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Research Article

Stability and Hopf Bifurcation in a Prey-Predator System with Disease in the Prey and Two Delays

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This paper is concerned with a prey-predator system with disease in the prey and two delays. Local stability of the positive equilibrium of the system and existence of local Hopf bifurcation are investigated by choosing different combinations of the two delays as bifurcation parameters. For further investigation, the direction and the stability of the Hopf bifurcation are determined by using the normal form method and center manifold theorem. Finally, some numerical simulations are given to support the theoretical analysis.

1. Introduction

The effect of disease in ecological system is an important issue from mathematical as well as ecological point of view. Therefore, the dynamics of epidemiological models have been investigated by many authors in recent years [1–6]. In [6], Jana and Kar proposed and investigated the following predator-prey system with disease in the prey:

$$\begin{aligned} \frac{dS(t)}{dt} &= rS\left(1-\frac{S}{K}\right)-\frac{\alpha SI(t-\tau)}{a+S}-\frac{\beta SP}{b+S}, \\ \frac{dI(t)}{dt} &= \frac{\alpha SI(t-\tau)}{a+S}-\gamma I, \\ \frac{dP(t)}{dt} &= \frac{m\beta SP}{b+S}-\delta P^2-\varepsilon P, \end{aligned} \tag{1}$$

where $S(t)$, $I(t)$ denote the population densities of the susceptible prey and the infected prey at time t, respectively. $P(t)$ denotes the population of the predator at time t. The susceptible prey grows logistically with the intrinsic growth rate r and the carrying capacity K. The conversion from the susceptible prey to the infected prey is governed by the response function $\alpha I/(a+S)$. The consumption of the susceptible prey by the predator is governed by the response function $\beta P/(b+S)$. γ is the removal rate of the infected prey biomass. δ is the intraspecific competition coefficient of the predator. ε is the removal rate of the predator due to natural death or harvesting. And the constant τ ($\tau \geq 0$) is the time delay due to susceptible prey which becomes the infected prey. The predator-prey system with single delay has been investigated by many researchers [7–11]. Jana and Kar [6] studied the boundedness of the solutions and stability of the positive equilibrium of system (1). Existence and properties of the Hopf bifurcation were also investigated.

In recent years, there are also some papers on the bifurcations of a prey-predator system with two or multiple delays [12–16]. As is known to all, the consumption of the susceptible prey by the predator throughout its past history governs the present birth rate of the predator. Therefore, it is reasonable to incorporate time delay due to the gestation of the predator into system (1). Based on this consideration, we consider the following system with two delays in this paper:

$$\begin{aligned} \frac{dS(t)}{dt} &= rS\left(1-\frac{S}{K}\right)-\frac{\alpha SI(t-\tau_1)}{a+S}-\frac{\beta SP}{b+S}, \\ \frac{dI(t)}{dt} &= \frac{\alpha SI(t-\tau_1)}{a+S}-\gamma I, \\ \frac{dP(t)}{dt} &= P\left[\frac{m\beta S(t-\tau_2)}{b+S(t-\tau_2)}-\delta P-\varepsilon\right], \end{aligned} \tag{2}$$

where $\tau_1 \geq 0$ is the time delay due to susceptible prey which becomes the infected prey and $\tau_2 \geq 0$ is the time delay due to the gestation of the predator.

This paper is organized as follows. In Section 2, we investigate local stability of the positive equilibrium and existence of local Hopf bifurcation of system (2) with respect to both delays. In Section 3, by using the normal form method and center manifold theorem, the properties of the Hopf bifurcation such as direction and stability are determined. Some numerical simulations are given for the support of the analytical findings in Section 4.

2. Local Stability and Hopf Bifurcation

According to the analysis in [6], system (2) has a unique positive equilibrium $E^*(S^*, I^*, P^*)$ if $\alpha > \gamma$, $am\beta\gamma/(b + (a - b)\gamma) > \varepsilon$ and $(1 - S^*/K) > \beta P^*/(b + S^*)$, where $S^* = a\gamma/(\alpha - \gamma)$, $I^* = (r(1 - S^*/K) - \beta P^*/(b + S^*))(a + S^*)/\alpha$, $P^* = (m\beta S^* - \varepsilon(b + S^*))/(\delta(b + S^*))$.

Let $\bar{S} = S - S^*$, $\bar{I} = I - I^*$, $\bar{P} = P - P^*$. Dropping the bars for convenience, system (2) becomes the following form:

$$\begin{aligned} \frac{dS(t)}{dt} &= a_1 S(t) + a_2 P(t) + b_1 I(t - \tau_1) + F_1, \\ \frac{dI(t)}{dt} &= a_3 S(t) + a_4 I(t) + b_2 I(t - \tau_1) + F_2, \\ \frac{dP(t)}{dt} &= a_5 P(t) + c_1 S(t - \tau_2) + F_3, \end{aligned} \tag{3}$$

where

$$\begin{aligned} &a_1 = -\frac{rS^*}{K} + \frac{\alpha S^* I^*}{(a + S^*)^2} + \frac{\beta S^* P^*}{(b + S^*)^2}, \qquad a_2 = -\frac{\beta S^*}{b + S^*}, \\ &a_3 = \frac{a\alpha I^*}{(a + S^*)^2}, \qquad a_4 = -\gamma, \qquad a_5 = -\delta P^*, \\ &b_1 = -\frac{\alpha S^*}{a + S^*}, \qquad b_2 = \frac{\alpha S^*}{a + S^*}, \qquad c_1 = \frac{mb\beta P^*}{(b + S^*)^2}, \end{aligned} \tag{4}$$

$$\begin{aligned} F_1 &= a_{21} S^2(t) + a_{22} S(t) I(t - \tau_1) + a_{23} S(t) P(t) + a_{31} S^3(t) \\ &\quad + s_{32} S^2(t) I(t - \tau_1) + a_{33} S^2(t) P(t) + \cdots, \\ F_2 &= b_{21} S^2(t) + b_{22} S(t) I(t - \tau_1) + b_{31} S^3(t) \\ &\quad + b_{32} S^2(t) I(t - \tau_1) + \cdots, \\ F_3 &= c_{21} P^2(t) + c_{22} P(t) S(t - \tau_2) + c_{23} S^2(t - \tau_2) \\ &\quad + c_{31} P(t) S^2(t - \tau_2) + c_{32} S^3(t - \tau_2) + \cdots, \end{aligned} \tag{5}$$

with

$$\begin{aligned} &a_{21} = -\frac{r}{K} + \frac{a\alpha I^*}{(a + S^*)^3} + \frac{b\beta P^*}{(b + S^*)^3}, \qquad a_{22} = -\frac{a\alpha}{(a + S^*)^3}, \\ &a_{23} = -\frac{b\beta}{(b + S^*)^2}, \qquad a_{31} = -\frac{a\alpha I^*}{(a + S^*)^4} - \frac{b\beta P^*}{(b + S^*)^4}, \\ &a_{32} = \frac{a\alpha}{(a + S^*)^3}, \qquad a_{33} = \frac{b\beta}{(b + S^*)^3}, \\ &b_{21} = -\frac{a\alpha I^*}{(a + S^*)^3}, \qquad b_{22} = \frac{a\alpha}{(a + S^*)^2}, \\ &b_{31} = -\frac{a\alpha I^*}{(a + S^*)^4}, \qquad b_{32} = -\frac{a\alpha}{(a + S^*)^4}, \\ &c_{21} = -\delta, \qquad c_{22} = \frac{mb\beta}{(b + S^*)^2}, \qquad c_{23} = -\frac{mb\beta P^*}{(b + S^*)^3}, \\ &c_{31} = -\frac{mb\beta}{(b + S^*)^3}, \qquad c_{32} = \frac{mb\beta P^*}{(b + S^*)^4}. \end{aligned} \tag{6}$$

