Research Article

Upper Bound of Second Hankel Determinant for Certain Subclasses of Analytic Functions

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In this present investigation, we first give a survey of the work done so far in this area of Hankel determinant for univalent functions. Then the upper bounds of the second Hankel determinant

\[
|a_2a_4 - a_2^3|
\]

for functions belonging to the subclasses \(S(\alpha, \beta), K(\alpha, \beta), S^*_{\alpha, \beta}, \) and \(K^*_{\alpha, \beta}\) of analytic functions are studied. Some of the results, presented in this paper, would extend the corresponding results of earlier authors.

1. Introduction

Let \(A\) denote the class of functions of the form

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]

which are analytic in the unit disc \(U = \{ z : |z| < 1 \}\), and let \(S\) denote the subclass of \(A\) that is univalent in \(U\). Suppose that \(f\) and \(g\) are analytic functions in \(U\); we say that \(f\) is subordinate to \(g\), written \(f < g\), if there exists a Schwarz function \(\omega\), which is analytic in \(U\) with \(\omega(0) = 0\) and \(|\omega(z)| < 1\) for all \(z \in U\), such that \(f(z) = g(\omega(z))\), \(z \in U\). In particular, if \(g\) is univalent in \(U\), then the subordination is equivalent to \(f(0) = g(0)\) and \(f(U) \subset g(U)\).

Let \(P\) be the family of all functions \(p\) analytic in \(U\) for which \(\Re\{p(z)\} > 0\) and

\[
p(z) = 1 + c_1 z + c_2 z^2 + \cdots
\]

for \(z \in U\).

It is well known that the following correspondence between the class \(P\) and the class of Schwarz functions \(\omega\) exists [1]:

\[
p \in P \iff p = \frac{1 + \omega}{1 - \omega}
\]

Let \(S^*\) denote the starlike subclass of \(S\). It is well known that \(f \in S^*\) if and only if

\[
\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \quad (z \in U).
\]

Let \(K\) denote the class of all functions \(f \in A\) that are convex. Further, \(f\) is convex if and only if \(zf'\) is starlike. Also we know that \(K \subset S^* \subset S\).

In 1959, Sakaguchi [2] introduced the class \(S^*_{\alpha, \beta}\) of functions starlike with respect to symmetric points, consisting of functions \(f \in S\) satisfying

\[
\Re \left\{ \frac{2zf'(z)}{f(z) - f(-z)} \right\} > 0 \quad (z \in U).
\]

In 1977, Das and Singh [3] introduced the class \(K^*_{\alpha, \beta}\) of functions convex with respect to symmetric points, which consists of functions \(f \in S\) satisfying

\[
\Re \left\{ \frac{2(zf'(z))'}{(f(z) - f(-z))'} \right\} > 0 \quad (z \in U).
\]

It is evident that \(f \in K^*_{\alpha, \beta}\) if and only if \(zf' \in S^*_{\alpha, \beta}\).


\[
\Re \left\{ \frac{2(zf'(z))'}{(f(z) - f(-z))'} \right\} > 0 \quad (z \in U).
\]
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Definition 1 (see [4]). Suppose that $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Let $S(\alpha, \beta)$ denote the class of functions $f$ in $\mathcal{A}$ satisfying the following inequality:

$$\left|zf' (z) \over f (z)-1 \right| < \beta \left| \alpha zf' (z) \over f (z) \right| + 1 \quad (z \in U). \tag{7}$$

From [4], one knows that the above condition is equivalent to

$$zf' (z) \over f (z)-1 < \beta \left| 1 + \beta z \over 1 - \alpha \beta z \right| \quad (z \in U), \tag{8}$$

which implies that

$$S(\alpha, \beta) \subset S^* \subset S. \tag{9}$$

If $\alpha = \beta = 1$, then the class $S(\alpha, \beta)$ reduces to the class $S^*$. In the similar way, one can easily get the following definitions.

Definition 2. Suppose that $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Let $K(\alpha, \beta)$ denote the class of functions $f$ in $\mathcal{A}$ satisfying the following inequality:

$$\left|\left( zf' (z) \right)' \over f' (z) -1 \right| < \beta \left| \alpha \left( zf' (z) \right)' \over f' (z) \right| + 1 \quad (z \in U). \tag{10}$$

It is evident that the above condition is equivalent to

$$\left( zf' (z) \right)' \over f' (z) < \beta \left| 1 + \beta z \over 1 - \alpha \beta z \right| \quad (z \in U), \tag{11}$$

which implies that

$$K(\alpha, \beta) \subset K \subset S. \tag{12}$$

If $\alpha = 1$ and $\beta = 1$, then the class $K(\alpha, \beta)$ reduces to the class $K$.

