Research Article

Distance from Bloch-Type Functions to the Analytic Space $F(p, q, s)$

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The analytic space $F(p, q, s)$ can be embedded into a Bloch-type space. We establish a distance formula from Bloch-type functions to $F(p, q, s)$, which generalizes the distance formula from Bloch functions to BMOA by Peter Jones, and to $F(p, p-2, s)$ by Zhao.

1. Introduction

Let $\mathbb{D}$ denote the unit disc $\{z \in \mathbb{C} : |z| < 1\}$ of the complex plane $\mathbb{C}$ and let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be its boundary. As usual, $H(\mathbb{D})$ denotes the space of all analytic functions on $\mathbb{D}$.

Recall that, for $0 < \alpha < \infty$, the Bloch-type space $\mathcal{B}_\alpha$ is the space of analytic functions on $\mathbb{D}$ satisfying

$$\|f\|_{\mathcal{B}_\alpha} = \sup_{z \in \mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty. \quad (1)$$

The little Bloch-type space $\mathcal{B}_\alpha^0$ is the subspace of all $f \in \mathcal{B}_\alpha$ with

$$\lim_{|z| \rightarrow 1} (1 - |z|^2)^\alpha |f'(z)| = 0. \quad (2)$$

It is well known that $\mathcal{B}_\alpha$ is a Banach space under the norm

$$\|f\|_{\mathcal{B}_\alpha}^* = |f(0)| + \|f\|_{\mathcal{B}_\alpha}. \quad (3)$$

In particular, when $\alpha = 1$, $\mathcal{B}_\alpha$ becomes the classic Bloch space $\mathcal{B}$, which is the maximal Möbius invariant Banach space that has a decent linear functional; see [1, 2] for more details on the Bloch spaces.

For $a \in \mathbb{D}$, the involution of the unit disk is denoted by $\sigma_a(z) = (a - z)/(1 - \bar{a}z)$. It is well known and easy to check that

$$1 - |\sigma_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2} = (1 - |a|^2) |\sigma'_a(z)|. \quad (4)$$

Let $0 < p < \infty$, $-2 < q < \infty$, $0 \leq s < \infty$, and $-1 < q + s < \infty$. The space $F(p, q, s)$, introduced by Zhao in [3] and known as the *general family of function spaces*, is defined as the set of $f \in H(\mathbb{D})$ for which

$$\begin{aligned} \|f\|_{F(p,q,s)}^p &= \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dA(z) < \infty, \end{aligned} \quad (5)$$

where $dA(z)$ is the normalized area measure on $\mathbb{D}$. The space $F_0(p, q, s)$ consists of all $f \in F(p, q, s)$ such that

$$\lim_{|a| \rightarrow 1} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^q (1 - |\sigma_a(z)|^2)^s dA(z) = 0. \quad (6)$$

For appropriate parameter values p , q , and s , $F(p, q, s)$ coincides with several classical function spaces. For instance, $F(p, q, s) = \mathcal{B}_{(q+2)/p}$ if $1 < s < \infty$. The space $F(p, p, 0)$ is the classical Bergman space $L_a^p(\mathbb{D})$, and $F(p, p-2, 0)$ is the classical Besov space B_p^p . The spaces $F(2, 0, s)$ are the Q_s spaces, in particular, $F(2, 0, 1) = \text{BMOA}$, and the function space of *bounded mean oscillation*. See [3–9] for these basic facts.

For $0 < s < \infty$, we say that a nonnegative Borel measure μ defined on $\mathbb{D}$ is an s -Carleson measure if

$$\|\mu\|_{\mathcal{C}_s} = \sup_{I \subset \mathbb{T}} \frac{\mu(S(I))}{|I|^s} < \infty, \quad (7)$$

where the supremum ranges over all subarcs I of $\mathbb{T}$, $|I|$ denotes the arc length of I , and

$$S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1 - |I| \leq r < 1, e^{i\theta} \in I\} \quad (8)$$

is the Carleson square based on a subarc $I \subseteq \mathbb{T}$. We write $\mathcal{CM}_s$ for the class of all s -Carleson measures. Moreover, μ is said to be a vanishing s -Carleson measure if

$$\lim_{|I| \rightarrow 0} \frac{\mu(S(I))}{|I|^s} = 0. \quad (9)$$

For f an analytic function on $\mathbb{D}$, we define

$$d\mu_f(z) = |f'(z)|^p (1 - |z|^2)^{q+s} dA(z). \quad (10)$$

It was proved in [3] that $f \in F(p, q, s)$ if and only if $d\mu_f$ is an s -Carleson measure and $f \in F_0(p, q, s)$ if and only if $d\mu_f$ is a vanishing s -Carleson measure.