The linearized system of (3) is

$$\begin{aligned} \frac{dS(t)}{dt} &= a_1 S(t) + a_2 P(t) + b_1 I(t - \tau_1), \\ \frac{dI(t)}{dt} &= a_3 S(t) + a_4 I(t) + b_2 I(t - \tau_1), \\ \frac{dP(t)}{dt} &= a_5 P(t) + c_1 S(t - \tau_2). \end{aligned} \tag{7}$$

Thus, we can get that the characteristic equation of system (7) is

$$\begin{aligned} &\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + \left(n_2\lambda^2 + n_1\lambda + n_0\right) e^{-\lambda\tau_1} \\ &\quad + \left(p_1\lambda + p_0\right) e^{-\lambda\tau_2} + q_0 e^{-\lambda(\tau_1+\tau_2)} = 0, \end{aligned} \tag{8}$$

where

$$\begin{aligned} &m_0 = -a_1 a_4 a_5, \qquad m_1 = a_1 a_4 + a_1 a_5 + a_4 a_5, \\ &m_2 = -\left(a_1 + a_4 + a_5\right), \qquad n_0 = a_5\left(a_1 b_2 - a_3 b_1\right), \\ &n_1 = \left(a_1 + a_5\right) b_2 - a_3 b_1, \qquad n_2 = -b_2, \\ &p_0 = a_2 a_4 c_1, \qquad p_1 = -a_2 c_1, \qquad q_0 = a_2 b_2 c_1. \end{aligned} \tag{9}$$

Case 1 ($\tau_1 = \tau_2 = 0$). Equation (8) becomes

$$\lambda^3 + m_{12}\lambda^2 + m_{11}\lambda + m_{10} = 0, \tag{10}$$

where

$$\begin{aligned} &m_{12} = m_2 + n_2, \qquad m_{11} = m_1 + n_1 + p_1, \\ &m_{10} = m_0 + n_0 + p_0 + q_0, \\ &A_{10} = A_0 + B_0 + C_0 + D_0 + E_0 + F_0. \end{aligned} \tag{11}$$

It follows from the Routh-Hurwitz criteria that all roots of (10) have negative real parts if the following condition holds: (H_{11}): $m_{12} > 0$ and $m_{12}m_{11} > m_{10}$. Then, the positive equilibrium of system (2) without delay is locally asymptotically stable.

Case 2 ($\tau_1 > 0, \tau_2 = 0$). On substituting $\tau_2 = 0$, (8) becomes

$$\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20} + \left(n_{22}\lambda^2 + n_{21}\lambda + n_{20}\right)e^{-\lambda\tau_1} = 0, \tag{12}$$

where

$$\begin{gathered} m_{22} = m_2, \qquad m_{21} = m_1 + p_1, \qquad m_{20} = m_0 + p_0, \\ n_{22} = n_2, \qquad n_{21} = n_1, \qquad n_{20} = n_0 + q_0. \end{gathered} \tag{13}$$

Let $\lambda = i\omega_1 (\omega_1 > 0)$ be a root of (12). Then, we have

$$\begin{gathered} n_{21}\omega_1 \sin\omega_1\tau_1 + \left(n_{20} - n_{22}\omega_1^2\right)\cos\omega_1\tau_1 = m_{22}\omega_1^2 - m_{20}, \\ n_{21}\omega_1 \cos\omega_1\tau_1 - \left(n_{20} - n_{22}\omega_1^2\right)\sin\omega_1\tau_1 = \omega_1^3 - m_{21}\omega_1, \end{gathered} \tag{14}$$

which implies that

$$\omega_1^6 + g_{22}\omega_1^4 + g_{21}\omega_1^2 + g_{20} = 0, \tag{15}$$

where

$$\begin{gathered} g_{20} = m_{20}^2 - n_{20}^2, \\ g_{21} = m_{21}^2 - n_{21}^2 - 2m_{20}m_{22} + 2n_{20}n_{22}, \\ g_{22} = m_{22}^2 - n_{22}^2 - 2m_{21}. \end{gathered} \tag{16}$$

Denote $\omega_1^2 = v_1$; then (15) becomes

$$v_1^3 + g_{22}v_1^2 + g_{21}v_1 + g_{20} = 0. \tag{17}$$

Let

$$f_1(v_1) = v_1^3 + g_{22}v_1^2 + g_{21}v_1 + g_{20}. \tag{18}$$

In [17], Song et al. obtained the following results on the distribution of roots of (17).

Lemma 1. *For* (17),

(1) *if* $g_{20} < 0$, *then* (17) *has at least one positive root;*

(2) *if* $g_{20} \geq 0$ *and* $g_{22}^2 - 3g_{21} \leq 0$, *then* (17) *has no positive roots;*

(3) *if* $g_{20} \geq 0$ *and* $g_{22}^2 - 3g_{21} < 0$, *then* (17) *has positive root if and only if* $v_1^* = (-g_{20} + \sqrt{g_{22}^2 + 3g_{21}})/3 > 0$ *and* $f_1(v_1^*) \leq 0$.

(H_{21}) *Suppose that* (17) *has at least one positive root.*

Without loss of generality, we assume that (17) *has three positive roots, which are denoted by* v_{11}, v_{12}, *and* v_{13}. *Then* (15) *has three positive roots* $\omega_{1k} = \sqrt{v_{1k}}$, $k = 1, 2, 3$. *For every fixed* ω_{1k}, *one can get*

$$\begin{aligned} \sin(\omega_{1k}\tau_1) &= \left(n_{22}\omega_{1k}^5 + \left(m_{22}n_{21} - m_{21}n_{22} - n_{20}\right)\omega_{1k}^3 \right. \\ &\quad \left. + \left(m_{21}n_{20} - m_{20}n_{21}\right)\omega_{1k}\right) \\ &\quad \times \left(n_{22}^2\omega_{1k}^4 + \left(n_{21}^2 - 2n_{20}n_{22}\right)\omega_{1k}^2 + n_{20}^2\right)^{-1} \\ &\triangleq T_{2s}(\omega_{1k}), \\ \cos(\omega_{1k}\tau_1) &= \left(\left(n_{21} - m_{22}n_{22}\right)\omega_{1k}^4 \right. \\ &\quad + \left(m_{20}n_{22} + m_{22}n_{20} - m_{21}n_{21}\right) \\ &\quad \left. \times\omega_{1k}^2 - m_{20}n_{20}\right) \\ &\quad \times \left(n_{22}^2\omega_{1k}^4 + \left(n_{21}^2 - 2n_{20}n_{22}\right)\omega_{1k}^2 + n_{20}^2\right)^{-1} \\ &\triangleq T_{2c}(\omega_{1k}). \end{aligned} \tag{19}$$

Thus,

$$\tau_{1k}^{(j)} = \begin{cases} \dfrac{1}{\omega_{1k}}\left(\arccos\left(T_{2c}(\omega_{1k})\right) + 2j\pi\right), & T_{2s}(\omega_{1k}) \geq 0, \\ \dfrac{1}{\omega_{1k}}\left(2\pi - \arccos\left(T_{2c}(\omega_{1k})\right) + 2j\pi\right), & T_{2s}(\omega_{1k}) < 0, \end{cases} \tag{20}$$

with $k = 1, 2, 3$; $j = 0, 1, 2, \ldots$.