Definition 3. Suppose that $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Let $S^*_f(\alpha, \beta)$ denote the class of functions $f$ in $\mathcal{A}$ satisfying the following inequality:

$$\left|zf' (z) \over f (z)-f (-z) \right| -1 < \beta \left| 2 \alpha zf' (z) \over f (z) -f (-z) \right| + 1 \quad (z \in U). \tag{13}$$

From [5], one knows that the above condition is equivalent to

$$zf' (z) \over f (z)-f (-z) < \beta \left| 1 + \beta z \over 1 - \alpha \beta z \right| \quad (z \in U). \tag{14}$$

The function class $S^*_f(\alpha, \beta)$ was introduced and investigated by Sudharsan et al. [6]. If $\alpha = 1$ and $\beta = 1$, then the class $S^*_f(\alpha, \beta)$ reduces to the class $S^*_f$.

Definition 4. Suppose that $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Let $K_f(\alpha, \beta)$ denote the class of functions $f$ in $\mathcal{A}$ satisfying the following inequality:

$$\left|2zf' (z) \over f (z) -f (-z) \right| -1 < \beta \left| 2 \alpha zf' (z) \over f (z) -f (-z) \right| + 1 \quad (z \in U). \tag{15}$$

It is evident that the above condition is equivalent to

$$2zf' (z) \over f (z) -f (-z) < \beta \left| 1 + \beta z \over 1 - \alpha \beta z \right| \quad (z \in U). \tag{16}$$

If $\alpha = 1$ and $\beta = 1$, then the class $K_f(\alpha, \beta)$ reduces to the class $K_f$.

In 1966, Pommerenke [7] stated the qth Hankel determinant for $q \geq 1$ and $n \geq 1$ as

$$H_q (n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}, \quad (a_1 = 1). \tag{17}$$

This Hankel determinant is useful and has also been considered by several authors. The growth rate of Hankel determinant $H_q (n)$ as $n \rightarrow \infty$ was investigated, respectively, when $f$ is a member of certain subclass of analytic functions, such as the class of p-valent functions [7, 8], the class of starlike functions [7], the class of univalent functions [9], the class of close-to-convex functions [10], the class of strong close-to-convex functions [11], a new class $V_k$ [12], and a new class $N_k (q, \rho, \beta)$ [13]. Similar to the above discussions, we can also refer to [14, 15]. Ehrenborg [16] studied the Hankel determinant of exponential polynomials. The Hankel transform of an integer sequence was defined and some of its properties were discussed by Layman [17]. Pommerenke [9] proved that the Hankel determinants of univalent function satisfy

$$|H_q (n)| \leq Kn^{-(1/2+\beta)q+3/2}. \tag{18}$$

Later, $|H_q (n)| \leq An^{1/2}$ was also proved by Hayman [18]. One can easily observe that the Fekete and Szegö functional is $H_2 (1) = a_2 - a_2^2$. For results related to the functional, see [19, 20]. Fekete and Szegö further generalized the estimate $|a_3 - \mu a_3^2|$, where $\mu$ is real and $f \in S$. For results related to the functional, see [21, 22]. In 2010, Hayami and Owa [21, 22] also generalized the estimate $|a_3a_{n-2} - \mu a_3^2|$ for analytic function. Later, in 2012, Krishna and Ramreddy [23] also generalized the estimate $|a_p a_{n-p} - \mu a_p^2|$ for p-valent analytic function; see also [24, 25].

For our discussion in this paper, we consider the second Hankel determinant in the case of $q = 2$ and $n = 2$, namely,

$$H_2 (2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2 a_4 - a_3^2. \tag{19}$$
Janteng et al. [26] have considered the functional $|H_2(2)|$ and found a sharp bound, the subclass of $S$ denoted by $R$, defined as $\mathcal{R}\{f'(z)\} > 0$. In their work, they have shown that if $f \in R$, then $|H_2(2)| \leq 4/9$. These authors [27, 28] also studied the second Hankel determinant and sharp bound for the classes of starlike and convex functions, close-to-starlike and close-to-convex functions with respect to symmetric points denoted by $S^*_s, S^*_c, K_s$, and $K_c$, and have shown that $|H_2(2)| \leq 1, |H_2(2)| \leq 1/8, |H_2(2)| \leq 1$, and $|H_2(2)| \leq 1/9$, respectively.

Singh [29] established the second Hankel determinant and sharp bound for the classes of close-to-starlike and close-to-convex functions with respect to conjugate and symmetric conjugate points denoted by $S^*_s, S^*_c, K_s$, and $K_c$, and has shown that $|H_2(2)| \leq 1, |H_2(2)| \leq 1, |H_2(2)| \leq 1/8$, and $|H_2(2)| \leq 1/9$, respectively.

