Research Article

Solving a Class of Singularly Perturbed Partial Differential Equation by Using the Perturbation Method and Reproducing Kernel Method

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Received 28 March 2014; Revised 14 May 2014; Accepted 28 May 2014; Published 17 June 2014

Academic Editor: Dragos-Patru Covei

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We give the analytical solution and the series expansion solution of a class of singularly perturbed partial differential equation (SPPDE) by combining traditional perturbation method (PM) and reproducing kernel method (RKM). The numerical example is studied to demonstrate the accuracy of the present method. Results obtained by the method indicate the method is simple and effective.

1. Introduction

Singularly perturbed problems (SPPs) arise very frequently in many branches of mathematics such as fluid mechanics and chemical reactor theory. It is well known that the solutions of SPPs exhibit a multiscale character. So there are some major computation difficulties. In recent years, many special methods have been developed to deal with SPPs. Many papers [1–4] are devoted to SPPs of ordinary differential equation and the authors discussed the situation and width of boundary layer(s) and give some effective numerical algorithms. But few papers [5–7] deal with SPPDE.

The reproducing kernel Hilbert function space has been shown in [8–10] to solve a large class of linear and nonlinear problems effectively. However, in [8–10], it cannot be used directly to SPPs. The aim of this work is to fill this gap. In this paper, we solve a class of SPPs in reproducing kernel space. By using a traditional perturbation method and RKM, the series expansion solution of a class of SPPDE is given. The main contribution of this paper is to use RKM in SPPDE. The reason why we use this method is that we aim to solve some problems in many areas of science and improve high precision.

Let us consider the following SPPDE:

\[ \varepsilon \frac{\partial U}{\partial t} + U \frac{\partial U}{\partial x} = F(x, t), \quad t \in [0, 1], \ x \in [0, 1], \]

\[ U(0, t) = 0, \ U(x, 0) = v(x), \quad x \in [0, 1], \]  

(1)

where \( \varepsilon \ll 1 \) is a positive number, functions \( f(x, t) \) and \( v(x) \) are sufficiently smooth, and \( v(0) = 0 \). Under suitable continuity and compatibility conditions, the problem (1) has a unique solution \( U(x, t) \). In [5–7], we notice that a small variation in the parameter \( \varepsilon \) produces a large variation in the solution. It is quite well known that solution of such problems involves boundary layers.

2. Perturbation Method

Let \( u(x, t) = U(x, t) - v(x) \); (1) can be equivalently turned into

\[ \varepsilon \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + u \frac{\partial v}{\partial x} + v \frac{\partial u}{\partial x} = f(x, t), \quad t \in [0, 1], \ x \in [0, 1], \]

\[ u(0, t) = u(x, 0) = 0, \quad t \in [0, 1], \ x \in [0, 1], \]  

(2)
3. Reproducing Kernel Method

For getting $u_0, u_1, u_2, \ldots$ from (6), (7), (8), we let

$$
(L_0u_0)(x,t) = u_0 \frac{dv}{dx},
$$

$$
f_0(x,t,u_0) = f(x,t)|_{t=0} - u_0 \frac{d^2u_0}{dx^2}.
$$

$$(L_j u_j)(x,t) = (u_0 + v) \frac{d^j u_j}{d x^j} + u_j \left( \frac{d^j u_j}{d x^j} + \frac{dv}{dx} \right),
$$

where $j = 1, 2, 3, \ldots$

Equations (7), (8), ..., can be converted into the following equivalent form:

$$(L_j u_j)(t,x) = f_j(x,t), \quad j = 1, 2, 3, \ldots$$

Be aimed at with the purpose of solving (11) and (12), we need to introduce the reproducing kernel space, previously. Like in [12], we give the reproducing kernel spaces $W^2_2[0,1]$:

$$W^2_2[0,1] = \{ u \mid u, u' \text{ is one-variable absolutely continuous function, } u'' \in L^2[0,1], \ u(0) = 0 \}.$$  

Then, we define the inner product of $W^2_2[0,1]$. Consider the following:

$$\langle u(x), v(x) \rangle = u(0)v(0) + u'(0)v'(0) + \int_0^b u''(x)v''(x)dy.$$  