Let $X \subset \mathcal{B}_\alpha$ be an analytic function space. The distance from a Bloch-type function f to X is defined by

$$\text{dist}_{\mathcal{B}_\alpha}(f, X) = \inf_{g \in X} \|f - g\|_{\mathcal{B}_\alpha}. \quad (11)$$

The following result is obtained by Zhao in [9].

Theorem 1. Suppose $1 \leq p < \infty$, $0 < s \leq 1$, and $f \in \mathcal{B}$. The following two quantities are equivalent:

- (1) $\text{dist}_{\mathcal{B}}(f, F(p, p-2, s))$;
- (2) $\inf\{\varepsilon : \chi_{\Omega_\varepsilon(f)}(1 - |z|^2)^{s-2} dA(z) \text{ is a Carleson measure}\}$,

where $\Omega_\varepsilon(f) = \{z \in \mathbb{D} : |f'(z)|(1 - |z|^2) \geq \varepsilon\}$ and χ denotes the characteristic function of a set.

When $p = 2$ and $s = 1$, the above characterization is Peter Jones's distance formula from a Bloch function to BMOA (Peter Jones never published his result but a proof was provided in [10]). Also, similar type results can be found in [11–13]. Specifically, distance from Bloch function to Q_K -type space is given in [11]; to the little Bloch space is obtained in [12], and to the Q_p space of the ball is characterized in [13]. All these spaces are Möbius invariant.

This paper is dedicated to characterize the distance from $f \in \mathcal{B}_{(q+2)/p}$ to $F(p, q, s)$, which extends Zhao's result. The main result is following.

Theorem 2. Suppose $1 \leq p < \infty$, $0 < s \leq 1$, $-1 < q + s < \infty$, and $f \in \mathcal{B}_{(2+q)/p}$. Then

$$\begin{aligned} & \text{dist}_{\mathcal{B}_{(q+2)/p}}(f, F(p, q, s)) \\ & \approx \inf \left\{ \varepsilon > 0 : \chi_{\Omega_\varepsilon(f)}(1 - |z|^2)^{s-2} dA(z) \in \mathcal{CM}_s \right\}, \end{aligned} \quad (12)$$

where

$$\Omega_\varepsilon(f) = \left\{ z \in \mathbb{D} : (1 - |z|^2)^{(q+2)/p} |f'(z)| \geq \varepsilon \right\}. \quad (13)$$

The strategy in this paper follows from Theorem 3.1.3 in [14]. The distance from a $\mathcal{B}_\alpha$ function to Campanato-Morrey space was given in [15] with similar idea.

Notation. Throughout this paper, we only write $U \leq V$ (or $V \geq U$) for $U \leq cV$ for a positive constant c , and moreover $U \approx V$ for both $U \leq V$ and $V \leq U$.

2. Preliminaries

We begin with a lemma quoted from Lemma 3.1.1 in [14].

Lemma 3. Let $s, \alpha \in (0, \infty)$, and μ be nonnegative Radon measures on $\mathbb{D}$. Then, $\mu \in \mathcal{CM}_s$ if and only if

$$\|\mu\|_{\mathcal{CM}_s, \alpha} = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha}{|1 - \bar{w}z|^{\alpha+s}} d\mu(z) < \infty. \quad (14)$$

According to Lemma 3 and the fact that $f \in F(p, q, s)$ if and only if $d\mu_f$ is an s -Carleson measure, we can easily get the following corollary.