Mishra and Gochhayat [30] obtained the sharp bound to $|H_2(2)|$ for the functions in the class denoted by $R_1(\alpha, \rho)$, $(0 \leq \lambda < 1, |\alpha| < \pi/2, 0 \leq \rho \leq 1)$ and defined as $\mathcal{R}\{e^{i\alpha}(\Omega_1 f(z)/z)\} \geq \rho \cos \alpha$, using the fractional differential operator denoted by $\Omega_1 f(z)$ and defined by Owa and Srivastava [31]. These authors have shown that if $f \in R_1(\alpha, \rho)$, then $|H_2(2)| \leq \{(1 - \rho)^2 (2 - \lambda)^2 (3 - \lambda)^2 \cos^2 \alpha)/9\}$.

Mohammed and Darus [32] have obtained a sharp upper bound to $|H_2(2)|$ for the functions in the class denoted by $S^*_m(\alpha, \sigma)$, $(|\alpha| < \pi/2, 0 \leq \sigma < 1)$ and defined as $\mathcal{R}\{e^{i\alpha}(\Omega_2 f(z)/z)\} > \sigma \cos \alpha$. These authors have proved that if $f \in S^*_m(\alpha, \sigma)$, then $|H_2(2)| \leq \{(4m^2(1 - \sigma)^2(1 + m)^2 \cos^2 \alpha)/(3\sigma^2(\lambda + 1)^2(\lambda + 2)^2)\}$.

Similar to the above discussions in a new subclass of analytic function with different operators, we can also refer to [33, 34]. Singh [35] also obtained a sharp upper bound for the functional $|H_2(2)|$ for the function $f \in M^\alpha$, where

\[ M^\alpha = \left\{ f \in \mathcal{A} : \mathcal{R}\left[ \frac{zf'(z) + \alpha z^2 f''(z)}{(1 - \alpha) f(z) + \alpha z f'(z)} \right] > 0, \quad 0 \leq \alpha \leq 1, z \in U \right\}. \tag{20} \]

and showed that if $f \in M^\alpha$, then $|H_2(2)| \leq 1/(1 + \alpha)(1 + 3\alpha)$.

Mehrok and Singh [36] have obtained a sharp upper bound to $|H_2(2)|$ for the function in the classes denoted by $M^\alpha$ and $C^\alpha_{\alpha(\alpha)}$ and defined as, respectively,

\[ C^\alpha_{\alpha(\alpha)} = \left\{ f \in \mathcal{A} : \mathcal{R}\left[ \left( \frac{zf'(z)}{f(z)} \right)^{1-\alpha} \left( \frac{(zf'(z))^\gamma}{f'(z)} \right)^\alpha \right] > 0, \quad 0 \leq \alpha \leq 1, z \in U \right\}. \tag{21} \]

In their work, they proved that if $f \in M^\alpha$, then

\[ |H_2(2)| \leq \frac{1}{(1 + 2\alpha)^2} \times \left( \alpha \left( 11 + 36\alpha + 38\alpha^2 + 12\alpha^3 - \alpha^4 \right) \times \left( 1 + 3\alpha \right) \left( -4 + 263\alpha + 603\alpha^2 + 253\alpha^3 + 37\alpha^4 \right) \times (1 + \alpha)^4 \right)^{-1} + 1, \tag{22} \]

and if $f \in C^\alpha_{\alpha(\alpha)}$, then $|H_2(2)| \leq 1/(1 + 2\alpha)^2$.

Shanmugam et al. [37] established the sharp upper bound of the second Hankel determinant for the classes of $S^*_m$ and $C_{\alpha}$, defined as, respectively,

\[ S^*_m = \left\{ f \in \mathcal{A} : \mathcal{R}\left[ \frac{zf'(z)}{f(z)} + \frac{\alpha z^2 f''(z)}{f'(z)} \right] > 0, z \in U \right\}, \tag{23} \]

\[ C_{\alpha} = \left\{ f \in \mathcal{A} : \mathcal{R}\left[ \left( \frac{zf'(z) + \alpha z^2 f''(z)}{f(z)} \right)^\gamma \right] > 0, z \in U \right\}. \tag{24} \]

These authors proved that if $f \in S^*_m$, then $|H_2(2)| \leq 1/(1 + 3\alpha)^2$ and if $f \in C_{\alpha}$, then

\[ |H_2(2)| \leq \frac{280\alpha^3 + 340\alpha^2 + 138\alpha + 18}{(1 + 2\alpha)^2(1 + 3\alpha)^2(1 + 4\alpha)}. \tag{25} \]

