Research Article

On the Tumura-Clunie Theorem and Its Application

Gaixian Xue\(^1\) and Jinjin Huang\(^2\)

\(^1\) School of Mathematics and Information Science, Henan University of Economics and Law, Zhengzhou 450046, China
\(^2\) College of Economics and Management, Zhoukou Normal University, Zhoukou 466001, China

Correspondence should be addressed to Gaixian Xue; qiaohuilei@163.com

Received 22 January 2014; Revised 15 March 2014; Accepted 26 March 2014; Published 22 April 2014

Abstract and Applied Analysis

\[ \text{Volume 2014, Article ID 615351, 6 pages} \]
\[ \text{http://dx.doi.org/10.1155/2014/615351} \]

1. Introduction and Main Results

A meromorphic function will always mean meromorphic in the complex plane \( \mathbb{C} \). We adopt the standard notation in the Nevanlinna value distribution theory of meromorphic functions such as \( T(r, f) \), \( m(r, f) \), \( N(r, f) \), and \( \overline{N}(r, f) \) as explained in [1, 2]. For any nonconstant meromorphic function \( f \), we denote by \( S(r, f) \) any quantity satisfying \( S(r, f) = o(T(r, f)) \) as \( r \to \infty \) possibly outside a set of finite linear measures that is not necessarily the same at each occurrence.

\begin{definition}[see [1]]\end{definition}

A meromorphic function “\( a(z) \)” is said to be a small function of \( f \) if \( T(r, a(z)) = S(r, f) \).

\begin{definition}
Throughout this paper one denotes by \( a_j(z) \) meromorphic functions satisfying \( r, a_j(z) = S(r, f) \)(\( j = 1, 2, \ldots, n \)). If \( a_0 \neq 0 \), we call \( P[f] = a_n f^n + a_{n-1} f^{n-1} + \cdots + a_1 f + a_0 \) a polynomial in \( f \) with degree \( n \). If \( n_0, n_1, \ldots, n_k \) are nonnegative integers, we call \( M[f] = f^{n_0}(f')^{n_1} \cdots (f^{(k)})^{n_k} \) a differential monomial in \( f \) of degree \( Y_M = n_0 + n_1 + \cdots + (k+1)n_k \).

\begin{definition}[Hua (see [3, page 69])]
Proved the following result.

\textbf{Theorem A.} Let \( f \) be a nonconstant meromorphic function and let \( f = f^n + Q[f] \)

\begin{equation}
\tag{1}
f = f^n + Q[f]
\end{equation}

be a differential polynomial, where \( Q[f] \) is also a differential polynomial and \( Y_Q \leq n - 1 \).

Hua (see [3, page 69]) proved the following result.

\begin{theorem}
Let \( f \) be a nonconstant meromorphic function and let \( f \) be given by (1) with \( Y_Q \leq n - 1 \). If

\begin{equation}
\tag{2}
N(r, f) + N\left( r, \frac{1}{f} \right) = S(r, f),
\end{equation}

then

\begin{equation}
\tag{3}
f = \left( f + \frac{a(z)}{n} \right)^n,
\end{equation}

where \( a(z) \) is a small function of \( f \).

Then \( f = g^n \), \( g = f + (a(z)/n) \), and \( a(z)g^{n-1} \) is obtained by substituting \( g \) for \( f \), \( g' \) for \( f' \), and so forth in the terms of degree \( n-1 \) in \( Q[f] \).
Remark 4. The conclusion still holds good if condition (2) is replaced with
\[ N(\nu f) + N\left(r, \frac{1}{f}\right) = S_\nu(r, f), \tag{4} \]
where \(S_\nu(r, f)\) denotes any quantity which satisfies \(S_\nu(r, f) = o(T(r, f))\) as \(r \to +\infty\) through a set of \(r\) of infinite measure.

Hua (see [3]) improved Theorem A and obtained the following result.