Let

$$\begin{gathered} \tau_{10} = \min\left\{\tau_{1k}^{(0)}\right\}, \quad k = 1, 2, 3, \\ \omega_{10} = \omega_{1k_0}. \end{gathered} \tag{21}$$

Differentiating both sides of (12) *with respect to* τ_1 *we can get*

$$\begin{aligned} \left[\frac{d\lambda}{d\tau_1}\right]^{-1} &= -\frac{3\lambda^2 + 2m_{22}\lambda + m_{21}}{\lambda\left(\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20}\right)} \\ &\quad + \frac{2n_{22}\lambda + n_{21}}{\lambda\left(n_{22}\lambda^2 + n_{21}\lambda + n_{20}\right)} - \frac{\tau_1}{\lambda}. \end{aligned} \tag{22}$$

Thus, we have

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_1}\right]^{-1}_{\tau=\tau_{10}} = \frac{f_1'(v_{1*})}{n_{22}^2\omega_{10}^4 + \left(n_{21}^2 - 2n_{20}n_{22}\right)\omega_{10}^2 + n_{20}^2}, \tag{23}$$

where $v_{1*} = \omega_{10}^2$. *Obviously, if the condition* (H_{22}) : $f_1'(v_{1*}) \neq 0$ *holds, then* $\operatorname{Re}[d\lambda/d\tau_1]^{-1}_{\tau_1=\tau_{10}} \neq 0$. *By the Hopf bifurcation theorem in [18], we have the following results.*

Theorem 2. *Suppose that conditions* (H_{21})-(H_{22}) *hold. The positive equilibrium* $E^*(S^*, I^*, P^*)$ *of system* (2) *is asymptotically stable for* $\tau_1 \in [0, \tau_{10})$ *and system* (2) *undergoes a Hopf bifurcation at* $E^*(S^*, I^*, P^*)$ *when* $\tau_1 = \tau_{10}$.

Case 3 ($\tau_2 > 0, \tau_1 = 0$). Substitute $\tau_1 = 0$ into (8), then (8) becomes

$$\lambda^3 + m_{32}\lambda^2 + m_{31}\lambda + m_{30} + (p_{31}\lambda + p_{30})e^{-\lambda\tau_2} = 0, \quad (24)$$

where

$$m_{32} = m_2 + n_2, \qquad m_{31} = m_1 + n_1,$$
$$m_{30} = m_0 + n_0, \qquad p_{31} = p_1, \quad (25)$$
$$p_{30} = p_0 + q_0.$$

Let $\lambda = i\omega_2$ ($\omega_2 > 0$) be a root of (24). Then, we get

$$p_{31}\omega_2 \sin\omega_2\tau_2 + p_{30}\cos\omega_2\tau_2 = m_{32}\omega_2^2 - m_{30},$$
$$p_{31}\omega_2 \cos\omega_2\tau_2 - p_{30}\sin\omega_2\tau_2 = \omega_2^3 - m_{31}\omega_2, \quad (26)$$

which follows that

$$\lambda_2^6 + g_{32}\lambda_2^4 + g_{31}\lambda_2 + g_{30} = 0, \quad (27)$$

where

$$g_{32} = m_{32}^2 - 2m_{31}, \qquad g_{31} = m_{31}^2 - 2m_{30}m_{32} - p_{31}^2,$$
$$g_{30} = m_{30}^2 - p_{30}^2. \quad (28)$$

Denote $\omega_2^2 = v_2$; then (27) becomes

$$v_2^3 + g_{32}v_2^2 + g_{31}v_2 + g_{30} = 0. \quad (29)$$

Let

$$f_2(v_2) = v_2^3 + g_{32}v_2^2 + g_{31}v_2 + g_{30} = 0. \quad (30)$$

Similarly as in Case 2, we suppose that (H_{31}), (29) has at least one positive root. Without loss of generality, we assume that it has three positive roots and we denote them by v_{21}, v_{22}, and v_{23}, respectively. Then (27) has three positive roots $\omega_{2k} = \sqrt{v_{2k}}$, $k = 1, 2, 3$. For every fixed ω_{2k},

$$\sin(\omega_{2k}\tau_2) = \frac{(m_{32}p_{31} - p_{30})\omega_{2k}^3 + (m_{31}p_{30} - m_{30}p_{31})\omega_{2k}}{p_{31}^2\omega_{2k}^2 + p_{30}^2} \triangleq T_{3s}(\omega_{2k}), \quad (31)$$

$$\cos(\omega_{2k}\tau_2) = \frac{p_{31}\omega_{2k}^4 + (m_{32}p_{30} - m_{31}p_{31})\omega_{2k}^2 - m_{30}p_{30}}{p_{31}^2\omega_{2k}^2 + p_{30}^2} \triangleq T_{3c}(\omega_{2k}). \quad (32)$$

Thus,

$$\tau_{2k}^{(j)} = \begin{cases} \dfrac{1}{\omega_{2k}}\left(\arccos(T_{3c}(\omega_{2k})) + 2j\pi\right), & T_{3s}(\omega_{2k}) \geq 0, \\ \dfrac{1}{\omega_{2k}}\left(2\pi - \arccos(T_{3c}(\omega_{2k})) + 2j\pi\right), & T_{3s}(\omega_{2k}) < 0, \end{cases} \quad (33)$$

with $k = 1, 2, 3$; $j = 0, 1, 2, \ldots$.

Let

$$\tau_{20} = \min\left\{\tau_{2k}^{(0)}\right\}, \quad k = 1, 2, 3,$$
$$\omega_{20} = \omega_{2k_0}. \quad (34)$$

Differentiating (24) regarding τ_2, we get

$$\left[\frac{d\lambda}{d\tau_2}\right]^{-1} = \frac{3\lambda^2 + 2m_{32}\lambda + m_{31}}{\lambda(\lambda^3 + m_{32}\lambda^2 + m_{31}\lambda + m_{30})} + \frac{p_{31}}{\lambda(p_{31}\lambda + p_{30})} - \frac{\tau_2}{\lambda}. \quad (35)$$

Then, we can get

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau_2}\right]^{-1}_{\tau=\tau_{20}} = \frac{f_2'(v_{2*})}{p_{31}^2\omega_{20}^2 + p_{30}^2}, \quad (36)$$

where $v_{2*} = \omega_{20}^2$. Therefore, if the condition (H_{32}): $f_2'(v_{2*}) \neq 0$ holds, then $\operatorname{Re}[d\lambda/d\tau_2]^{-1}_{\tau_2=\tau_{20}} \neq 0$. Thus, by the Hopf bifurcation theorem in [18], we have the following results.