Krishna and Ramreddy [38] obtained a sharp upper bound to the nonlinear functional $|H_2(2)|$ for a new subclass of analytic functions $Q(\alpha, \beta, \gamma)$, $(\alpha, \beta > 0, 0 \leq \gamma < \alpha + \beta \leq 1)$, defined by

\[ Q(\alpha, \beta, \gamma) = \left\{ f \in \mathcal{A} : \mathcal{R}\left[ \alpha \frac{zf'(z)}{z} + \beta f'(z) \right] \geq \gamma, z \in U \right\}. \tag{26} \]

These authors proved that if $f \in Q(\alpha, \beta, \gamma)$, then $|H_2(2)| \leq [4(\alpha + \beta - \gamma)^2/(\alpha + 3\beta)]^2$.

Similar to the above discussions defined as different classes of analytic functions, we can also refer to [39–49].
Raza and Malik [50] studied the third Hankel determinant $H_3(1)$ of analytic functions related with lemniscate of Bernoulli; see also [51].

Motivated by the above-mentioned results obtained by different authors in this direction, in this present investigation, we determine the upper bounds of the second Hankel determinant $H_2(2)$ for functions belonging to these classes $S(\alpha, \beta)$, $K(\alpha, \beta)$, $S_\eta^\gamma(\alpha, \beta)$, and $K_\gamma(\alpha, \beta)$.

### 2. Preliminary Results

In order to prove our main results, we need the following lemmas.

**Lemma 5** (see [52]). If the function $p \in \mathcal{P}$ is given by the power series (2), then $|c_k| \leq 2 \ (k = 1, 2, \ldots)$.

**Lemma 6** (see [53, 54]). If the function $p \in \mathcal{P}$ is given by the power series (2), then

\[ 2c_2 = c_1^2 + (4 - c_1^2)x \]  

for some $x$ with $|x| \leq 1$ and

\[ 4c_3 = c_1^3 + 2c_1 (4 - c_1^2)x - c_1 (4 - c_1^2)x^2 + 2 (4 - c_1^2) (1 - |x|^2)z \]  

for some $z$ with $|z| \leq 1$.

### 3. Main Results

**Theorem 7.** Let $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Suppose that the function $f$ given by (1) is in the class $S(\alpha, \beta)$. Then

\[ |a_2a_4 - a_2^2| \leq \frac{1}{4} \beta^2 (1 + \alpha)^2. \]  

The result is sharp, with the extremal function

\[
f_1(z) = \begin{cases} 
  z^\left(-1 + \alpha / 2 \right) & , \quad 0 < \alpha \leq 1, \\
  z & , \quad \alpha = 0.
\end{cases}
\]

**Proof.** Since $f \in S(\alpha, \beta)$, it follows from (8) that there exists a Schwarz function $\omega$, which is analytic in $U$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ in $U$, such that

\[ \frac{zf'(z)}{f(z)} = \phi(\omega(z)) \quad (z \in U), \]  

where

\[
\phi(z) = \frac{1 + \beta z}{1 - \alpha \beta z} = 1 + \beta (1 + \alpha) z + \alpha^2 \beta^2 (1 + \alpha) z^2 + \alpha^2 \beta^3 (1 + \alpha) z^3 + \cdots.
\]

Define the function $p$ by

\[ p(z) = \frac{1 + \omega(z)}{1 - \omega(z)} = 1 + c_1 z + c_2 z^2 + \cdots. \]  

From (3), we get $p \in \mathcal{P}$ and

\[
\omega(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{c_1 z + \frac{1}{2} c_3 + \frac{1}{4} c_4}{1 + c_1 z + \frac{1}{2} c_3 c_4 z + \frac{1}{4} c_4 c_5 z^2 + \cdots}.
\]

In view of (30), (31), and (33), we have

\[
\frac{zf'(z)}{f(z)} = \phi(\omega(z))
\]

\[
= \phi\left(\frac{1}{2} c_1 z + \frac{1}{2} c_3 + \frac{1}{4} c_4 \right) z^2 + \frac{1}{2} c_2 c_4 z^3 + \cdots
\]

\[
= 1 + \frac{1}{2} \beta (1 + \alpha) c_1 z
\]

\[
+ \left[\frac{1}{2} \beta (1 + \alpha) \left(c_3 - \frac{1}{2} c_1^2\right) + \frac{1}{4} \alpha \beta^2 (1 + \alpha) c_3^2\right] z^2
\]

\[
+ \left[\frac{1}{2} \beta (1 + \alpha) \left(c_2 - \frac{1}{2} c_1 c_3 + \frac{1}{4} c_4\right) + \frac{1}{8} \alpha^2 \beta^3 (1 + \alpha) c_3\right] z^3 + \cdots.
\]  

