$
$$\leq \rho_0 \frac{g_1(t^*)}{1 - g_3(t^*)} \tag{60}$$
$$< \frac{t^* g_1(t^*)}{1 - g_3(t^*)} = 1,$$

it follows by Banach lemma that $F'(\xi)$ is invertible and hence $x^{**} = x^*$. This ends the proof. □

4. Numerical Tests

In this section, we present some numerical results for the method given by (4) (NTM) and compare it with PPM on their numerical behavior. We also test the composite methods combining PPM with some known vector extrapolation methods mentioned in Section 1, which are indicated as VEA-PPM, MPE-PPM, and RRE-PPM, respectively. We use $\|F_k\|_2$ to denote the value of $\|F(x)\|_2$ at the kth approximate solution x_k.

We consider the nonlinear elliptic differential equation:

$$-\nabla \cdot K(\Theta(\psi)) \nabla \psi = 0. \tag{61}$$

This equation often arises from the flow model in porous media and in this case, ψ is the pressure, Θ the fluid saturation, and K the conductivity. The boundary conditions can be given by

$$-K(\Theta(\psi)) \nabla \psi = V_B, \quad \text{on } \Gamma_N,$$
$$\psi = \psi_B, \quad \text{on } \Gamma_D. \tag{62}$$

TABLE 1: Results of the case $m = 100$.

x_0	PPM	VEA-PPM	MPE-PPM	RRE-PPM	NTM
$0.5S$	D	D	8	8	5
$1S$	D	6	7	7	5
$1.5S$	22	6	6	6	5
$2S$	D	5	6	6	4
$3S$	D	5	6	6	4

TABLE 2: Results of the case $m = 1000$.

x_0	PPM	VEA-PPM	MPE-PPM	RRE-PPM	NTM
$0.1S$	D	D	10	10	7
$0.5S$	D	D	9	9	6
$1S$	D	30	8	8	6
$1.5S$	D	8	7	7	5
$2S$	D	7	7	7	5
$3S$	D	6	7	7	5

In this test, we consider the one-dimensional case. The uniform cell-centered finite difference (CCFD) approximation method is used to discretize the boundary value problem. For the detailed CCFD formulations, we refer to [30] or the references therein. The values $K(\Theta(\psi))$ on the faces of each cell are taken as the harmonic mean of cell-central ones. Here, we take $K(\Theta(\psi)) = k\psi$ where k is a positive real constant. The input boundary condition is given by $V_B = 1$, while the output boundary condition is $\psi_B = 1$.

The discrete scheme leads to a nonlinear equation system with m variables. We test two cases with the sizes $m = 100$ and 1000, respectively. We take $\omega = 2$ in our method. All methods start from the initial approximate solutions and stop when they satisfy the given criteria. For the case $m = 100$, the stopping criterion is $\|F\|_2 < 1e - 12$, while it is taken as $\|F\|_2 < 1e - 11$ for $m = 1000$. In these tables, we show the iteration number cost by various methods.

The computational results are displayed in Tables 1 and 2. In the tables, denote $S = (1, 1, \ldots, 1)^T$ and "D" indicates that the method is divergent or cannot converge in 50 steps. We use NTM to represent the proposed method.

From the numerical results, we can know that the performance of NTM is more efficient and robust than PPM.

5. Conclusions

We establish the convergence of a third-order method for systems of nonlinear equations; an existence-uniqueness theorem and the error estimate for this method are also obtained. Numerical results show that this method is more robust and efficient than a number of Newton-type methods with the other vector extrapolation algorithms.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.
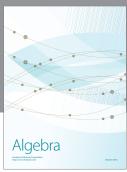
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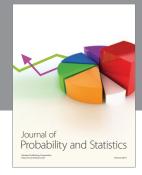
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[30] J. Kou, S. Sun, and B. Yu, "Multiscale time-splitting strategy for multiscale multiphysics processes of two-phase flow in fractured media," *Journal of Applied Mathematics*, vol. 2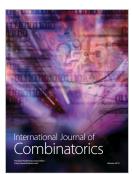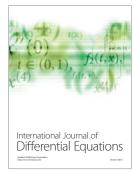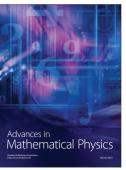011, Ar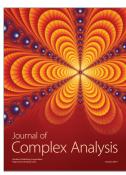ticle ID 861905, 24 pages, 2011.