Theorem 3. *Suppose that conditions (H_{31})-(H_{32}) hold. The positive equilibrium $E^*(S^*, I^*, P^*)$ of system (2) is asymptotically stable for $\tau_1 \in [0, \tau_{20})$ and system (2) undergoes a Hopf bifurcation at $E^*(S^*, I^*, P^*)$ when $\tau_2 = \tau_{20}$.*

Case 4 ($\tau_1 = \tau_2 = \tau > 0$). Let $\tau_1 = \tau_2 = \tau$; then (8) becomes

$$\lambda^3 + m_{42}\lambda^2 + m_{41}\lambda + m_{40} + \left(n_{42}\lambda^2 + n_{41}\lambda + n_{40}\right)e^{-\lambda\tau} + q_{40}e^{-2\lambda\tau} = 0, \quad (37)$$

where

$$m_{42} = m_2, \qquad m_{41} = m_1, \qquad m_{40} = m_0, \qquad n_{42} = n_2,$$
$$n_{41} = n_1 + p_1, \qquad n_{40} = n_0 + p_0, \qquad q_{40} = q_0. \quad (38)$$

Multiplying (37) by $e^{\lambda\tau}$, then (37) becomes

$$n_{42}\lambda^2 + n_{41}\lambda + n_{40} + \left(\lambda^3 + m_{42}\lambda^2 + m_{41}\lambda + m_{40}\right)e^{\lambda\tau} + q_{40}e^{-\lambda\tau} = 0. \quad (39)$$

Let $\lambda = i\omega(\omega > 0)$ be the root of (39); then we can get

$$\left(m_{40} + q_{40} - m_{42}\omega^2\right)\cos\tau\omega + \left(\omega^3 - m_{41}\omega\right)\sin\tau\omega = n_{32}\omega^2 - n_{40},$$
$$\left(m_{40} - q_{40} - m_{42}\omega^2\right)\sin\tau\omega - \left(\omega^3 - m_{41}\omega\right)\cos\tau\omega = -n_{41}\omega, \quad (40)$$

which follows that

$$\sin(\tau\omega) = \frac{g_5\omega^5 + g_3\omega^3 + g_1\omega}{\omega^6 + h_4\omega^4 + h_2\omega^2 + h_0} \triangleq T_{4s}(\omega),$$
$$\cos(\tau\omega) = \frac{g_4\omega^4 + g_2\omega^2 + g_0}{\omega^6 + h_4\omega^4 + h_2\omega^2 + h_0} \triangleq T_{4c}(\omega), \quad (41)$$

where

$$\begin{gathered} g_0 = n_{40}q_{40} - m_{40}n_{40}, \\ g_1 = m_{41}n_{40} - m_{40}n_{41} - n_{41}q_{40}, \\ g_2 = m_{40}n_{42} - m_{41}n_{41} + m_{42}n_{40} - n_{42}q_{40}, \\ g_3 = m_{42}n_{41} - m_{41}n_{42} - n_{40}, \\ g_4 = n_{41} - m_{42}n_{42}, \qquad g_5 = n_{42}, \qquad h_0 = m_{40}^2 - q_{40}^2, \\ h_2 = m_{41}^2 - 2m_{40}m_{42}, \qquad h_4 = m_{42}^2 - 2m_{41}. \end{gathered} \tag{42}$$

Then, we can obtain

$$\omega^{12} + g_{45}\omega^{10} + g_{44}\omega^8 + g_{43}\omega^6 + g_{42}\omega^4 + g_{41}\omega^2 + g_{40} = 0, \tag{43}$$

where

$$\begin{gathered} g_{40} = h_0^2 - g_0^2, \qquad g_{41} = 2h_0h_2 - 2g_0g_2 - g_1^2, \\ g_{42} = h_2^2 + 2h_0h_4 - g_2^2 - 2g_0g_4 - 2g_1g_3, \\ g_{43} = 2h_0 + 2h_2h_4 - g_3^2 - 2g_1g_5 - 2g_2g_4, \\ g_{44} = h_4^2 + 2h_2 - g_4^2 - 2g_3g_5, \qquad g_{45} = 2h_4 - g_5^2. \end{gathered} \tag{44}$$

Let $\omega^2 = v_3$; then (43) becomes

$$v_3^6 + g_{45}v_3^5 + g_{44}v_3^4 + g_{43}v_3^3 + g_{42}v_3^2 + g_{41}v_3 + g_{40} = 0. \tag{45}$$

If we know all the coefficients of system (2), then we can get all the coefficients of (45) and then all the roots of (45) can be obtained by Matlab. Therefore, we give the following assumption.

Suppose that (H_{41}): (45) has at least one positive root.

Without loss of generality, we assume that (45) has six positive roots, which are denoted by $v_{31}, v_{32}, \ldots, v_{36}$, respectively. Then, (43) has six positive roots $\omega_k = \sqrt{v_{3k}}$, $k = 1, 2, \ldots, 6$. For every ω_k,

$$\tau_k^{(j)} = \begin{cases} \dfrac{1}{\omega_k}\left(\arccos\left(T_{4c}\left(\omega_k\right)\right) + 2j\pi\right), & T_{4s}\left(\omega_k\right) \geq 0, \\ \dfrac{1}{\omega_k}\left(2\pi - \arccos\left(T_{4c}\left(\omega_k\right)\right) + 2j\pi\right), & T_{4s}\left(\omega_k\right) < 0, \end{cases} \tag{46}$$

with $k = 1, 2, 3, \ldots, 6$; $j = 0, 1, 2, \ldots$.

Let

$$\begin{gathered} \tau_0 = \min\left\{\tau_k^{(0)}\right\}, \quad k = 1, 2, \ldots, 6, \\ \omega_0 = \omega_{k_0}. \end{gathered} \tag{47}$$

Next, taking the derivative of λ with respect to τ in (39), we have

$$\begin{aligned} \left[\frac{d\lambda}{d\tau}\right]^{-1} &= \left(2n_{42}\lambda + n_{41} + \left(3\lambda^2 + 2m_{42}\lambda + m_{41}\right)e^{\lambda\tau}\right) \\ &\quad \times \left(q_{40}\lambda e^{-\lambda\tau} - \left(\lambda^4 + m_{42}\lambda^3 + m_{41}\lambda^2 \right.\right. \\ &\quad \left.\left. + m_{40}\lambda\right)e^{\lambda\tau}\right)^{-1} - \frac{\tau}{\lambda}. \end{aligned} \tag{48}$$

Then we have

$$\operatorname{Re}\left[\frac{d\lambda}{d\tau}\right]^{-1}_{\tau=\tau_0} = \frac{P_RQ_R + P_IQ_I}{Q_R^2 + Q_I^2}, \tag{49}$$

where

$$\begin{aligned} P_R &= \left(m_{41} - 3\omega_0^2\right)\cos\tau_0\omega_0 - 2m_{42}\omega_0\sin\tau_0\omega_0 + n_{41}, \\ P_I &= \left(m_{41} - 3\omega_0^2\right)\sin\tau_0\omega_0 + 2m_{42}\omega_0\cos\tau_0\omega_0 + 2n_{42}\omega_0, \\ Q_R &= \left(m_{41}\omega_0^2 - \omega_0^4\right)\cos\tau_0\omega_0 \\ &\quad - \left(m_{42}\omega_0^3 - m_{40}\omega_0 - q_{40}\omega_0\right)\sin\tau_0\omega_0, \\ Q_I &= \left(m_{41}\omega_0^2 - \omega_0^4\right)\sin\tau_0\omega_0 \\ &\quad + \left(m_{42}\omega_0^3 - m_{40}\omega_0 + q_{40}\omega_0\right)\cos\tau_0\omega_0. \end{aligned} \tag{50}$$

Obviously, if condition (H_{42}): $P_RQ_R + P_IQ_I \neq 0$ holds, then $\operatorname{Re}\left[d\lambda/d\tau\right]^{-1}_{\tau=\tau_0} \neq 0$. Thus, by the discussion above and the Hopf bifurcation theorem in [18], we have the following results.