Corollary 4. Let f be an analytic function on $\mathbb{D}$. $f \in F(p, q, s)$ if and only if there exists an $\alpha > 0$ such that

$$\begin{aligned} & \|f\|_{F(p, q, s), \alpha}^p \\ & = \sup_{w \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |w|^2)^\alpha}{|1 - \bar{w}z|^{\alpha+s}} |f'(z)|^p (1 - |z|^2)^{q+s} dA(z) < \infty. \end{aligned} \quad (15)$$

We will also need the following standard result from [16].

Lemma 5. Suppose $t > -1$ and $c > 0$. Then,

$$\int_{\mathbb{D}} \frac{(1 - |w|^2)^t}{|1 - \bar{w}z|^{2+t+c}} dA(w) \approx \frac{1}{(1 - |z|^2)^c} \quad (16)$$

for all $z \in \mathbb{D}$.

The following lemma, quoted from Lemma 1 in [9], is an extension of Lemma 5. See also [17].

Lemma 6. Suppose $s > -1$ and $r, t > 0$. If $t < s + 2 < r$, then

$$\int_{\mathbb{D}} \frac{(1 - |w|^2)^s}{|1 - \bar{w}z|^r |1 - \bar{w}\zeta|^t} dA(w) \leq \frac{1}{(1 - |z|^2)^{r-s-2} |1 - \bar{\zeta}z|^t}. \quad (17)$$

Next, we see that $F(p, q, s)$ is contained in $\mathcal{B}_{(2+q)/p}$. We thank Zhao for pointing out that the following result is firstly proved in [3]. Here, we give another proof with a different approach.

Lemma 7. For $1 \leq p < \infty$, $-2 < q < \infty$, and $0 \leq s < \infty$, $F(p, q, s) \subset \mathcal{B}_{(2+q)/p}$. In particular, if $s > 1$, then $F(p, q, s) = \mathcal{B}_{(2+q)/p}$.

Proof. We can use the reproducing formula for f' to get that

$$f'(z) = C \int_{\mathbb{D}} \frac{(1 - |w|^2)^{b-1} f'(w)}{(1 - \bar{w}z)^{b+1}} dA(w) \quad (18)$$

for some constant C , where b is a real number greater than $1 + (q + s)/p$; see, for example, [14, page 55].

Let $0 < \alpha < 2 + q$. If $p > 1$, denote $p' = p/(p - 1)$; it follows from the Hölder's inequality and (15) that

$$\begin{aligned} & |f'(z)| \\ & \leq \int_{\mathbb{D}} \frac{(1 - |w|^2)^{(q+s)/p} (1 - |z|^2)^{\alpha/p} |f'(w)|}{|1 - \bar{w}z|^{(s+\alpha)/p}} \\ & \quad \times \frac{(1 - |w|^2)^{b-1-(q+s)/p} dA(w)}{(1 - |z|^2)^{\alpha/p} |1 - \bar{w}z|^{b+1-(s+\alpha)/p}} \\ & \leq \left(\int_{\mathbb{D}} \frac{(1 - |z|^2)^{\alpha}}{|1 - \bar{w}z|^{s+\alpha}} |f'(w)|^p (1 - |w|^2)^{q+s} dA(w) \right)^{1/p} \\ & \quad \times \left(\int_{\mathbb{D}} \frac{(1 - |w|^2)^{p'(b-1-(q+s)/p)} dA(w)}{(1 - |z|^2)^{p'(\alpha/p)} |1 - \bar{w}z|^{p'(b+1-(s+\alpha)/p)}} \right)^{1/p'} \\ & \leq \frac{\|f\|_{F(p,q,s),\alpha}}{(1 - |z|^2)^{\alpha/p}} \\ & \quad \times \left(\int_{\mathbb{D}} \frac{(1 - |w|^2)^{p'(b-1-(q+s)/p)} dA(w)}{|1 - \bar{w}z|^{p'(b+1-(s+\alpha)/p)}} \right)^{1/p'} \\ & \leq \|f\|_{F(p,q,s),\alpha} \frac{1}{(1 - |z|^2)^{\alpha/p}} \\ & \quad \times \left(\frac{1}{(1 - |z|^2)^{(2-\alpha+q)/(p-1)}} \right)^{1/p'} \\ & = \|f\|_{F(p,q,s),\alpha} \frac{1}{(1 - |z|^2)^{(2+q)/p}}. \end{aligned} \quad (19)$$

Apparently, we have used Lemma 5 in the last inequality. This gives that $F(p, q, s) \subset \mathcal{B}_{(q+2)/p}$ when $1 < p < \infty$.