(34)

Similarly,

\[
\frac{zf'(z)}{f(z)} = 1 + a_2 z + (2a_2 - a_2^2) z^2 + \left(3a_4 - 3a_2 a_3 + a_3^2\right) z^3 + \cdots.
\]  

(35)

Comparing the coefficients of $z$, $z^2$, and $z^3$ in (34) and (35), we obtain

\[
a_2 = \frac{1}{2} \beta (1 + \alpha) c_1,
\]

\[
a_3 = \frac{1}{8} \beta (1 + \alpha) \left[2c_2 + (\beta + 2\alpha \beta - 1) c_3\right],
\]

\[
a_4 = \frac{1}{8} \beta (1 + \alpha)
\]

\[
\times \left(1 - \frac{1}{2} \beta - \frac{7}{6} \alpha \beta + \frac{5}{6} \alpha \beta^2 + \frac{1}{6} \beta^3\right) c_3
\]

\[
- \frac{1}{2} \beta (1 + \alpha) \left(\frac{1}{3} - \frac{1}{4} \beta - \frac{7}{12} \alpha \beta\right) c_2 c_3 + \frac{1}{6} \beta (1 + \alpha) c_3.
\]  

(36)
Thus we have
\[ a_2a_4 - a_3^2 = -\frac{1}{192} \beta^2 (1 + \alpha)^2 \]
\[ \times \left[ (2\alpha^2 + 2\alpha\beta + \beta^2 - 1) \right. \\
\[ \left. c_4^4 - 4(\alpha\beta - 1) c_1^2 c_2 - 16c_1c_3 + 12c_2^2 \right] , \]
\[ (37) \]

\[ |a_2a_4 - a_3^2| = \frac{1}{192} \beta^2 (1 + \alpha)^2 \]
\[ \times \left[ (2\alpha^2 + 2\alpha\beta + \beta^2 - 1) \right. \\
\[ \left. c_4^4 - 4(\alpha\beta - 1) c_1^2 c_2 - 16c_1c_3 + 12c_2^2 \right] . \]
\[ (38) \]

Since the functions \( p(z) \) and \( p(e^{i\theta}z) \) \( (\theta \in \mathbb{R}) \) are members of the class \( \mathcal{P} \) simultaneously, we assume without loss of generality that \( c_1 > 0 \). For convenience of notation, we take \( c_1 = c \) \( (c \in [0,2]) \). By substituting the values of \( c_2 \) and \( c_3 \), respectively, from (26) and (27) in (38), we have
\[ |a_2a_4 - a_3^2| \leq \frac{1}{192} \beta^2 (1 + \alpha)^2 \]
\[ \times \left[ (2\alpha + 1) \beta^2 c^4 - 2\alpha\beta c^2 (4 - c^2) \right. \\
\[ \left. + (12 + c^2) (4 - c^2) |x|^2 \\
\[ - 8c (4 - c^2) (1 - |x|^2) \right] . \]
\[ (39) \]

Using the triangle inequality and \( |z| \leq 1 \), we have
\[ |a_2a_4 - a_3^2| \leq \frac{1}{192} \beta^2 (1 + \alpha)^2 \]
\[ \times \left[ (2\alpha + 1) \beta^2 c^4 - 2\alpha\beta c^2 (4 - c^2) |x| \\
\[ + (12 + c^2) (4 - c^2) |x|^2 \\
\[ + 8c (4 - c^2) (1 - |x|^2) \right] \]
\[ = \frac{1}{192} \beta^2 (1 + \alpha)^2 \]
\[ \times \left[ (2\alpha + 1) \beta^2 c^4 + 2\alpha\beta c^2 (4 - c^2) |x| \\
\[ + (12 + c^2) (4 - c^2) |x|^2 \\
\[ + 8c (4 - c^2) (1 - |x|^2) \right] \\
\[ = F(c, \mu) , \quad (say), \]
\[ (40) \]

where \( \mu = |x| \leq 1 \).

We next maximize the function \( F(c, \mu) \) on the closed square \([0,2] \times [0,1]\). Differentiating \( F(c, \mu) \) in (40) partially with respect to \( \mu \), we get
\[ \frac{\partial F(c, \mu)}{\partial \mu} = \frac{1}{96} \beta^2 (1 + \alpha)^2 \]
\[ \times \left[ (2\alpha + 1) \beta^2 (4 - c^2) + (c - 2) (c - 6) (4 - c^2) \mu \right] . \]
\[ (41) \]