Theorem 4. *Suppose that conditions (H_{41})-(H_{42}) hold. The positive equilibrium $E^*(S^*, I^*, P^*)$ of system (2) is asymptotically stable for $\tau_1 \in [0, \tau_0)$ and system (2) undergoes a Hopf bifurcation at $E^*(S^*, I^*, P^*)$ when $\tau = \tau_0$.*

Case 5 ($\tau_2 > 0$ and $\tau_1 \in (0, \tau_{10})$). Let $\lambda = i\omega_2^*(\omega_2^*)$ be the root of (8). Then, we get

$$\begin{aligned} M_{51}\sin\tau_2\omega_2^* + M_{52}\cos\tau_2\omega_2^* &= N_{51}, \\ M_{51}\cos\tau_2\omega_2^* - M_{52}\sin\tau_2\omega_2^* &= N_{52}, \end{aligned} \tag{51}$$

where

$$M_{51} = p_1\omega_2^* - q_0\sin\tau_1\omega_2^*, \qquad N_{52} = p_0 + q_0\cos\tau_1\omega_2^*, \tag{52}$$

$$\begin{aligned} N_{51} &= m_2\left(\omega_2^*\right)^2 - m_0 - \left(n_0 - n_2\left(\omega_2^*\right)^2\right)\cos\tau_1\omega_2^* \\ &\quad - n_1\omega_2^*\sin\tau_1\omega_2^*, \end{aligned} \tag{53}$$

$$\begin{aligned} N_{52} &= \left(\omega_2^*\right)^3 - m_1\omega_2^* + \left(n_0 - n_2\left(\omega_2^*\right)^2\right)\sin\tau_1\omega_2^* \\ &\quad - n_1\omega_2^*\cos\tau_1\omega_2^*. \end{aligned} \tag{54}$$

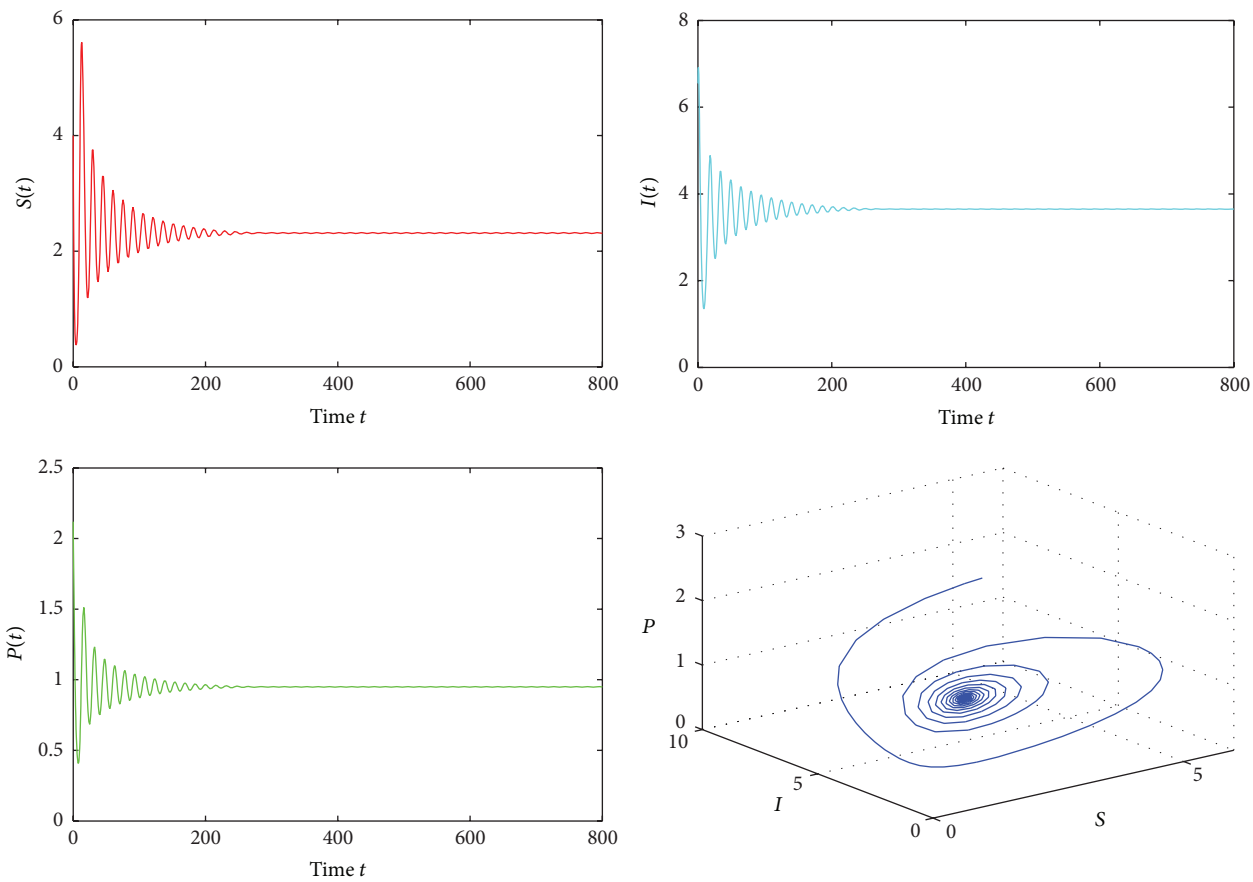

FIGURE 1: E^* is asymptotically stable when $\tau_1 = 0.45 < 0.5148 = \tau_{10}$.