If $p = 1$, then

$$\begin{aligned} & (1 - |z|^2)^{2+q} |f'(z)| \\ & \leq \int_{\mathbb{D}} \frac{(1 - |w|^2)^{q+s} (1 - |z|^2)^{\alpha} |f'(w)|}{|1 - \bar{w}z|^{\alpha+s}} \\ & \quad \times \frac{(1 - |w|^2)^{b-1-q-s} dA(w)}{(1 - |z|^2)^{\alpha-2-q} |1 - \bar{w}z|^{b+1-s-\alpha}} \end{aligned}$$

$$\begin{aligned} & \leq \int_{\mathbb{D}} |f'(w)| (1 - |w|^2)^{q+s} \frac{(1 - |z|^2)^{\alpha}}{|1 - \bar{w}z|^{\alpha+s}} dA(w) \\ & \quad \times \sup_{w \in \mathbb{D}} \frac{(1 - |w|^2)^{b-1-q-s} (1 - |z|^2)^{2+q-\alpha}}{|1 - \bar{w}z|^{b+1-\alpha-s}} \\ & \leq \|f\|_{F(p,q,s),\alpha} \sup_{w \in \mathbb{D}} \frac{(1 - |w|^2)^{b-1-q-s} (1 - |z|^2)^{2+q-\alpha}}{|1 - \bar{w}z|^{b+1-\alpha-s}}. \end{aligned} \quad (20)$$

Recall that $b > 1 + q + s$ and $0 < \alpha < 2 + q$. We can easily use (4) to check that

$$\sup_{z, w \in \mathbb{D}} \frac{(1 - |w|^2)^{b-1-q-s} (1 - |z|^2)^{2+q-\alpha}}{|1 - \bar{w}z|^{b+1-\alpha-s}} \leq 1. \quad (21)$$

Thus, $F(p, q, s) \subset \mathcal{B}_{(q+2)/p}$ when $p = 1$.

Now, suppose $s > 1$ and let $f \in \mathcal{B}_{(q+2)/p}$, then

$$|f'(z)| (1 - |z|^2)^{(q+2)/p} \leq \|f\|_{(q+2)/p} < \infty \quad (22)$$

for all $z \in \mathbb{D}$. It follows that

$$\begin{aligned} & \|f\|_{F(p,q,s)}^p \\ & = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{q+s} \left(\frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^s dA(z) \\ & = \sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^{q+2} \\ & \quad \times (1 - |z|^2)^{s-2} \left(\frac{1 - |a|^2}{|1 - \bar{a}z|^2} \right)^s dA(z) \\ & \leq \|f\|_{\mathcal{B}_{(q+2)/p}}^p \sup_{a \in \mathbb{D}} (1 - |a|^2)^s \int_{\mathbb{D}} \frac{(1 - |z|^2)^{s-2}}{|1 - \bar{a}z|^{2s}} dA(z) \\ & \approx \|f\|_{\mathcal{B}_{(q+2)/p}}^p. \end{aligned} \quad (23)$$

Again, the above inequality follows from Lemma 5. This completes the proof. $\square$

Our strategy relies on an integral operator preserving the s -Carleson measures. For $a, b > 0$, we define the integral operator $T_{a,b}$ as

$$T_{a,b} f(z) = \int_{\mathbb{D}} \frac{(1 - |w|^2)^{b-1}}{|1 - \bar{w}z|^{a+b}} f(w) dA(w) \quad \forall z \in \mathbb{D}. \quad (24)$$

The following lemma is similar to Theorem 2.5 in [18]. Indeed, Qiu and Wu proved the case $1 < p < \infty$. Specially, the $p = 2$ case is just Lemma 3.1.2 in [14].