**Theorem B.** Let \(f\) be a nonconstant meromorphic function and let \(f\) be given by (1) with \(Y_Q \leq \nu - 1\). If
\[ N(\nu f) + N\left(r, \frac{1}{f}\right) = S(\nu f), \tag{5} \]
then
\[ f = (f + \frac{a(z)}{n})^n, \tag{6} \]
where \(a(z)\) is a small function of \(f\).

Another theorem is due to Zhang and Li (see [4]), which can be stated as follows.

**Theorem C.** Let \(f\) be a nonconstant meromorphic function and let \(f\) be given by (1), where \(n(\geq Y_Q + 1)\) is an integer. Then one of the following occurs.
(i) If \(\Gamma_Q > \nu - 1\), then
\[ T(\nu f) \leq [1 + 2(\Gamma_Q - \nu + 1)] N(\nu f) \]
\[ + (\Gamma_Q - \nu + 2) N\left(r, \frac{1}{f}\right) + S(\nu f). \tag{7} \]
Or there exists a small proximity function \(a(z)\) of \(f\) such that
\[ f = (f + \frac{a(z)}{n})^n, \tag{8} \]
and \(N(r, a(z)) \leq (\Gamma_Q - \nu + 1)[N(\nu f) + N(\nu, 1/f)] + S(\nu f).
(ii) If \(\Gamma_Q \leq \nu - 1\), then
\[ T(\nu f) \leq 2N(\nu f) + N\left(r, \frac{1}{f}\right) + S(\nu f), \tag{9} \]
or
\[ f = (f + \frac{a(z)}{n})^n, \tag{10} \]
where \(a(z)\) is a small function of \(f\).
(iii) In the special case, if \(Q[f] = a_{n-1} f^{n-1} + P[f]\), where \(\Gamma_p \leq \nu - 2\), then
\[ T(\nu f) \leq N(\nu f) + N\left(r, \frac{1}{f}\right) + S(\nu f), \tag{11} \]
or
\[ f = (f + \frac{a(z)}{n})^n, \tag{12} \]
where \(a(z)\) is a small function of \(f\).

**Corollary 5.** From Theorem C we know that if condition (2) is replaced with \(N(\nu f) + N(\nu, 1/f) = S(\nu f)\) in Theorem A, then the conclusion remains valid.

In this direction Ren (see [5]) also generalized Tumura-Clunie’s theorem concerning differential polynomials.

Combining the methods used in their proofs we show the following theorem.

**Theorem 6.** Let \(f\) be a nonconstant meromorphic function and let \(f\) be given by (1), where \(n(\geq Y_Q + 1)\) is an integer and \(\Gamma_f(\not= 2)\) is the weight of \(f\).
\[ N(\nu, 1/f) = N(\nu f), \tag{13} \]
then
\[ f = \left( f + \frac{a(z)}{n^\nu} \right)^n, \tag{14} \]
where \(a(z)\) is a small function of \(f\).

It is easily seen from the following example that \(\Gamma_f \not= 2\) in Theorem 6 is necessary.

**Example 7.** Let \(f = \tan z\) and \(f = f^2 + 1\). Obviously, (13) is obtained but (14) does not hold.

### 2. Some Lemmas

To prove our results, we need some lemmas.

**Lemma 8** (see [1]). Let \(f_1\) and \(f_2\) be two nonzero meromorphic functions in the complex plane; then
\[ N(\nu, f_1 f_2) = N\left(r, \frac{1}{f_1 f_2}\right) \]
\[ = N(\nu, f_1) + N(\nu, f_2) - N\left(r, \frac{1}{f_1}\right) - N\left(r, \frac{1}{f_2}\right). \tag{15} \]

**Lemma 9.** If \(N(\nu, f^{(k)} f | f \not= 0)\) denotes the counting functions of those zeros of \(f^{(k)}\) which are not the zeros of \(f\), where a zero of \(f^{(k)}\) is counted according to its multiplicity, then
\[ N(\nu, f^{(k)} f | f \not= 0) \leq k N(\nu, f) + N(\nu, 0; f | 0) \]
\[ + k N(\nu, 0; f | 0) + S(\nu, f). \tag{16} \]

**Lemma 10.** Suppose that \(Q[f] = 0\) is given in Definition 2. Let \(z_0\) be a pole of \(f\) of order \(p\) and neither a zero nor a pole of coefficients of \(Q[f]\). Then \(z_0\) is a pole of \(Q[f]\) of order at most \(p Y_Q + (\Gamma_Q - Y_Q)\).

