Research Article

Riemann-Liouville and Higher Dimensional Hardy Operators for NonNegative Decreasing Function in $L^{p(\cdot)}$ Spaces

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One-weight inequalities with general weights for Riemann-Liouville transform and n -dimensional fractional integral operator in variable exponent Lebesgue spaces defined on $\mathbb{R}^n$ are investigated. In particular, we derive necessary and sufficient conditions governing one-weight inequalities for these operators on the cone of nonnegative decreasing functions in $L^{p(x)}$ spaces.

1. Introduction

We derive necessary and sufficient conditions governing the one-weight inequality for the Riemann-Liouville operator

$$R_\alpha f(x) = \frac{1}{x^\alpha} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad 0 < \alpha < 1 \quad (1)$$

and n -dimensional fractional integral operator

$$I_\alpha g(x) = \frac{1}{|x|^\alpha} \int_{|y| < |x|} \frac{g(t)}{|x-t|^{n-\alpha}} dt \quad 0 < \alpha < n, \quad (2)$$

on the cone of nonnegative decreasing function in $L^{p(x)}$ spaces.

In the last two decades a considerable interest of researchers was attracted to the investigation of the mapping properties of integral operators in so-called Nakano spaces $L^{p(\cdot)}$ (see, e.g., the monographs [1, 2] and references therein). Mathematical problems related to these spaces arise in applications to mechanics of the continuum medium. For example, Ružicka [3] studied the problems in the so-called rheological and electrorheological fluids, which lead to spaces with variable exponent.

Weighted estimates for the Hardy transform

$$(H_1 f)(x) = \int_0^x f(t) dt, \quad x > 0, \quad (3)$$

in $L^{p(\cdot)}$ spaces were derived in the papers [4] for power-type weights and in [5–9] for general weights. The Hardy inequality for nonnegative decreasing functions was studied in [10, 11]. Furthermore Hardy type inequality was studied in [12, 13] by Rafeiro and Samko in Lebesgue spaces with variable exponent.

Weighted problems for the Riemann-Liouville transform in $L^{p(x)}$ spaces were explored in the papers [5, 14–16] (see also the monograph [17]).

Historically, one and two weight Hardy inequalities on the cone of nonnegative decreasing functions defined on $\mathbb{R}_+$ in the classical Lebesgue spaces were characterized by Arino and Muckenhoupt [18] and Sawyer [19], respectively.

It should be emphasized that the operator $I_\alpha f(x)$ is the weighted truncated potential. The trace inequity for this operator in the classical Lebesgue spaces was established by Sawyer [20] (see also the monograph [21], Ch.6 for related topics).

In general, the modular inequality

$$\int_0^1 \left| \int_0^x f(t) dt \right|^{q(x)} v(x) dx \leq c \int_0^1 |f(t)|^{p(t)} w(t) dt \quad (*)$$

for the Hardy operator is not valid (see [22], Corollary 2.3, for details). Namely, the following fact holds: if there exists a positive constant c such that inequality $(*)$ is true for all

$f \geq 0$, where q ; p ; w ; and v are nonnegative measurable functions, then there exists $b \in [0, 1]$ such that $w(t) > 0$ for almost every $t < b$; $v(x) = 0$ for almost every $x > b$, and $p(t)$ and $q(x)$ take the same constant values a.e. for $t \in (0; b)$ and $x \in (0; b) \cap \{v \neq 0\}$.

To get the main result we use the following pointwise inequalities:

$$\begin{aligned} c_1 (Tf)(x) &\leq (R_\alpha f)(x) \leq c_2 (Tf)(x), \\ c_3 (Hg)(x) &\leq (I_\alpha g)(x) \leq c_4 (Hg)(x), \end{aligned} \quad (4)$$

for nonnegative decreasing functions, where c_1, c_2, c_3 , and c_4 are constants and are independent of f, g , and x , and

$$\begin{aligned} Tf(x) &= \frac{1}{x} \int_0^x f(t) dt, \\ Hg(x) &= \frac{1}{|x|^n} \int_{|y| < |x|} g(y) dy. \end{aligned} \quad (5)$$

In the sequel by the symbol $Tf \approx Tg$ we mean that there are positive constants c_1 and c_2 such that $c_1 Tf(x) \leq Tg(x) \leq c_2 Tf(x)$. Constants in inequalities will be mainly denoted by c or C ; the symbol $\mathbb{R}_+$ means the interval $(0, +\infty)$.