Lemma 8. Assume $0 < s \leq 1$, $1 \leq p < \infty$, and $\alpha > -1$. Let $b > (\alpha + 1)/p$, let $a > 1 - (\alpha + 1)/p$, and let f be Lebesgue measurable on $\mathbb{D}$. If $|f(z)|^p (1 - |z|^2)^{\alpha} dA(z)$ belongs to $\mathcal{CM}_s$, then $|T_{a,b} f(z)|^p (1 - |z|^2)^{p(a-1)+\alpha} dA(z)$ also belongs to $\mathcal{CM}_s$.

Proof. We firstly prove the case $p = 1$ and then sketch the outline argument of the case $1 < p < \infty$ modified from [18] for the completeness.

When $p = 1$, according to Lemma 3, it is sufficient to show that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^x}{|1 - \bar{a}z|^{x+s}} |T_{a,b}f(z)| (1 - |z|^2)^{a-1+\alpha} dA(z) < \infty \quad (25)$$

for some $x > 0$. That is to show

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^x}{|1 - \bar{a}z|^{x+s}} \left| \int_{\mathbb{D}} \frac{(1 - |w|^2)^{b-1} f(w)}{|1 - \bar{w}z|^{a+b}} dA(w) \right| \times (1 - |z|^2)^{a-1+\alpha} dA(z) \quad (26)$$

is finite. By Fubini's theorem, it is enough to verify that

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} (1 - |a|^2)^x \int_{\mathbb{D}} \frac{(1 - |z|^2)^{a-1+\alpha} dA(z)}{|1 - \bar{w}z|^{a+b} |1 - \bar{a}z|^{x+s}} \times |f(w)| (1 - |w|^2)^{b-1} dA(w) \quad (27)$$

is finite.

Choosing x such that $x+s < a+1+\alpha$, we can use Lemma 6 to control the last integral by

$$\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} \frac{(1 - |a|^2)^x}{|1 - \bar{a}w|^{x+s}} |f(w)| (1 - |w|^2)^{\alpha} dA(w). \quad (28)$$

Since $|f(z)|(1 - |z|^2)^{\alpha} dA(z)$ is an s -Carleson measure, we can complete the proof by using Lemma 3 again.

When $1 < p < \infty$, we need to verify that

$$\frac{1}{|I|^s} \int_{S(I)} |T_{a,b}f(z)|^p (1 - |z|^2)^{p(a-1+\alpha)} dA(z) \lesssim 1 \quad (29)$$

holds for any arc $I \subset \mathbb{T}$. In order to make this estimate, let N_I be the biggest integer satisfying $N_I \leq -\log_2 |I|$, and let I_n , $n = 0, 1, 2, \dots, N_I$, denotes the arcs on $\mathbb{T}$ with the same center as I and length $2^n |I|$, and I_{N_I+1} is just $\mathbb{T}$. We can control and decompose the integral as

$$\begin{aligned} & \int_{S(I)} |T_{a,b}f(z)|^p (1 - |z|^2)^{p(a-1+\alpha)} dA(z) \\ & \leq \int_{S(I)} \left(\int_{S(I_1)} \frac{(1 - |w|^2)^{b-1} (1 - |z|^2)^{(a-1)+\alpha/p}}{|1 - \bar{w}z|^{a+b}} \right. \\ & \quad \left. \times |f(w)| dA(w) \right)^p dA(z) \end{aligned}$$

$$\begin{aligned} & + \int_{S(I)} \left(\int_{\mathbb{D} \setminus S(I_1)} \frac{(1 - |w|^2)^{b-1} (1 - |z|^2)^{(a-1)+\alpha/p}}{|1 - \bar{w}z|^{a+b}} \right. \\ & \quad \left. \times |f(w)| dA(w) \right)^p dA(z) \\ & = \text{Int}_1 + \text{Int}_2. \end{aligned} \quad (30)$$

In order to estimate Int_1 , we define the linear operator $B : L^p(\mathbb{D}) \rightarrow L^p(\mathbb{D})$ as

$$B(f)(z) = \int_{\mathbb{D}} K(z, w) f(w) dA(w), \quad (31)$$

where

$$K(z, w) = \frac{(1 - |w|^2)^{b-1} (1 - |z|^2)^{(a-1)+\alpha/p}}{|1 - \bar{w}z|^{a+b}}. \quad (32)$$