For \( 0 < \mu < 1 \) and for any fixed \( c \) with \( 0 < c < 2 \), from (41), we observe that \( \partial F(c, \mu)/\partial \mu > 0 \). Consequently, \( F(c, \mu) \) is an increasing function of \( \mu \) and hence it cannot have a maximum value at any point in the interior of the closed square \([0,2] \times [0,1]\). Moreover, for fixed \( c \in [0,2] \), we have
\[ \max_{0 \leq \mu \leq 1} F(c, \mu) = F(c, 1) = G(c) \quad (say) . \]
\[ (42) \]

From the relations (40) and (42), upon simplification, we obtain
\[ G(c) = F(c, 1) = \frac{1}{192} \beta^2 (1 + \alpha)^2 \]
\[ \times \left[ (2\alpha + 1) \beta^2 (4 - c^2) + 8(\alpha\beta - 1) c^2 + 48 \right] . \]
\[ (43) \]

Next, since
\[ G'(c) = \frac{1}{48} \beta^2 (1 + \alpha)^2 c \]
\[ \times \left[ (2\alpha + 1) \beta^2 (4 - c^2) + 4(\alpha\beta - 1) \right] , \]
\[ (44) \]

we get that \( G'(c) \leq 0 \) for \( 0 < c \leq 2 \) and \( G(c) \) has real critical point at \( c = 0 \). Therefore, the maximum of \( G(c) \) occurs at \( c = 0 \). Thus, the upper bound of \( F(c, \mu) \) corresponds to \( \mu = 1 \) and \( c = 0 \). Hence,
\[ |a_2a_4 - a_3^2| \leq \frac{1}{4} \beta^2 (1 + \alpha)^2 . \]
\[ (45) \]

Equality holds for the function
\[ f_1(z) = \begin{cases} 
(1 - \alpha\beta z^2)^{-\frac{(1+\alpha)/2}{\alpha z}}, & 0 < \alpha \leq 1, \\
(1 - \alpha\beta z^2)^{-\frac{1}{2}}, & \alpha = 0.
\end{cases} \]
\[ (46) \]
By calculating, we have

\[
\frac{zf_1'(z)}{f_1(z)} = \frac{1 + \beta z^2}{1 - \alpha \beta z^2} < \frac{1 + \beta z}{1 - \alpha \beta z}
\]  

(47)

and \(a_2 = 0, a_3 = (1/2)\beta(1 + \alpha), \) and \(a_4 = 0.\) So \(f_1(z) \in S(\alpha, \beta)\) and equality holds. This shows that the result is sharp, and the proof of Theorem 7 is complete.

Setting \(\alpha = \beta = 1\) in Theorem 7, we obtain the following result due to Janteng et al. [27].

**Corollary 8.** If \(f(z) \in S^*,\) then

\[
|a_2a_4 - a_3^2| \leq 1.
\]

(48)

The result is sharp, with the extremal function

\[
f_2(z) = \frac{z}{1 - z^2}.
\]

(49)

By using the similar method as in the proof of Theorem 7, one can similarly prove Theorem 9.

**Theorem 9.** Let \(0 \leq \alpha \leq 1\) and \(0 < \beta \leq 1.\) Suppose that the function \(f\) given by (1) is in the class \(K(\alpha, \beta)\). Then

\[
|a_2a_4 - a_3^2| \leq \frac{5\alpha \beta + \beta - 2}{576\beta^2(1 + \alpha)^2}, \quad \text{if} \quad 5\alpha \beta + \beta - 2 \leq 0,
\]

\[
\text{or}
\]

\[
\frac{1}{36\beta^2(1 + \alpha)^2}, \quad \text{if} \quad 5\alpha \beta + \beta - 2 > 0.
\]

(50)

The results are sharp, with the extremal function

\[
f_3(z) = \int_0^z (1 - \alpha \beta \mu^2)^{-\alpha/2} d\mu, \quad 0 < \alpha \leq 1,
\]

\[
\int_0^z e^{\beta \mu^2/2} d\mu, \quad \alpha = 0
\]

(51)

for the case \(5\alpha \beta + \beta - 2 \leq 0,\) and there is no extremal function for the case \(5\alpha \beta + \beta - 2 > 0.\)

Setting \(\alpha = \beta = 1\) in Theorem 9, one obtains the following result due to Janteng et al. [27].

**Corollary 10.** If \(f(z) \in K,\) then

\[
|a_2a_4 - a_3^2| \leq \frac{1}{8}.
\]

(52)

The result is sharp.