2. Preliminaries

We say that a radial function $f : \mathbb{R}^n \rightarrow \mathbb{R}_+$ is decreasing if there is a decreasing function $g : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $g(|x|) = f(x)$, $x \in \mathbb{R}^n$. We will denote g again by f . Let $p : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a measurable function, satisfying the conditions $p^- = \operatorname{essinf}_{x \in \mathbb{R}^n} p(x) > 0$, $p^+ = \operatorname{esssup}_{x \in \mathbb{R}^n} p(x) < \infty$.

Given $p : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$ and a nonnegative measurable function (weight) u in $\mathbb{R}^n$, let us define the following local oscillation of p :

$$\varphi_{p(\cdot), u}(\delta) = \operatorname{esssup}_{x \in B(0, \delta) \cap \operatorname{supp} u} p(x) - \operatorname{essinf}_{x \in B(0, \delta) \cap \operatorname{supp} u} p(x), \quad (6)$$

where $B(0, \delta)$ is the ball with center 0 and radius δ .

We observe that $\varphi_{p(\cdot), u}(\delta)$ is nondecreasing and positive function such that

$$\lim_{\delta \rightarrow \infty} \varphi_{p(\cdot), u}(\delta) = p_u^+ - p_u^-, \quad (7)$$

where p_u^+ and p_u^- denote the essential infimum and supremum of p on the support of u , respectively.

By the similar manner (see [10]) the function $\psi_{p(\cdot), u}(\eta)$ is defined for an exponent $p : \mathbb{R}_+ \mapsto \mathbb{R}_+$ and weight v on $\mathbb{R}_+$:

$$\psi_{p(\cdot), v}(\eta) = \operatorname{esssup}_{x \in (0, \eta) \cap \operatorname{supp} v} p(x) - \operatorname{essinf}_{x \in (0, \eta) \cap \operatorname{supp} v} p(x). \quad (8)$$

Let $D(\mathbb{R}_+)$ be the class of nonnegative decreasing functions on $\mathbb{R}_+$ and let $DR(\mathbb{R}^n)$ be the class of all nonnegative radially decreasing functions on $\mathbb{R}^n$. Suppose that u is measurable a.e. positive function (weight) on $\mathbb{R}^n$. We denote by $L^{p(x)}(u, \mathbb{R}^n)$ the class of all nonnegative functions on $\mathbb{R}^n$ for which

$$S_p(f) = \int_{\mathbb{R}^n} |f(x)|^{p(x)} u(x) d\mu(x) < \infty. \quad (9)$$

For essential properties of $L^{p(x)}$ spaces we refer to the papers [23, 24] and the monographs [1, 2].

Under the symbol $L_{\text{dec}}^{p(x)}(u, \mathbb{R}_+)$ we mean the class of nonnegative decreasing functions on $\mathbb{R}_+$ from $L^{p(x)}(u, \mathbb{R}^n) \cap DR(\mathbb{R}^n)$.

Now we list the well-known results regarding one-weight inequality for the operator T . For the following statement we refer to [18].

Theorem A. *Let r be constant such that $0 < r < \infty$. Then the inequity*

$$\int_0^\infty v(x) (Tf(x))^r dx \leq c \int_0^\infty v(x) (f(x))^r dx, \quad (10)$$

$$f \in L^r(v, \mathbb{R}_+), \quad f \downarrow$$

for a weight v holds, if and only if there exists a positive constant C such that for all $s > 0$

$$\int_s^\infty \left(\frac{s}{x}\right)^r v(x) dx \leq C \int_0^s v(x) dx. \quad (11)$$

Condition (11) is called B_r condition and was introduced in [18].

Theorem B (see [10]). *Let v be a weight on $(0, \infty)$ and $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and assume that $\psi_{p(\cdot), v(0^+)} = 0$. The following facts are equivalent:*

(a) *there exists a positive constant c such that, for any $f \in D(\mathbb{R}_+)$,*

$$\int_0^\infty (Tf(x))^{p(x)} v(x) dx \leq C \int_0^\infty (f(x))^{p(x)} v(x) dx; \quad (12)$$

(b) *for any $r, s > 0$,*

$$\int_r^\infty \left(\frac{r}{sx}\right)^{p(x)} v(x) dx \leq C \int_0^r \frac{v(x)}{s^{p(x)}} dx; \quad (13)$$

(c) $p|_{\operatorname{supp} v} \equiv p_0$ a.e. and $v \in B_{p_0}$.