**Lemma 11** (see [6]). Let \(f\) be a nonconstant meromorphic function and let \(Q[f] = 0\) be given in Definition 2. Then
\[ m(\nu, Q[f]) \leq Y_Q m(\nu, f) + \sum_{j=1}^n m(\nu, a_j) + S(\nu, f), \tag{17} \]
\[ N(\nu, Q[f]) \leq \nu Q N(\nu, f) + \sum_{j=1}^n N(\nu, a_j) + S(\nu, f). \]
Lemma 12. Suppose that \( f \) is a nonconstant meromorphic function and \( Q[f] \) is given in Definition 2. Then \( S(r, Q) = S(r, f) \).

Proof. It is straightforward by Lemma 11.

Lemma 13 (see [7]). Let \( f \) be a nonconstant meromorphic function in the complex plane and let \( Q_1[f] \) and \( Q_2[f] \) be quasi-differential polynomials in \( f \). If \( \Upsilon_{Q_2} \leq n \) and \( f^n Q_1[f] = Q_2[f] \), then \( m(r, Q_1[f]) = S(r, f) \).

Lemma 14. Let \( f \) be a nonconstant meromorphic function and let \( \varphi \) be given by (1). Then
\[
(\Gamma_f - 2) N_1(r, f) \leq 2N_{12}(r, f)
\]
\[
+ 2N \left( r, \frac{1}{f} \right) + S(r, f).
\]

Proof. If \( \Gamma_f \leq 2 \), the conclusion of Lemma 14 holds obviously. In the following we suppose that \( \Gamma_f > 2 \).

With \( f = f^n + Q[f] \), we set
\[
g(z) = \frac{\frac{f'}{f}}{\frac{f'}{f} + 1}.
\]
(19)

Let \( z_0 \) be a simple pole of \( f \) and not a zero of coefficients of \( Q[f] \); then
\[
f(z) = \frac{a}{z - z_0} + O(1), \quad a \neq 0 \text{ as } z \to z_0.
\]
(20)

From Lemma 10 we know that \( z_0 \) is a pole of \( f \) of order at most \( \Gamma_f \); then we have
\[
f(z) = \frac{b}{z - z_0}^{\Gamma_f} + O(1),
\]
(21)

\[
f'(z) = - \frac{b \Gamma_f}{(z - z_0)^{\Gamma_f + 1}} + O(1),
\]
where \( b \neq 0 \).

Then
\[
f(z) = \frac{b}{z - z_0}^{\Gamma_f} \left[ 1 + O(z - z_0)^{\Gamma_f} \right],
\]
\[
f'(z) = - \frac{b \Gamma_f}{(z - z_0)^{\Gamma_f + 1}} \left[ 1 + O(z - z_0)^{\Gamma_f + 1} \right],
\]
(22)

So \( g(z) \neq 0, \infty \). But \( z_0 \) is a zero of \( g'(z) \) of order at least \( \Gamma_f - 1 \). Then
\[
(\Gamma_f - 1) N_1(r, f) \leq N_0 \left( r, \frac{1}{g} \right),
\]
(23)

where \( N_0(r, 1/g') \) denotes the counting function of the zeros of \( g' \), not of \( g \).