**Theorem 11.** Let \(0 \leq \alpha \leq 1\) and \(0 < \beta \leq 1.\) Suppose that the function \(f\) given by (1) is in the class \(S^*_s(\alpha, \beta)\). Then

\[
|a_2a_4 - a_3^2| \leq \frac{1}{4\beta^2(1 + \alpha)^2}.
\]

(53)

The result is sharp, with the extremal function

\[
f_4(z) = \begin{cases} 
\int_0^z (1 - \alpha \beta \mu^2)^{-\alpha/2} \left(1 + \beta \mu^2 \right) d\mu, \quad 0 < \alpha \leq 1, \\
\int_0^z e^{\beta \mu^2/2} \left(1 + \beta \mu^2 \right) d\mu, \quad \alpha = 0.
\end{cases}
\]

(54)

**Proof.** Since \(f \in S^*_s(\alpha, \beta),\) it follows from (14) that there exists a Schwarz function \(\omega,\) which is analytic in \(U\) with \(\omega(0) = 0\) and \(|\omega(z)| < 1\) in \(U,\) such that

\[
\frac{2zf'(z)}{f(z) - f(-z)} = \phi(\omega(z)) \quad (z \in U),
\]

(55)

where \(\phi\) was defined by (31).

In view of (31), (33), and (55), we have

\[
\frac{2zf'(z)}{f(z) - f(-z)} = \phi(\omega(z)) \quad (z \in U)
\]

(56)

\[
= \phi \left( \frac{1}{2} c_1 z + \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \frac{1}{2} \left( c_2 - c_1 c_2 + \frac{c_1^4}{4} \right) z^3 + \cdots \right)
\]

\[
= 1 + \frac{1}{2} \beta (1 + \alpha) c_1 z
\]

\[
+ \left[ \frac{1}{2} \beta (1 + \alpha) \left( c_2 - \frac{c_1^2}{2} \right) + \frac{1}{4} \alpha \beta^2 (1 + \alpha) c_1^2 \right] z^2
\]

\[
+ \left[ \frac{1}{2} \beta (1 + \alpha) \left( c_3 - c_1 c_2 + \frac{c_1^4}{4} \right) + \frac{1}{2} \alpha \beta^2 (1 + \alpha) c_1 c_2 \right] z^3 + \cdots.
\]

(57)
Similarly,
\[
\frac{2zf'(z)}{f(z) - f(-z)} = 2a_2z + 2a_3z^2 + 2(2a_4 - a_2a_1)z^3 + \cdots.
\] 
(57)

Comparing the coefficients of \(z\), \(z^2\), and \(z^3\) in (56) and (57), we obtain
\[
\begin{align*}
a_2 &= \frac{1}{4} \beta (1 + \alpha) c_1, \\
a_3 &= \frac{1}{4} \beta (1 + \alpha) \left((\alpha \beta - 1) c_1^2 + 2c_2\right), \\
a_4 &= \frac{1}{64} \beta (1 + \alpha) \\
&\quad \times \left(2 - 4\alpha \beta + 3\alpha^2 \beta^2 + \alpha^3 \beta^3\right) c_1^3 \\
&\quad + \frac{1}{32} \beta (1 + \alpha) (5\alpha \beta + \beta - 4) c_1 c_2 \\
&\quad + \frac{1}{8} \beta (1 + \alpha) c_3.
\end{align*}
\] 
(58)

Thus we have
\[
\begin{align*}
|a_2a_4 - a_3^2| &= \frac{1}{256} \beta^2 (1 + \alpha)^2 \\
&\quad \times \left((\alpha^2 \beta^2 - \alpha^2 \beta^2 - 4\alpha \beta + 2) c_1^4 \\
&\quad + (6\alpha \beta - 2\beta - 8) c_1^2 c_2 - 8c_1 c_3 + 16c_2^2\right).
\end{align*}
\] 
(59)

Using the triangle inequality and \(|z| < 1\), we have
\[
|a_2a_4 - a_3^2| \leq \frac{1}{256} \beta^2 (1 + \alpha)^2 \\
&\quad \times \left[(\beta + \alpha \beta + \alpha^2 \beta^2 - \alpha^2 \beta^2)c^4 \\
&\quad + (3\alpha \beta - \beta + 4)c^2 (4 - c^2) |x| + 2 (4 - c^2) \right. \\
&\quad \times \left(8 - c^2\right) |x|^2 + 4c (4 - c^2) \left(1 - |x|^2\right) \right]
\] 
(60)

where \(\mu = |x| \leq 1\).