Proposition 1. *For the operators T, H, R_α , and I_α , the following relations hold:*

(a)

$$R_\alpha f \approx Tf, \quad 0 < \alpha < 1, \quad f \in D(\mathbb{R}_+); \quad (14)$$

(b)

$$I_\alpha g \approx Hg, \quad 0 < \alpha < n, \quad g \in DR(\mathbb{R}^n). \quad (15)$$

Proof. (a) Upper estimate: represent $R_\alpha f$ as follows:

$$\begin{aligned} R_\alpha f(x) &= \frac{1}{x^\alpha} \int_0^{x/2} \frac{f(t)}{(x-t)^{1-\alpha}} dt \\ &\quad + \frac{1}{x^\alpha} \int_{x/2}^x \frac{f(t)}{(x-t)^{1-\alpha}} dt \\ &= S_1(x) + S_2(x). \end{aligned} \quad (16)$$

Observe that if $t < x/2$, then $x/2 < x - t$. Hence

$$S_1(x) \leq c \frac{1}{x} \int_0^{x/2} f(t) dt \leq cTf(x), \quad (17)$$

where the positive constant c does not depend on f and x . Using the fact that f is decreasing we find that

$$S_2(x) \leq cf\left(\frac{x}{2}\right) \leq cTf(x). \quad (18)$$

Lower estimate follows immediately by using the fact that f is nonnegative and the obvious estimate $x - t \leq x$ and $0 < t < x$.

(b) Upper estimate: let us represent the operator I_α as follows:

$$\begin{aligned} I_\alpha g(x) &= \frac{1}{|x|^\alpha} \int_{|y| < |x|/2} \frac{g(y)}{|x-y|^{n-\alpha}} dy \\ &\quad + \frac{1}{|x|^\alpha} \int_{|x|/2 < |y| < |x|} \frac{g(y)}{|x-y|^{n-\alpha}} dy \\ &=: S'_1(x) + S'_2(x). \end{aligned} \quad (19)$$

Since $|x|/2 \leq |x-y|$ for $|y| < |x|/2$ we have that

$$S'_1(x) \leq \frac{c}{|x|^n} \int_{|y| < |x|/2} g(y) dy \leq cHg(x). \quad (20)$$

Taking into account the fact that f is radially decreasing on $\mathbb{R}^n$ we find that there is a decreasing function $f: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that

$$S'_2(x) \leq f\left(\frac{|x|}{2}\right) \cdot \frac{1}{|x|^\alpha} \int_{|x|/2 < |y| < |x|} |x-y|^{\alpha-n} dy. \quad (21)$$

Let $F_x = \{y: |x|/2 < |y| < |x|\}$. Then we have

$$\begin{aligned} &\int_{F_x} |x-y|^{\alpha-n} dy \\ &= \int_0^\infty \left| \{y \in F_x : |x-y|^{\alpha-n} > t\} \right| dt \\ &\leq \int_0^{|x|^{\alpha-n}} \left| \{y \in F_x : |x-y|^{\alpha-n} > t\} \right| dt \\ &\quad + \int_{|x|^{\alpha-n}}^\infty \left| \{y \in F_x : |x-y|^{\alpha-n} > t\} \right| dt \\ &=: I_1 + I_2. \end{aligned} \quad (22)$$

It is easy to see that

$$I_1 \leq \int_0^{|x|^{\alpha-n}} |B(0, |x|)| dt = c|x|^\alpha; \quad (23)$$

while using the fact that $n/(n-\alpha) > 1$ we find that

$$\begin{aligned} I_2 &\leq \int_{|x|^{\alpha-n}}^\infty \left| \{y \in F_x : |x-y| \leq t^{1/(\alpha-n)}\} \right| dt \\ &\leq c \int_{|x|^{\alpha-n}}^\infty t^{n/(\alpha-n)} dt = c_{\alpha,n} |x|^\alpha. \end{aligned} \quad (24)$$

Finally we conclude that

$$S'_2(x) \leq cf\left(\frac{|x|}{2}\right) \leq cHf(x). \quad (25)$$

Lower estimate follows immediately by using the fact that f is nonnegative and the obvious estimate $|x-y| \leq |x|$, where $0 < |y| < |x|$. $\square$

We will also need the following statement.