Then, we can have

$$g_{51}(\omega_2^*) + 2g_{52}(\omega_2^*)\cos\tau_1\omega_2^* + 2g_{53}(\omega_2^*)\sin\tau_1\omega_2^* = 0, \tag{55}$$

where

$$\begin{aligned} g_{51}(\omega_2^*) &= (\omega_2^*)^6 + \left(m_2^2 + n_2^2 - 2m_1\right)(\omega_2^*)^4 \\ &\quad + \left(m_1^2 + n_1^2 - p_1^2 - 2m_0m_2 - 2n_0n_2\right)(\omega_2^*)^2 \\ &\quad + m_0^2 + n_0^2 - p_0^2 - q_0^2, g_{52}(\omega_2^*) \\ &= (m_2n_2 - n_1)(\omega_2^*)^4 \\ &\quad + (m_1n_1 - m_0n_2 - m_2n_0)(\omega_2^*)^2 \\ &\quad + m_0n_0 - p_0q_0, g_{53}(\omega_2^*) \\ &= -n_2(\omega_2^*)^5 + (n_0 - m_2n_1 + m_1n_2)(\omega_2^*)^3 \\ &\quad + (m_0n_1 - m_1n_0 + p_1q_0)\omega_2^*. \end{aligned} \tag{56}$$

We suppose that (H_{51}), (55) has at least finite positive roots. And we denote the positive roots of (55) by $\omega_{21}^*, \omega_{22}^*, \ldots, \omega_{2k}^*$. Then, for every fixed ω_{2i}^* $(i = 1, 2, \ldots, k)$,

$$\begin{aligned} \sin(\tau_2\omega_{2i}^*) &= \left.\frac{M_{51}N_{51} - M_{52}N_{52}}{M_{51}^2 + M_{52}^2}\right|_{\omega_2^*=\omega_{2i}^*} \triangleq T_{5s}(\omega_{2i}^*), \\ \cos(\tau_2\omega_{2i}^*) &= \left.\frac{M_{51}N_{52} + M_{52}N_{51}}{M_{51}^2 + M_{52}^2}\right|_{\omega_2^*=\omega_{2i}^*} \triangleq T_{5c}(\omega_{2i}^*). \end{aligned} \tag{57}$$

Thus,

$$\tau_{2i}^{*(j)} = \begin{cases} \dfrac{1}{\omega_{2i}^*}\left(\arccos\left(T_{5c}(\omega_{2i}^*)\right) + 2j\pi\right), & T_{5s}(\omega_k) \geq 0, \\ \dfrac{1}{\omega_{2i}^*}\left(2\pi - \arccos\left(T_{5c}(\omega_{2i}^*)\right) + 2j\pi\right), & T_{5s}(\omega_k) < 0, \end{cases} \tag{58}$$

with $i = 1, 2, \ldots, k$; $j = 0, 1, 2, \ldots$.

Let $\tau_{20}^* = \min\{\tau_{2i}^{*(0)} \mid i = 1, 2, \ldots, k\}$. When $\tau_2 = \tau_{20}^*$, (8) has a pair of purely imaginary roots $\pm i\omega_{2*}$ for $\tau_1 \in (0, \tau_{10})$. Next, in order to give the main results with respect to $\tau_2 > 0$, $\tau_1 \in (0, \tau_{10})$, we give the following assumption: (H_{52}): $\operatorname{Re}[d\lambda/d\tau_2]^{-1}_{\tau=\tau_{20}^*} \neq 0$.

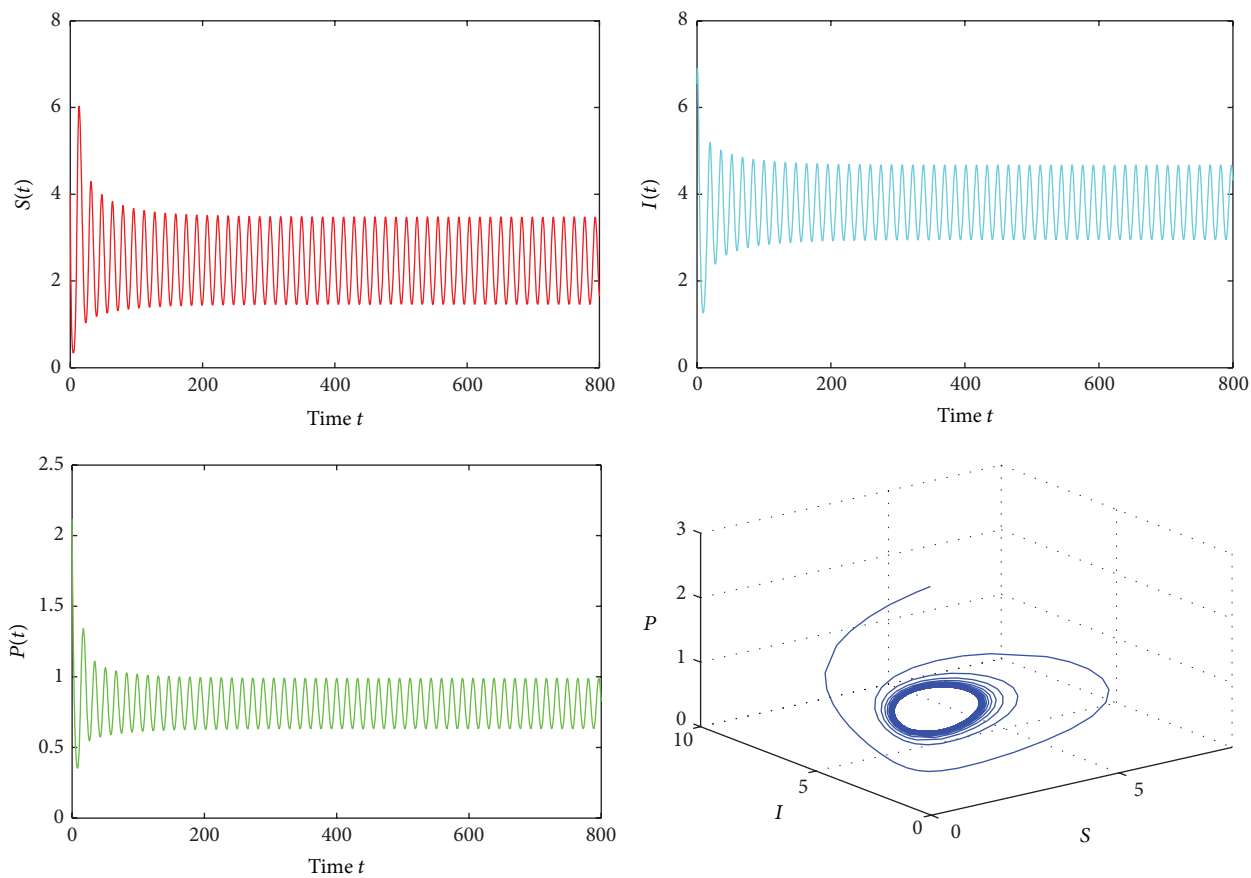

FIGURE 2: E^* is unstable when $\tau_1 = 0.575 > 0.5148 = \tau_{10}$.

Through the analysis above and the Hopf bifurcation theorem in [18], we have the following results.

Theorem 5. *If conditions* (H_{51})-(H_{52}) *hold and* $\tau_1 \in (0, \tau_{10})$, *then the positive equilibrium* $E^*(S^*, I^*, P^*)$ *of system* (2) *is asymptotically stable for* $\tau_2 \in [0, \tau_{20}^*)$ *and system* (2) *undergoes a Hopf bifurcation at* $E^*(S^*, I^*, P^*)$ *when* $\tau_2 = \tau_{20}^*$.

3. Stability of Bifurcating Periodic Solutions

In the previous section, it is shown that system (2) undergoes a Hopf bifurcation for different combinations of τ_1 and τ_2 under certain conditions. In this section, the properties of Hopf bifurcation such as direction and stability are investigated with respect to τ_2 for $\tau_1 \in (0, \tau_{10})$ by using the normal form method and center manifold theorem in [18]. Throughout this section, we assume that $\tau_{10}^* < \tau_{20}^*$ where $\tau_{10}^* \in (0, \tau_{10})$.