We next maximize the function \(F(c, \mu)\) on the closed square \([0, 2] \times [0, 1]\). Differentiating \(F(c, \mu)\) in (62) partially with respect to \(\mu\), we get
\[
\frac{\partial F (c, \mu)}{\partial \mu} = \frac{1}{256} \beta^2 (1 + \alpha)^2 \\
&\quad \times \left[(2\beta - 2\alpha \beta + \alpha^2 \beta^2 - \alpha^2 \beta^2 - 2) c^2 \\
&\quad + 2(3\alpha \beta - \beta - 2)ight]
\] 
(63)

For \(0 < \mu < 1\) and for any fixed \(c\) with \(0 < c < 2\), from (63), we observe that \(\partial F (c, \mu)/\partial \mu > 0\). Consequently, \(F(c, \mu)\) is an increasing function of \(\mu\) and hence it cannot have a maximum value at any point in the interior of the closed square \([0, 2] \times [0, 1]\). Moreover, for fixed \(c \in [0, 2]\), we have
\[
\max_{0 \leq \mu \leq 1} F (c, \mu) = F (c, 1) = G (c) \quad \text{say}.
\] 
(64)

From the relations (62) and (64), upon simplification, we obtain
\[
G (c) = F (c, 1)
\] 
(65)

Next, since
\[
G' (c) = \frac{1}{64} \beta^2 (1 + \alpha)^2 c \\
&\quad \times \left[(2\beta - 2\alpha \beta + \alpha^2 \beta^2 - \alpha^2 \beta^2 - 2)c^2 \\
&\quad + 2(3\alpha \beta - \beta - 2)ight],
\] 
(66)
we get that $G'(c) \leq 0$ for $0 < c \leq 2$ and $G(c)$ has real critical point at $c = 0$. Therefore, the maximum of $G(c)$ occurs at $c = 0$. Thus, the upper bound of $F(c, \mu)$ corresponds to $\mu = 1$ and $c = 0$. Hence,

$$|a_2a_4 - a_3^2| \leq \frac{1}{4}\beta^2(1 + \alpha)^2.$$  \hfill (67)

Equality holds for the function

$$f_4(z) = \begin{cases} \int_0^z \left(1 - \alpha\beta\mu^2\right)^{(1+\alpha)/2\alpha} \times \left(1 + \beta\mu^2\right) d\mu, & 0 < \alpha \leq 1, \\ \int_0^z e^{\beta\mu^2/2} \left(1 + \beta\mu^2\right) d\mu, & \alpha = 0. \end{cases} \hfill (68)$$

By calculating, we have

$$\frac{2zf_4'(z)}{f_4(z) - f_4(-z)} = \frac{1 + \beta z^2}{1 - \alpha\beta z^2} \leq \frac{1 + \beta z}{1 - \alpha\beta z} \hfill (69)$$

and $a_2 = 0, a_3 = -(1/2)\beta(1+\alpha)$, and $a_4 = 0$. So $f_4(z) \in S(\alpha, \beta)$ and equality holds. This shows that the result is sharp, and the proof of Theorem 11 is complete. \hfill \square

Setting $\alpha = \beta = 1$ in Theorem 11, we obtain the following result due to Janteng et al. [28].

**Corollary 12.** If $f(z) \in S^*_1$, then

$$|a_2a_4 - a_3^2| \leq 1.$$ \hfill (70)

The result is sharp, with the extremal function

$$f_5(z) = \int_0^z \frac{1 + \mu^2}{(1 - \mu^2)^2} d\mu. \hfill (71)$$

By using the similar method as in the proof of Theorem 11, one can similarly prove Theorem 13.

**Theorem 13.** Let $0 \leq \alpha \leq 1$ and $0 < \beta \leq 1$. Suppose that the function $f(z)$ given by (1) is in the class $K_\alpha(\alpha, \beta)$. Then

$$|a_2a_4 - a_3^2| \leq \frac{1}{36}\beta^2(1 + \alpha)^2.$$ \hfill (72)

The result is sharp, with the extremal function

$$f_6(z) = \begin{cases} \int_0^z 1 \omega \left\{ \int_0^\omega \left( \frac{2}{\alpha - \alpha\beta\mu^2} \right)^{(1+\alpha)/2\alpha} \times \left(1 + \beta\mu^2\right) d\mu \right\} d\omega, & 0 < \alpha \leq 1, \\ \int_0^z \omega \left\{ \int_0^\omega \frac{2 + \beta\mu^2}{2 - \alpha\beta\mu^2} d\mu \right\} d\omega, & \alpha = 0. \end{cases} \hfill (73)$$

Setting $\alpha = \beta = 1$ in Theorem 13, one obtains the following result due to Janteng et al. [28].

**Corollary 14.** If $f(z) \in K_\alpha$, then

$$|a_2a_4 - a_3^2| \leq \frac{1}{9}.$$ \hfill (74)

The result is sharp, with the extremal function

$$f_7(z) = 2 \int_0^z 1 \omega \left\{ \int_0^\omega \frac{2 + \mu^2}{(2 - \mu^2)^2} d\mu \right\} d\omega. \hfill (75)$$

**Conflict of Interests**

The authors declare that they have no conflict of interests regarding the publication of this paper.

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