Lemma 2. Let r be a constant such that $0 < r < \infty$. Then the inequality

$$\begin{aligned} \int_{\mathbb{R}^n} (Hf(x))^r u(x) dx &\leq C \int_{\mathbb{R}^n} (f(x))^r u(x) dx, \\ f &\in L_{dec}^r(u, \mathbb{R}^n), \end{aligned} \quad (26)$$

holds, if and only if there exists a positive constant C such that, for all $s > 0$,

$$\begin{aligned} \int_{|x| > s} \left(\frac{s}{|x|}\right)^r |x|^{r(1-n)} u(x) dx \\ \leq C \int_{|x| < s} |x|^{r(1-n)} u(x) dx. \end{aligned} \quad (27)$$

Proof. We will see that inequality (26) is equivalent to the inequality

$$\int_0^\infty \tilde{u}(t) (T\bar{f}(t))^r dt \leq C \int_0^\infty \tilde{u}(t) (\bar{f}(t))^r dt, \quad (28)$$

where $\tilde{u}(t) = t^{(n-1)(1-r)} \bar{u}(t)$, $\bar{f}(t) = t^{n-1} f(t)$, and $\bar{u}(t) = \int_{S^{n-1}} u(t\bar{x}) d\sigma(\bar{x})$.

Indeed, using polar coordinates in $\mathbb{R}^n$ we have

$$\begin{aligned} &\int_{\mathbb{R}^n} (Hf(x))^r u(x) dx \\ &= \int_{\mathbb{R}^n} u(x) \left(\frac{1}{t^n} \int_{|y| < t} f(y) dy \right)^r dx \\ &= \int_0^\infty t^{n-1} \left(\frac{1}{t^n} \int_{|y| < t} f(y) dy \right)^r \left(\int_{S^{n-1}} u(t\bar{x}) d\sigma(\bar{x}) \right) dt \\ &= C \int_0^\infty t^{n-1} t^{-nr} t^r \left(\frac{1}{t} \int_0^t \tau^{n-1} f(\tau) d\tau \right)^r \bar{u}(t) dt \\ &= C \int_0^\infty t^{n-1} t^{r(1-n)} \bar{u}(t) \left(\frac{1}{t} \int_0^t \bar{f}(\tau) d\tau \right)^r dt \\ &\leq C \int_0^\infty \tilde{u}(t) (\bar{f}(t))^r dt \\ &= C \int_0^\infty t^{(n-1)(1-r)} t^{(n-1)r} (f(t))^r dt \\ &= C \int_{\mathbb{R}^n} (f(x))^r u(x) dx. \end{aligned} \quad (29)$$

Conversely taking the test function $f_r(x) = \chi_{B(0,r)}(x)|x|^{1-n}$, $r > 0$, in modular inequality (26), one can easily obtain inequality (27). $\square$

3. The Main Results

To formulate the main results we need to prove the following proposition.

Proposition 3. *Let u be a weight on $\mathbb{R}^n$ and $p : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and assume that $\varphi_{p(\cdot), u(0^+)} = 0$. The following statements are equivalent:*

(a) *there exists a positive constant C such that, for any $f \in DR(\mathbb{R}^n)$,*

$$\int_{\mathbb{R}^n} (Hf(x))^{p(x)} u(x) dx \leq C \int_{\mathbb{R}^n} (f(x))^{p(x)} u(x) dx; \quad (30)$$

(b) *for any $r, s > 0$,*

$$\int_{|x|>r} \left(\frac{r}{s|x|^n} \right)^{p(x)} u(x) dx \leq C \int_{B(0,r)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}} dx; \quad (31)$$

(c) $p_{\sup u} \equiv p_0$ a.e. and $u \in B_{p_0}$.

Proof. We use the arguments of [10]. To show that (a) implies (b) it is enough to test the modular inequality (30) for the function $f_{r,s}(x) = (1/s)\chi_{B(0,r)}(x)|x|^{1-n}$, $s, r > 0$. Indeed, it can be checked that

$$Hf_{r,s}(x) = \begin{cases} \frac{1}{|x|^n s} \int_{|y| \leq |x|} |y|^{1-n} dy, & \text{if } |x| \leq r; \\ \frac{1}{|x|^n s} \int_{|y| \leq r} |y|^{1-n} dy, & \text{if } |x| > r. \end{cases} \quad (32)$$

Further, we find that

$$\begin{aligned} & \int_{|x|>r} u(x) (Hf_{r,s})^{p(x)} dx \\ & \leq \int_{\mathbb{R}^n} u(x) (Hf_{r,s})^{p(x)} dx \\ & \leq C \int_{\mathbb{R}^n} u(x) \left(\frac{1}{s} \chi_{B(0,r)}(x) |x|^{1-n} \right)^{p(x)} dx. \end{aligned} \quad (33)$$

Therefore

$$\int_{|x|>r} u(x) \left(\frac{r}{s|x|^n} \right)^{p(x)} dx \leq C \int_{B(0,r)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}} dx. \quad (34)$$