If we choose a test function $g(z) = (1 - |z|^2)^{-1/pp'}$, then Schur's lemma combines with Lemma 5 implying that

$$\begin{aligned} & \int_{\mathbb{D}} K(w, z) g^p(w) dA(w) \leq g^p(z), \\ & \int_{\mathbb{D}} K(w, z) g^{p'}(z) dA(z) \leq g^{p'}(w). \end{aligned} \quad (33)$$

Hence, B is a bounded operator. Letting $h(w) = |f(w)|(1 - |w|^2)^{\alpha/p} \chi_{S(I_1)}(w)$, then $h \in L^p(\mathbb{D})$ with

$$\|h\|_{L^p}^p = \int_{S(I_1)} |f(w)|^p (1 - |w|^2)^{\alpha} dA(w) \lesssim |I|^s. \quad (34)$$

Thus,

$$\text{Int}_1 \lesssim \int_{\mathbb{D}} |B(h)(z)|^p dA(z) = \|B(h)\|_{L^p}^p \lesssim \|h\|_{L^p}^p \lesssim |I|^s. \quad (35)$$

To handle Int_2 , first note that, for $n = 0, 1, \dots, N_I$, if $z \in S(I)$ and $w \in S(I_{n+1}) \setminus S(I_n)$, then $|1 - \bar{w}z| \gtrsim 2^n |I|$. Further, it is easy to check that, for any fixed $\beta > -1$,

$$\int_{S(I_n)} (1 - |w|^2)^{\beta} dA(w) \lesssim (2^n |I|)^{\beta+2}, \quad n = 0, 1, \dots, N_I. \quad (36)$$

Now, splitting $\mathbb{D} \setminus S(I_1)$ as

$$\bigcup_{n=1}^{N_I} S(I_{n+1}) \setminus S(I_n) = \bigcup_{n=1}^{N_I} \tilde{S}_{n+1}, \quad (37)$$

we have

$$\begin{aligned} \text{Int}_2 &\leq \int_{S(I)} \left| \sum_{n=1}^{N_I} \int_{\tilde{S}_{n+1}} \frac{(1-|w|^2)^{b-1} |f(w)|}{|1-\bar{w}z|^{a+b}} dA(w) \right|^p \\ &\quad \times (1-|z|^2)^{p(a-1)+\alpha} dA(z) \\ &\lesssim |I|^{p(a-1)+\alpha+2} \\ &\quad \times \left(\sum_{n=1}^{N_I} \frac{1}{(2^n |I|)^{a+b}} \right. \\ &\quad \left. \times \int_{S(I_{n+1})} (1-|w|^2)^{b-1} |f(w)| dA(w) \right)^p. \end{aligned} \quad (38)$$

Recall that $|f(z)|^p (1-|z|^2)^\alpha dA(z) \in \mathcal{CM}_s$. It follows from Hölder's inequality that

$$\begin{aligned} &\int_{S(I_{n+1})} (1-|w|^2)^{b-1} |f(w)| dA(w) \\ &\lesssim |I_{n+1}|^{s/p} \cdot (2^{n+1} |I|)^{b-1-\alpha/p+2/p'}. \end{aligned} \quad (39)$$

Now, an easy computation gives that

$$\text{Int}_2 \lesssim \left(\sum_{n=1}^{N_I} 2^{-n(a-1+(\alpha+2-s)/p)} \right)^p |I|^s \lesssim |I|^s, \quad (40)$$

since $a > 1 - (\alpha + 1)/p$ and $0 < s \leq 1$. This completes the proof. $\square$

3. Proof of the Main Result

Proof of Theorem 2. For $f \in \mathcal{B}_{(q+2)/p}$, it is easy to establish the following formula (see, e.g., [19, (1.1)] or [14, page 55]. Notice that it is a special case of the α -order derivative of f , as $\alpha = 0$ in [14], which holds for all holomorphic f on $\mathbb{D}$). Consider

$$f(z) = f(0) + \int_{\mathbb{D}} \frac{(1-|w|^2)^{(q+2)/p} f'(w)}{\bar{w}(1-\bar{w}z)^{1+(q+2)/p}} dA(w) \quad \forall z \in \mathbb{D}. \quad (41)$$