By Lemma 8 and Nevanlinna first fundamental theorem, we get
\[
N \left( r, \frac{g}{g'} \right) - N \left( r, \frac{g'}{g} \right) = N \left( r, \frac{1}{g} \right) + N \left( r, g \right) - N \left( r, g' \right) - N \left( r, \frac{1}{g} \right) = N_0 \left( r, \frac{1}{g} \right) - N(r, g) - N \left( r, \frac{1}{g} \right),
\]
\[
N \left( r, \frac{g}{g'} \right) - N \left( r, \frac{g'}{g} \right) = N_0 \left( r, \frac{1}{g} \right) - N(r, g) - m \left( r, \frac{g'}{g} \right) + O(1).
\]
(24)

From (24), we have
\[
N_0 \left( r, \frac{1}{g} \right) \leq N \left( r, \frac{1}{g} \right) + N(r, g) + m \left( r, \frac{g'}{g} \right) + O(1)
\]
\[
\leq N \left( r, \frac{1}{g} \right) + N(r, g) + S(r, f).
\]
(25)

From (19), we know that the poles and zeros of \( g(z) \) can only occur at the multiple zeros of \( f(z) \), the zeros of \( f \), and the zeros of \( f' \). Hence
\[
N(r, g) + N \left( r, \frac{1}{g} \right) \leq N_{12}(r, f) + N \left( r, \frac{1}{f} \right)
\]
\[
+ N_0 \left( r, \frac{1}{f} \right) + S(r, f),
\]
(26)

where \( N_0(r, 1/f') \) denotes the counting function of the zeros of \( f' \), not of \( f \).

By Lemmas 9 and 12, we obtain
\[
N_0 \left( r, \frac{1}{f} \right) \leq N(r, f) + N \left( r, \frac{1}{f} \right) + S(r, f)
\]
\[
\leq N(r, f) + N \left( r, \frac{1}{f} \right) + S(r, f),
\]
(27)

Combining (23), (25), (26), and (27), we obtain (18). This completes the proof of Lemma 14.

Proof of Theorem 6. We consider two cases.

Case 1. If \( \Gamma_f = 1 \), (14) holds obviously.

Case 2. If \( \Gamma_f > 2 \), by Lemma 14 and (13) we have
\[
N(r, f) = N_1(r, f) + N_{12}(r, f)
\]
\[
\leq \frac{\Gamma_f}{\Gamma_f - 2} N_{12}(r, f) + \frac{2}{\Gamma_f - 2} N \left( r, \frac{1}{f} \right) + S(r, f)
\]
\[
\leq S(r, f).
\]
(28)
This shows that
\[ \overline{N}(r, f) = S(r, f). \quad (29) \]
Suppose that \( f \equiv 0 \).
So we have \( f^n = -Q[f] \) and \( Q[f] \neq 0 \); moreover
\( T(r, Q[f]) = nT(r, f) + S(r, f) \).
By Lemma II we get \( m(r, Q[f]) \leq Y_Q m(r, f) + S(r, f) \).
On the other hand, we have
\[ \begin{align*}
    mn(r, f) &= m(r, f^n) = m(r, f - Q[f]) \\
    &\leq m(r, f) + m(r, Q[f]) + S(r, f) \\
    &\leq Y_Q m(r, f) + S(r, f). 
\end{align*} \quad (30) \]
It follows that \( m(r, f) = S(r, f) \), which is impossible.
Therefore, \( f \not\equiv 0 \).

From (29) and the condition of the theorem, we know
\[ T(r, f'/f) = S(r, f). \]
By \( f = f^n + Q[f] \), we have
\[ f' = \frac{f'}{f} f^n + \frac{f'}{f} Q[f], \quad f' = nf^{n-1} f' + Q'[f]. \quad (32) \]
And hence
\[ f^{n-1} \left( f' - nf' \right) = Q[f] \left( \frac{Q'[f]}{Q[f]} - \frac{f'}{f} \right). \quad (33) \]
Let
\[ \begin{align*}
    \Omega_1[f] &= f^{n-1} f' - nf' \\
    \Omega_2[f] &= Q[f] \left( \frac{Q'[f]}{Q[f]} - \frac{f'}{f} \right). \quad (34) 
\end{align*} \]
Then
\[ f^{n-1} \Omega_1[f] = \Omega_2[f], \quad (35) \]
where \( \Omega_1[f] \) and \( \Omega_2[f] \) are quasi-differential polynomials.
By Lemma 13 we have
\[ m(r, \Omega_1[f]) = S(r, f). \quad (36) \]
By Lemma 10 and (35) we obtain
\[ \begin{align*}
    N(r, \Omega_1[f]) &= N(r, \Omega_2[f]) - (n - 1) N(r, f) + S(r, f) \\
    &\leq Y_Q N(r, f) + (Y_Q - Y_Q + 1) N(r, f) - (n - 1) N(r, f) + S(r, f) \\
    &\leq (Y_Q - Y_Q + 1) N(r, f) + S(r, f). \quad (37) 
\end{align*} \]
Note that \( \overline{N}(r, f) = S(r, f) \).
So \( T(r, \Omega_1[f]) = S(r, f) \).