To obtain (c) from (b) we are going to prove that condition (b) implies that $\varphi_{p(\cdot), u(\delta)}$ is a constant function; namely, $\varphi_{p(\cdot), u(\delta)} = p_u^+ - p_u^-$ for all $\delta > 0$. This fact and the hypothesis on $\varphi_{p(\cdot), u(0^+)}$ imply that $\varphi_{p(\cdot), u(0^+)} \equiv 0$, and hence, due to (7),

$$p_{\sup u} \equiv p_u^+ - p_u^- \equiv p_0 \quad \text{a.e.} \quad (35)$$

Finally (31) means that $u \in B_{p_0}$. Let us suppose that $\varphi_{p(\cdot), u}$ is not constant. Then one of the following conditions holds:

(i) there exists $\delta > 0$ such that

$$\alpha = \operatorname{esssup}_{x \in B(0,\delta) \cap \operatorname{supp} u} p(x) < p_u^+ < \infty, \quad (36)$$

and, hence, there exists $\epsilon > 0$ such that

$$\{|x| > \delta : p(x) \geq \alpha + \epsilon\} \cap \operatorname{supp} u \neq \emptyset, \quad (37)$$

or

(ii) there exists $\delta > 0$ such that

$$\beta = \operatorname{essinf}_{x \in B(0,\delta) \cap \operatorname{supp} u} p(x) > p_u^- > 0, \quad (38)$$

and then, for some $\epsilon > 0$,

$$\{|x| > \delta : p(x) \leq \beta - \epsilon\} \cap \operatorname{supp} u \neq \emptyset. \quad (39)$$

In case (i) we observe that condition (b), for $r = \delta$, implies that

$$\int_{|x|>\delta} \left(\frac{\delta}{s} \right)^{p(x)} \frac{u(x)}{|x|^{np(x)}} dx \leq C \int_{B(0,\delta)} \frac{|x|^{(1-n)p(x)} u(x)}{s^{p(x)}} dx. \quad (40)$$

Then using (36) we obtain, for $s < \min(1, \delta)$,

$$\begin{aligned} & \left(\frac{\delta}{s} \right)^{\alpha+\epsilon} \int_{\{|x| \geq \delta : p(x) \geq \alpha+\epsilon\}} \frac{u(x)}{|x|^{np(x)}} dx \\ & \leq \frac{C}{s^\alpha} \int_{B(0,\delta)} u(x) |x|^{(1-n)p(x)} dx, \end{aligned} \quad (41)$$

which is clearly a contradiction if we let $s \downarrow 0$. Similarly in case (ii) let us consider the same condition (b), for $r = \delta$, and fix now $s > 1$. Taking into account (38) we find that

$$\begin{aligned} & \frac{1}{s^{\beta-\epsilon}} \int_{\{|x| \geq \delta : p(x) \leq \beta-\epsilon\}} \left(\frac{\delta}{|x|^n} \right)^{p(x)} u(x) dx \\ & \leq \frac{C}{s^\beta} \int_{B(0,\delta)} |x|^{(1-n)p(x)} u(x) dx, \end{aligned} \quad (42)$$

which is a contradiction if we let $s \uparrow \infty$.

Finally, the fact that condition (c) implies (a) follows from [18, Theorem 1.7]. $\square$

Theorem 4. *Let u be a weight on $(0, \infty)$ and $p : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$. Assume that $\psi_{p(\cdot), v(0^+)} = 0$. The following facts are equivalent:*

(i) *there exists a positive constant C such that, for any $f \in D(\mathbb{R}_+)$,*

$$\begin{aligned} & \int_{\mathbb{R}_+} (R_\alpha f(x))^{p(x)} u(x) dx \\ & \leq C \int_{\mathbb{R}_+} (f(x))^{p(x)} u(x) dx; \end{aligned} \quad (43)$$

(ii) condition (13) holds;

(iii) condition (c) of Theorem B is satisfied.

Proof. Proof follows by using Theorem B and Proposition 1(a). $\square$

Theorem 5. Let u be a weight on $\mathbb{R}^n$ and $p : \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that $0 < p^- \leq p^+ < \infty$, and assume that $\varphi_{p(\cdot), u(0^+)} = 0$. The following facts are equivalent:

(i) there exists a positive constant C such that, for any $f \in DR(\mathbb{R}^n)$,

$$\begin{aligned} & \int_{\mathbb{R}^n} (I_\alpha f(x))^{p(x)} u(x) dx \\ & \leq C \int_{\mathbb{R}^n} (f(x))^{p(x)} u(x) dx; \end{aligned} \quad (44)$$

(ii) condition (31) holds;

(iii) condition (c) of Proposition 3 holds.

Proof. Proof follows by using Propositions 3 and 1(b). $\square$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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