Define, for each $\varepsilon > 0$,

$$\begin{aligned} f_1(z) &= f(0) + \int_{\tilde{\Omega}_\varepsilon(f)} \frac{(1-|w|^2)^{(q+2)/p} f'(w)}{\bar{w}(1-\bar{w}z)^{1+(q+2)/p}} dA(w), \\ f_2(z) &= \int_{\mathbb{D} \setminus \tilde{\Omega}_\varepsilon(f)} \frac{(1-|w|^2)^{(q+2)/p} f'(w)}{\bar{w}(1-\bar{w}z)^{1+(q+2)/p}} dA(w). \end{aligned} \quad (42)$$

Then,

$$\begin{aligned} &|f'_1(z)| \\ &\lesssim \|f\|_{\mathcal{B}_{(q+2)/p}} \int_{\mathbb{D}} \frac{\chi_{\tilde{\Omega}_\varepsilon(f)}(w)}{|1-\bar{w}z|^{2+(q+2)/p}} dA(w) \\ &= \|f\|_{\mathcal{B}_{(q+2)/p}} \int_{\mathbb{D}} \frac{(1-|w|^2)^{2/p}}{|1-\bar{w}z|^{2+(q+2)/p}} \\ &\quad \times \frac{\chi_{\tilde{\Omega}_\varepsilon(f)}(w)}{(1-|w|^2)^{2/p}} dA(w). \end{aligned} \quad (43)$$

Write

$$g(w) = \frac{\chi_{\tilde{\Omega}_\varepsilon(f)}(w)}{(1-|w|^2)^{2/p}}. \quad (44)$$

Then,

$$|g(w)|^p (1-|w|^2)^s dA(w) = \chi_{\tilde{\Omega}_\varepsilon(f)}(w) (1-|w|^2)^{s-2} dA(w). \quad (45)$$

So, if

$$\chi_{\tilde{\Omega}_\varepsilon(f)}(z) (1-|z|^2)^{s-2} dA(z) \quad (46)$$

is in $\mathcal{CM}_s$, Lemma 8 implies that

$$|f'_1(z)|^p (1-|z|^2)^{q+s} dA(z) \in \mathcal{CM}_s. \quad (47)$$

By Corollary 4, $f_1 \in F(p, q, s)$. Meanwhile, recall that, for $w \in \mathbb{D} \setminus \tilde{\Omega}_\varepsilon(f)$ and $(1-|w|^2)^{(q+2)/p} |f'(w)| < \varepsilon$, we can use Lemma 5 to obtain

$$\begin{aligned} &|f'_2(z)| \leq \int_{\mathbb{D} \setminus \tilde{\Omega}_\varepsilon(f)} \frac{(1-|w|^2)^{(q+2)/p} |f'(w)|}{|1-\bar{w}z|^{2+(q+2)/p}} dA(w) \\ &< \varepsilon \int_{\mathbb{D}} \frac{1}{|1-\bar{w}z|^{2+(q+2)/p}} dA(w) \\ &\approx \frac{\varepsilon}{(1-|z|^2)^{(2+q)/p}}. \end{aligned} \quad (48)$$

This means that

$$(1-|z|^2)^{(2+q)/p} |f'_2(z)| \leq \varepsilon. \quad (49)$$

To summarize the above argument, we have $f = f_1 + f_2$, $f_1 \in F(p, q, s)$ (by (47)), and $f_2 \in \mathcal{B}_{(2+q)/p}$ (by (49)), and $\chi_{\tilde{\Omega}_\varepsilon(f)}(z) (1-|z|^2)^{s-2} dA(z)$ is an s -Carleson measure for each $\varepsilon > 0$. Thus,

$$\begin{aligned} &\text{dist}_{\mathcal{B}_{(2+q)/p}}(f, F(p, q, s)) \\ &\leq \inf \left\{ \varepsilon > 0 : \chi_{\tilde{\Omega}_\varepsilon(f)}(z) (1-|z|^2)^{s-2} dA(z) \in \mathcal{CM}_s \right\}. \end{aligned} \quad (50)$$