For convenience, let $\tau_2 = \mu + \tau_{20}^*$, so that $\mu = 0$ is the Hopf bifurcation value of system (2). Let $u_1(t) = S(t) - S^*$, $u_2(t) = I(t) - I^*$, $u_3(t) = P(t) - P^*$ and rescale the time delay $t \to (t/\tau_2)$; then system (2) can be rewritten as

$$\dot{u}(t) = L_\mu u_t + F(\mu, u_t), \tag{59}$$

where

$$L_\mu \phi = (\tau_{20}^* + \mu)\left(A'\phi(0) + B'\phi\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + C'\phi(-1)\right),$$
$$F(\mu, \phi) = (\tau_{20}^* + \mu)(F_1, F_2, F_3)^T, \tag{60}$$

with

$$\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \phi_3(\theta))^T \in C([-1, 0], R^3),$$
$$A' = \begin{pmatrix} a_1 & 0 & a_2 \\ a_3 & a_4 & 0 \\ 0 & 0 & a_5 \end{pmatrix}, \qquad B' = \begin{pmatrix} 0 & b_1 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$C' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c_1 & 0 & 0 \end{pmatrix},$$
$$F_1 = a_{21}\phi_1^2(0) + a_{22}\phi_1(0)\phi_2\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + a_{23}\phi_1(0)\phi_3(0) + a_{31}\phi_1^3(0) + a_{32}\phi_1^2(0)\phi_2\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + a_{33}\phi_1^2(0)\phi_3(0) + \cdots,$$
$$F_2 = b_{21}\phi_1^2(0) + b_{22}\phi_1(0)\phi_2\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right)$$

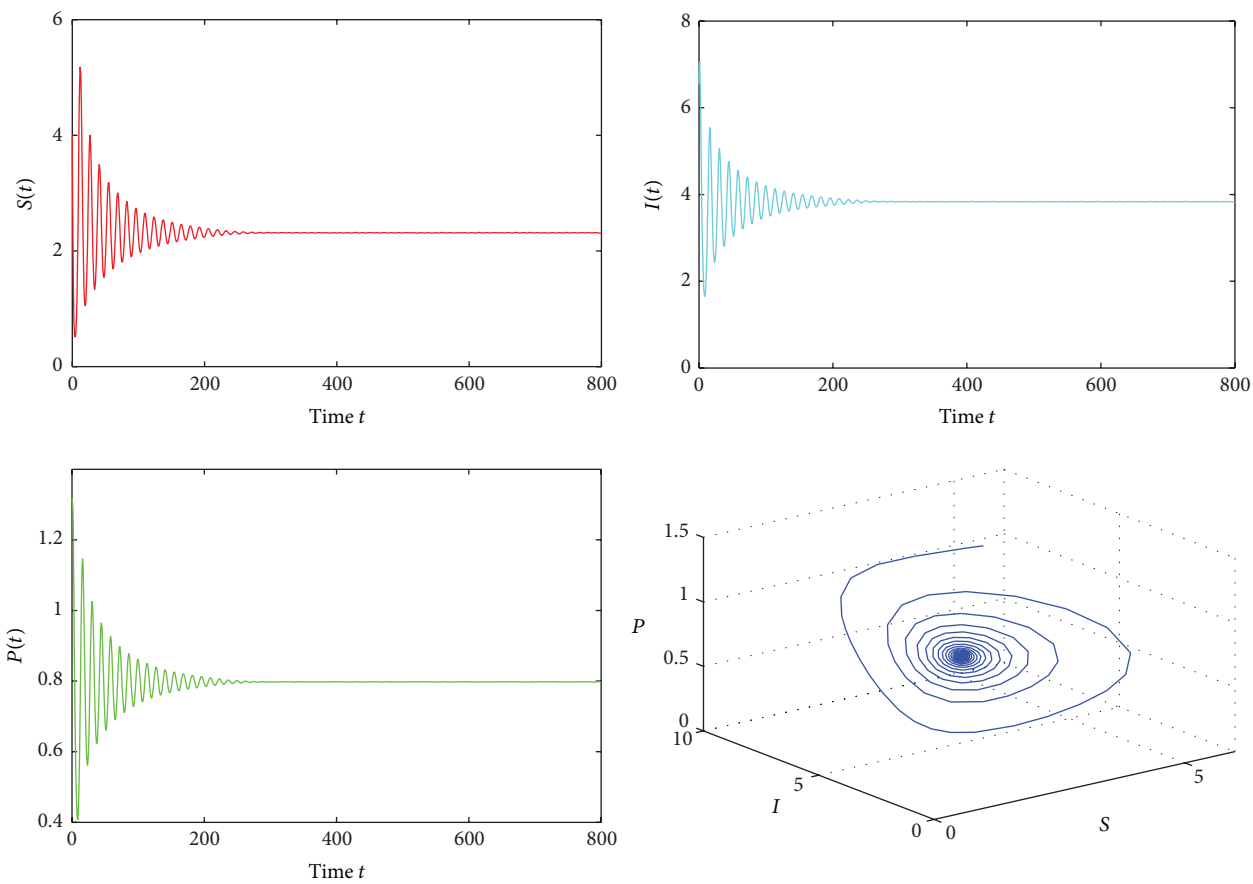

FIGURE 3: E^* is asymptotically stable when $\tau_2 = 1.275 < 1.8829 = \tau_{20}$.

$$+ b_{31}\phi_1^3(0) + b_{32}\phi_1^2(0)\phi_2\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + \cdots,$$

$$F_3 = c_{21}\phi_3^2(0) + c_{22}\phi_1(-1)\phi_3(0) + c_{23}\phi_1^2(-1) + c_{31}\phi_1^2(-1)\phi_3(0) + c_{32}\phi_1^3(-1) + \cdots. \tag{61}$$

Therefore, according to the Riesz representation theorem, there exists a 3×3 matrix function $\eta(\theta,\mu) : [-1,0] \to R^3$ whose elements are of bounded variation such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), \quad \phi \in C([-1,0], R^3). \tag{62}$$

In fact, we choose

$$\eta(\theta,\mu) = \begin{cases} (\tau_{20}^* + \mu)(A' + B' + C'), & \theta = 0, \\ (\tau_{20}^* + \mu)(B' + C'), & \theta \in \left[-\frac{\tau_{10}^*}{\tau_{20}^*}, 0\right), \\ (\tau_{20}^* + \mu)C', & \theta \in \left(-1, -\frac{\tau_{10}^*}{\tau_{20}^*}\right), \\ 0, & \theta = -1. \end{cases} \tag{63}$$

For $\phi \in C([-1,0], R^3)$, we define

$$A(\mu)\phi = \begin{cases} \dfrac{d\phi(\theta)}{d\theta}, & -1 \le \theta < 0, \\ \displaystyle\int_{-1}^{0} d\eta(\theta,\mu)\phi(\theta), & \theta = 0, \end{cases} \tag{64}$$

$$R(\mu)\phi = \begin{cases} 0, & -1 \le \theta < 0, \\ F(\mu,\phi), & \theta = 0. \end{cases}$$

Then system (59) can be transformed into the following operator equation:

$$\dot{u}(t) = A(\mu)u_t + R(\mu)u_t, \tag{65}$$

where $u_t = u(t+\theta) = (u_1(t+\theta), u_2(t+\theta), u_3(t+\theta))$ for $\theta \in [-1,0]$.