From (34) we know that \( Q[f] \) is a polynomial and \( Y_Q \leq n - 1 \).
Set
\[ Q[f] = b(z) f^{n-1} + P[f], \quad (38) \]
where \( P[f] \) is a polynomial and \( b(z) \) is a small function of \( f \); moreover \( Y_P \leq n - 2 \).
Set \( g = f + (b(z)/n) \); we have
\[ f = g^n + R[g], \quad (39) \]
where \( R[g] \) is a polynomial and \( Y_R \leq n - 2 \).
Now proceeding as the above proof, we get
\[ g^{n-1} \left( g' - nf' \right) = R[g] \left( \frac{R'[g]}{R[g]} - \frac{f'}{f} \right). \quad (40) \]
By Lemma 13 we obtain
\[ \begin{align*}
m(r, (g - nf') g) &= S(r, f), \\
m(r, g - nf') &= S(r, f). \quad (41) 
\end{align*} \]
Therefore we have
\[ \begin{align*}
T(r, (g - nf') g) &= S(r, f), \\
T(r, g - nf') &= S(r, f). \quad (42) 
\end{align*} \]
Notice that \( T(r, g) = T(r, f) + S(r, f) \neq S(r, f) \).
We can get \( g'(f') - nf' \equiv 0 \).
So \( f \not\equiv g^n \), where \( c \) is a constant. Obviously \( c = 1 \).
This proves Theorem 6.

3. Application

Very recently, Yi (see [8, 9]) proved the following result.

**Theorem D.** Let \( f \) be a transcendental meromorphic function and let \( p(z) \) be a polynomial, \( p(z) \neq 0 \). If \( f \) and \( f' \) share 0 in \( C \), then \( f' - p(z) \) has infinitely many zeros.

**Remark 15.** From the hypothesis of Theorem E, it can be easily seen that all zeros of \( f \) have multiplicity at least two.

Ren and Yang 2013 (see [10]) obtained the following result.

**Theorem E.** Let \( f \) be a transcendental meromorphic function and let \( R \) be a rational function, \( R \not\equiv 0 \). Suppose that, with the exception of possibly finitely many, all zeros and poles of \( f \) are multiple. Then \( f' - R \) has infinitely many zeros.

It is natural to ask the following question: what can we say if \( f' \) is replaced by \( f^{(k)} \) and \( p(z) \) and \( R \) are replaced by a small function relative to \( f \) in Theorems D and E?

Later, Yang (see [11]) answered the above question and obtained the following result.
Theorem F. Let $f$ be a transcendental meromorphic function satisfying

$$N(r, \frac{1}{f}) = S(r, f). \quad (43)$$

Then, for any $k \geq 1$ and any small function $a(z) ( \neq 0, \infty)$ of $f$,

$$N \left( r, \frac{1}{f^{(k)} - a(z)} \right) \neq S(r, f). \quad (44)$$

We supplement Theorems D and E, improve Theorem F, and obtain the following result.

Theorem 16. Let $h$ be a transcendental meromorphic function satisfying

$$N_{(2)} \left( r, \frac{1}{h} \right) = S(r, h). \quad (45)$$

Then, for any $n \geq 2$ and any small function $a(z) ( \neq 0, \infty)$ of $h$,

$$N \left( r, \frac{1}{h^{(n)} - a(z)} \right) \neq S(r, h). \quad (46)$$

The method of our proof essentially belongs to Yang. For the completeness, we give the proof here.