From [13], we can prove $W^2_2[0,1]$ is a reproducing kernel Hilbert space, and the reproducing kernel of it is

$$R^{(2)}_x(y) = \begin{cases} 
1 - \frac{y^3}{6} + \frac{xy}{2}(2 + y), & x < y, \\
1 - \frac{x^3}{6} + \frac{x^2y}{2} + xy, & y < x.
\end{cases}$$
After all of these, we introduce the reproducing kernel space \( W_2(D) \) \[^{[14]}\]

\[
W_2^2(D) = W_2^2[0,1] \otimes W_2^2[0,1]
\]

\[
= \left\{ u(x,t) \mid \frac{\partial^{p+q}}{\partial x^p \partial t^q} u(x,t) \text{ are two-variable complete continuous functions, } n=0,1, \right. \\
m=0,1,2, \right. \\
q=0,1,2, u(x,0) = u(0,t) = 0 \right\} 
\]

(16)

and the inner product of it; see \[^{[15]}\], and the reproducing kernel of \( W_2(D) \) is

\[
K_{\hat{f},\tilde{f}}(t,x) = R_{\eta}^{(2)}(t) R_{\eta}^{(1)}(x). 
\]

(17)

Similar to the definition of \( W_2(D) \), we can define \( W_1(D) \) and it is the reproducing kernel \( K_{\hat{f},\tilde{f}}(t,x) = R_{\eta}^{(1)}(t) R_{\eta}^{(1)}(x) \), where \( W_2^2[0,1] \) is also a reproducing kernel space with the reproducing kernel \( K_{\hat{f},\tilde{f}}(y) \) (see \[^{[16–18]}\]).

It is easy to prove \( L_j \), \((j = 0,1,2,\ldots) \) is a linear bounded operator, because the problem (1) has a unique solution \( U(x,t) \); in other words, \( L_j \) is also an invertible operator, so \[^{[19]}\] if \( L(u(x,t)) = f_j(x,t,u) \) \((j = 0,1,2,\ldots) \), where \( u(x,t) \in W_2(D) \) and \( f_j(x,t,u) \in W_1(D) \), \( L_j^{-1} \) is existent and \( \{x_l, t_l\}_{l=1}^{\infty} \) is countable dense points in \( D \). Let \( \tilde{\psi}_j(x,t) = \sum_{k=1}^{i} \beta_{jk} \psi_k(x,t) \), where \( \beta_{jk} \) are the coefficients resulting from Gram-Schmidt orthonormalization and \( \psi_k(x,t) = (L_{j\eta})(K_{\eta})(y,s)(x_l,t_l), i=1,2,\ldots; \)

\[
u(x,t) = \sum_{i=1}^{k=1} \beta_{jk} f_j(x_k,t_k,u(x_k,t_k)) \tilde{\psi}_j(x,t) 
\]

(18)

is an analytical solution of equation \( L_j u(x,t) = f_j(x,t,u) \).

(i) Linear Problem. Suppose equation \( L_j u(x,t) = f_j(x,t,u) \) is a linear problem; that is, \( f_j(x,t,u) = f_j(x,t) \); we define an approximate solution \( u_n(x,t) \) by

\[
u_n(x,t) = \sum_{i=1}^{k=1} \beta_{jk} f_j(x_k,t_k) \tilde{\psi}_j(x,t) .
\]

(19)

Theorem 1 (see \[^{[20–22]}\] convergence analysis). Let \( \varepsilon_n^2 = \|u(x,t) - u_n(x,t)\|^2 \); then the sequence of real numbers \( \varepsilon_n \) is monotonously decreasing and \( \varepsilon_n \to 0 \) and the sequence \( u_n(x,t) \) is convergent uniformly to \( u(x,t) \).

(ii) Nonlinear Problem (see \[^{[23]}\]). Suppose equation \( L_j \mu(x,t) = f_j(x,t,u) \) is a nonlinear problem; that is, \( f_j(x,t,u) = N(u) + F_j(x,t) \), where \( N : W_2(D) \to W_1(D) \) is a nonlinear operator; we give an iterative sequence \( u_n(x,t) \):

\[
u_0(x,t) = \text{the solution of the linear equation } L_j u = F_j(x,t); \]

\[
u_{n+1}(x,t) = \text{the solution of the linear equation } L_j u = N(u_n(x,t)) + F_j(x,t), n = 0,1,2,\ldots.
\]

Lemma 2. If \( u_{n+1}(x,t) \to u(x,t) \), then \( u(x,t) \) is the solution of equation \( L_j u(x,t) = f_j(x,t,u) \).