In order to prove the other direction of the inequality, we assume that ε_0 equals the right-hand quantity of the last inequality and

$$\text{dist}_{\mathcal{B}_{(2+q)/p}}(f, F(p, q, s)) < \varepsilon_0. \quad (51)$$

We only consider the case $\varepsilon_0 > 0$. Then, there exists an ε_1 such that

$$0 < \varepsilon_1 < \varepsilon_0, \quad \text{dist}_{\mathcal{B}_{(2+q)/p}}(f, F(p, q, s)) < \varepsilon_1. \quad (52)$$

Hence, by definition, we can find a function $h \in F(p, q, s)$ such that

$$\|f - h\|_{\mathcal{B}_{(2+q)/p}} < \varepsilon_1. \quad (53)$$

Now, for any $\varepsilon \in (\varepsilon_1, \varepsilon_0)$, we have that

$$\chi_{\widetilde{\Omega}_\varepsilon(f)}(z) (1 - |z|^2)^{s-2} dA(z) \quad (54)$$

is not in $\mathcal{CM}_s$. But, according to (53), we get

$$(1 - |z|^2)^{(2+q)/p} |h'(z)| > (1 - |z|^2)^{(2+q)/p} |f'(z)| - \varepsilon_1 \quad (55)$$

$$\forall z \in \mathbb{D},$$

and so

$$\chi_{\widetilde{\Omega}_\varepsilon(f)}(z) \leq \chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z) \quad \forall z \in \mathbb{D}. \quad (56)$$

This implies that

$$\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z) (1 - |z|^2)^{s-2} dA(z) \quad (57)$$

does not belong to $\mathcal{CM}_s$. But, it follows from (13) that $\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h) = \{z \in \mathbb{D} : (1 - |z|^2)^{(q+2)/p} |h'(z)| \geq \varepsilon - \varepsilon_1\}$. Therefore,

$$\begin{aligned} & \chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z) (1 - |z|^2)^{s-2} dA(z) \\ &= \chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z) \frac{(1 - |z|^2)^{q+s}}{(1 - |z|^2)^{q+2}} dA(z) \\ &\leq \frac{|h'(z)|^p}{(\varepsilon - \varepsilon_1)^p} (1 - |z|^2)^{q+s} \chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z) dA(z) \\ &\leq \frac{1}{(\varepsilon - \varepsilon_1)^p} |h'(z)|^p (1 - |z|^2)^{q+s} dA(z). \end{aligned} \quad (58)$$

Since $h \in F(p, q, s)$, Corollary 4 implies that

$$|h'(z)|^p (1 - |z|^2)^{q+s} dA(z) \quad (59)$$

is in $\mathcal{CM}_s$. This means that

$$(\varepsilon - \varepsilon_1)^p \chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z) (1 - |z|^2)^{s-2} dA(z) \quad (60)$$

is in $\mathcal{CM}_s$, and so is $\chi_{\widetilde{\Omega}_{\varepsilon-\varepsilon_1}(h)}(z) (1 - |z|^2)^{s-2} dA(z)$. This contradicts (57). Thus, we must have

$$\text{dist}_{\mathcal{B}_{(2+q)/p}}(f, F(p, q, s)) \geq \varepsilon_0 \quad (61)$$

as required. $\square$

Remark 9. Theorem 2 characterizes the closure of $F(p, q, s)$ in the $\mathcal{B}_{(q+2)/p}$ norm. That is, for $f \in \mathcal{B}_{(q+2)/p}$, f is in the closure of $F(p, q, s)$ in the $\mathcal{B}_{(q+2)/p}$ norm if and only if, for every $\varepsilon > 0$,

$$\int_{\widetilde{\Omega}_\varepsilon(f) \cap S(I)} (1 - |z|^2)^{s-2} dA(z) \leq |I|^s \quad (62)$$

for any Carleson square $S(I)$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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