Proof. Set

$$h = \frac{1}{f}. \quad (47)$$

Then

$$T(r, f) = T(r, h) + O(1), \quad (48)$$

$$N_{(2)} \left( r, \frac{1}{h} \right) = N_{(2)} (r, f). \quad (49)$$

Obviously

$$S(r, f) = S(r, h). \quad (50)$$

Now

$$h'' = -ff' + 2(f')^2, \quad (51)$$

$$h''' = -6(f')^2 - 2f''f' + 2f(f')^2 + 4ff'f'' \cdots. \quad (52)$$

Thus, in general,

$$h^{(n)} = \frac{Q_n(f)}{f^{n+1}}, \quad (53)$$

where $Q_n(f)$ denotes a homogeneous differential polynomial in $f$ of degree $n$. So

$$h^{(n)} - a(z) = \frac{Q_n(f) - a(z) f^{n+1}}{f^{n+1}}. \quad (54)$$

If the assertion of the theorem was false, that is,

$$N \left( r, \frac{1}{h^{(n)} - a(z)} \right) = S(r, f), \quad (55)$$

then from (52) we have

$$f = f^{(n+1)} - \frac{Q_n(f)}{a(z)}. \quad (56)$$

Thus from (48), (53), and (54), we obtain

$$N_{(2)} (r, f) + N \left( r, \frac{1}{f} \right) = S(r, f). \quad (57)$$

Combining Theorem 6, (55) gives

$$f = \left( f + \frac{c}{n+1} \right)^{n+1}, \quad (58)$$

where $c$ (a small function of $f$) is determined by the two equations: $g = f + (c/(n+1))$ and $cg^n = -(Q_n(g)/a(z))$.

We may claim that

(i) $S(r, f) = S(r, g)$;
(ii) $N(r, g) = S(r, g)$;
(iii) $T(r, g^{(k)}/g) = S(r, g)$ for all $k \in \mathbb{N}$.

In fact, from the definition of $g$ we know that the claim (i) above holds.

By (54) we have $\Gamma_f > 2$.

From $g = f + (c/(n+1))$, $\Gamma_g > 2$, and (29) we get

$$N(r, g) = N(r, f) + N(r, c) = S(r, f) = S(r, g). \quad (59)$$

That is, the claim (ii) above holds.

Combining (53) and the claims (i) and (ii), we may deduce

$$T \left( r, \frac{g^{(k)}}{g} \right) = N \left( r, \frac{g^{(k)}}{g} \right) + m \left( r, \frac{g^{(k)}}{g} \right) \leq kN(r, g) + N \left( r, \frac{1}{g} \right) + S(r, g) \quad (60)$$

Then the claim (iii) is true also.

Thus, by (54) and (56), we obtain

$$\left( f + \frac{c}{n+1} \right)^{n+1} = f^{n+1} + \sum_{k=2}^{n+1} \binom{c}{n+1} k^{k-1} \quad (61)$$

where $Q_n(f)$ denotes a homogeneous differential polynomial in $f$ of degree $n$. So
Since \( c f^n \equiv -(Q_n(f)/a(z)) \), it follows that

\[
\sum_{k=2}^{n+1} C_k^{n+1} \left( \frac{c}{n+1} \right)^k f^{n+1-k} = 0,
\]

which is impossible unless \( c \equiv 0 \).

But then, from (59), \(-(Q_n(f)/a(z)) \equiv 0\) and we have \( h^{(n)} \equiv 0 \) which contradicts the fact that \( h \) is a transcendental meromorphic function.

This completes the proof of Theorem 16.

\( \square \)

**Remark 17.** For \( n = 1 \), from the proof of Theorem 16 and Corollary 5, we know that if the condition \( \overline{N}_{i\omega}(r,1/h) = S(r,h) \) is replaced with \( \overline{N}(r,1/h) = S(r,h) \) in Theorem 16, then the conclusion still holds.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

**Acknowledgment**

This work was supported by the National Natural Science Foundation of China under Grant Nos. 11301140 and U1304102.

**References**