Theorem 3. Suppose the nonlinear operator \( A \triangleq (L_j^{-1}N) : W_1(D) \to W_2(D) \) satisfies contractive mapping principle; that is,

\[
\|A(u) - A(v)\| \leq \lambda \|u - v\|, \quad \lambda < 1; \quad (20)
\]

and then \( u_{n+1}(x,t) \) is convergent.

Using reproducing kernel method, we can get

\[
u_0(x,t) = \sum_{i=1}^{k=1} \beta_{jk} f_j(x_k,t_k, u_0(x_k,t_k)) \tilde{\psi}_j(x,t), \quad (21)
\]

\[
u_j(x,t) = \sum_{i=1}^{k=1} \beta_{jk} f_j(x_k,t_k, \tilde{\psi}_j(x,t), \quad j = 1,2,\ldots.
\]

Therefore, the analytical solution of (2) is obtained.

\[
u(x,t) = \sum_{j=0}^{\infty} \tilde{\psi}_j(x,t). \quad (22)
\]

In calculation, we use

\[
u_{n+1}(x,t) = \sum_{i=1}^{k=1} \beta_{jk} f_j(x_k,t_k, \tilde{\psi}_j(x,t) \quad (23)
\]

as the approximation solution of (2).

4. Numerical Experiment

Example 4. Considering a nonlinear advection equation with perturbation term

\[
\varepsilon \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(x,t), \quad (x,t) \in D = [0,1] \times [0,1],
\]

(24)

\[
u(0,t) = et, \quad u(0,0) = x,
\]

where \( f(x,t) = e^2 + et + x, u_{T}(x,t) = et + x \) is the true solution, and \( u_{n+1}(x,t) \) is the approximate solution (Table 2). When we take \( m = 3, n = 2, l = 2 \), the numerical results are given in Table 1.

5. Conclusions and Remarks

In this paper, the combination of traditional perturbation and reproducing kernel space methods was employed successfully for solving nonlinear advection equation with singular term. The numerical results show that the present method is an accurate and reliable. Moreover, the method is also effective solving other nonlinear singular perturbation problems.
Table 1: Comparison of the absolute error $\varepsilon = 1 \times 10^{-3}$.

<table>
<thead>
<tr>
<th>$(x, t)$</th>
<th>$u_f(t, x)$</th>
<th>Approximate solution</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0.0001, 0.0001)</td>
<td>0.0000001</td>
<td>0.0001</td>
<td>$1 \times 10^{-7}$</td>
</tr>
<tr>
<td>(0.0200, 0.0200)</td>
<td>0.0200200</td>
<td>0.0200</td>
<td>$2 \times 10^{-5}$</td>
</tr>
<tr>
<td>(0.0050, 0.0050)</td>
<td>0.0050505</td>
<td>0.0050</td>
<td>$5 \times 10^{-6}$</td>
</tr>
<tr>
<td>(0.8100, 0.8100)</td>
<td>0.8108100</td>
<td>0.8100</td>
<td>$8 \times 10^{-4}$</td>
</tr>
<tr>
<td>(0.2000, 0.2000)</td>
<td>0.2020200</td>
<td>0.2000</td>
<td>$2 \times 10^{-4}$</td>
</tr>
<tr>
<td>(0.5500, 0.5500)</td>
<td>0.5505500</td>
<td>0.5500</td>
<td>$5 \times 10^{-4}$</td>
</tr>
<tr>
<td>(0.0330, 0.0330)</td>
<td>0.0330330</td>
<td>0.0330</td>
<td>$3 \times 10^{-5}$</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 2: Comparison of the absolute error $\varepsilon = 1 \times 10^{-4}$.

<table>
<thead>
<tr>
<th>$(x, t)$</th>
<th>$u_f(t, x)$</th>
<th>Approximate solution</th>
<th>Absolute error</th>
</tr>
</thead>
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<td>0</td>
<td>0</td>
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<tr>
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<tr>
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<td>$5 \times 10^{-6}$</td>
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<tr>
<td>(0.06, 0.06)</td>
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<td>$6 \times 10^{-6}$</td>
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<tr>
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<tr>
<td>(0.10, 0.10)</td>
<td>0.100010</td>
<td>0.10</td>
<td>$1 \times 10^{-5}$</td>
</tr>
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</table>

Conflict of Interests
The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments
This paper is supported by the Natural Science Foundation of China (11361037), the Natural Science Foundation of Inner Mongolia (2013M50109), and the Project Application Technology Research and Development Foundation of Inner Mongolia (no. 20120312).

References