For $\varphi \in C^1([0,1], (R^3)^*)$, where $(R^3)^*$ is the 3-dimensional space of row vectors, we define the adjoint operator A^* of A:

$$A^*(\varphi) = \begin{cases} -\dfrac{d\varphi(s)}{ds}, & 0 < s \le 1, \\ \displaystyle\int_{-1}^{0} d\eta^T(s,0)\varphi(-s), & s = 0, \end{cases} \tag{66}$$

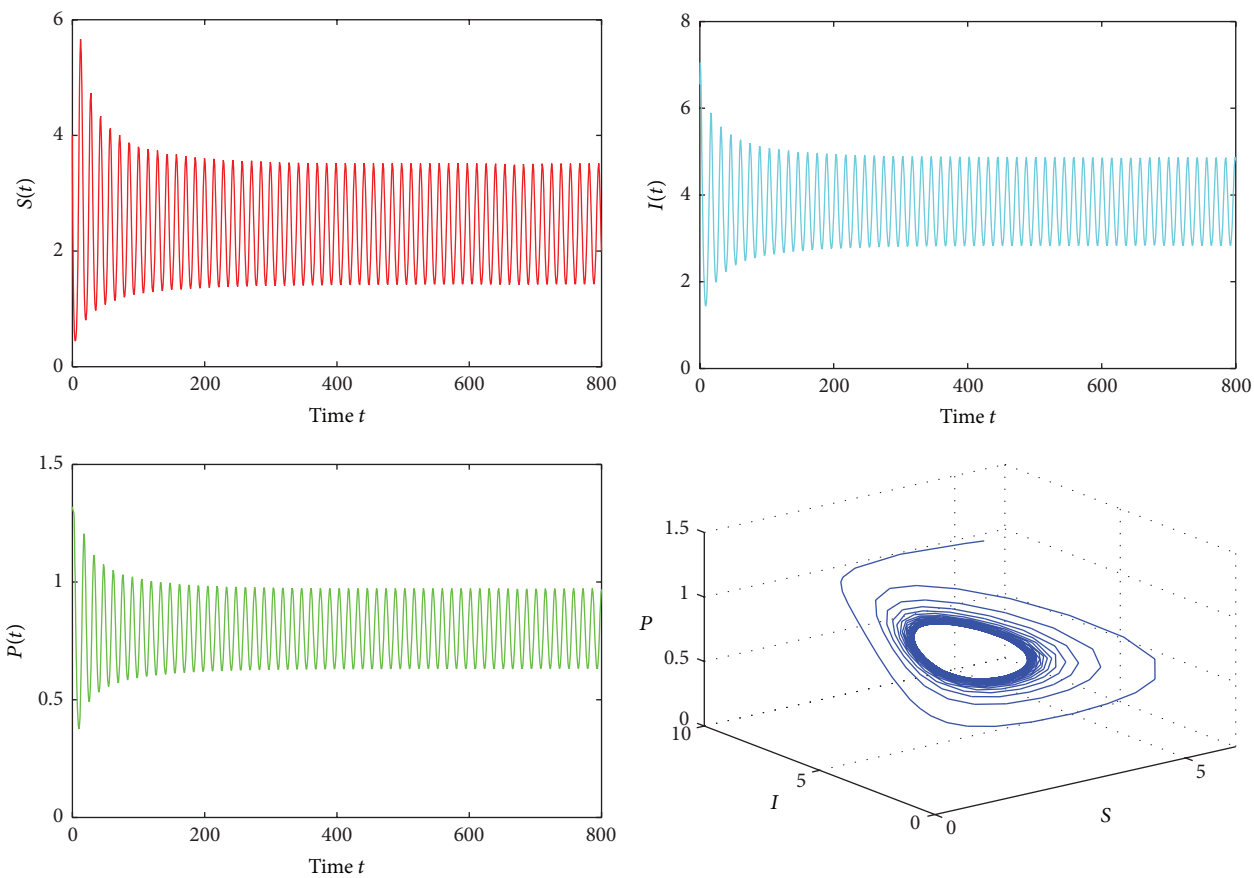

FIGURE 4: E^* is unstable when $\tau_2 = 2.25 > 1.8829 = \tau_{20}$.

and a bilinear inner product

$$\langle \varphi(s), \phi(\theta) \rangle = \bar{\varphi}(0)\phi(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{\varphi}(\xi-\theta)\, d\eta(\theta)\, \phi(\xi)\, d\xi, \tag{67}$$

where $\eta(\theta) = \eta(\theta, 0)$.

Then $A(0)$ and $A^*(0)$ are adjoint operators. From the discussion above, we know that $\pm i\omega_{2*}\tau_{20}^*$ are eigenvalues of $A(0)$ and they are also eigenvalues of $A^*(0)$.

Let $q(\theta) = (1, q_2, q_3)^T e^{i\omega_{2*}\tau_{20}^*\theta}$ be the eigenvectors of $A(0)$ corresponding to the eigenvalue $+i\omega_{2*}\tau_{20}^*$ and $q^*(s) = D(1, q_2^*, q_3^*)e^{i\omega_{2*}\tau_{20}^* s}$ the eigenvectors of $A^*(0)$ corresponding to the eigenvalue $-i\omega_{2*}\tau_{20}^*$.

It is not difficult to verify that

$$q_2 = \frac{c_1 e^{-i\omega_{2*}\tau_{20}^*}}{i\omega_{2*} - a_5}, \qquad q_3 = \frac{a_3}{i\omega_{2*} - a_4 - b_2 e^{-i\omega_{2*}\tau_{20}^*}},$$
$$q_2^* = -\frac{b_1 e^{i\omega_{2*}\tau_{20}^*}}{i\omega_{2*} + a_4 + b_2 e^{i\omega_{2*}\tau_{20}^*}}, \qquad q_3^* = -\frac{a_2}{i\omega_{2*} + a_5}. \tag{68}$$

From (67), we can get

$$\bar{D} = \Big[1 + q_2\bar{q}_2^* + q_3\bar{q}_3^* + b_{11}\tau_{10}^* e^{-i\omega_1^*\tau_{10}^*} + \tau_2^* e^{-i\omega_1^*\tau_{2*}} \left(\bar{q}_2^*\left(c_{21} + c_{22}q_2\right) + \bar{q}_3^*\left(c_{32}q_2 + c_{33}q_3\right)\right)\Big]^{-1}, \tag{69}$$

such that $\langle q^*, q\rangle = 1$, $\langle q^*, \bar{q}\rangle = 0$.

In the remainder of this section, we obtain the coefficients used to determine the properties of the periodic solution by the algorithms given in [18] and using a computation process similar to that in [11]:

$$g_{20} = 2\tau_{20}^*\bar{D}\Big[a_{21} + a_{22}q^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + a_{23}q^{(3)}(0) + \bar{q}_2^*\left(b_{21} + b_{22}q^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right)\right) + \bar{q}_3^*\Big(c_{21}\left(q^{(3)}(0)\right)^2 + c_{22}q^{(1)}(-1)\,q^{(3)}(0) + c_{23}\left(q^{(1)}(-1)\right)^2\Big)\Big],$$

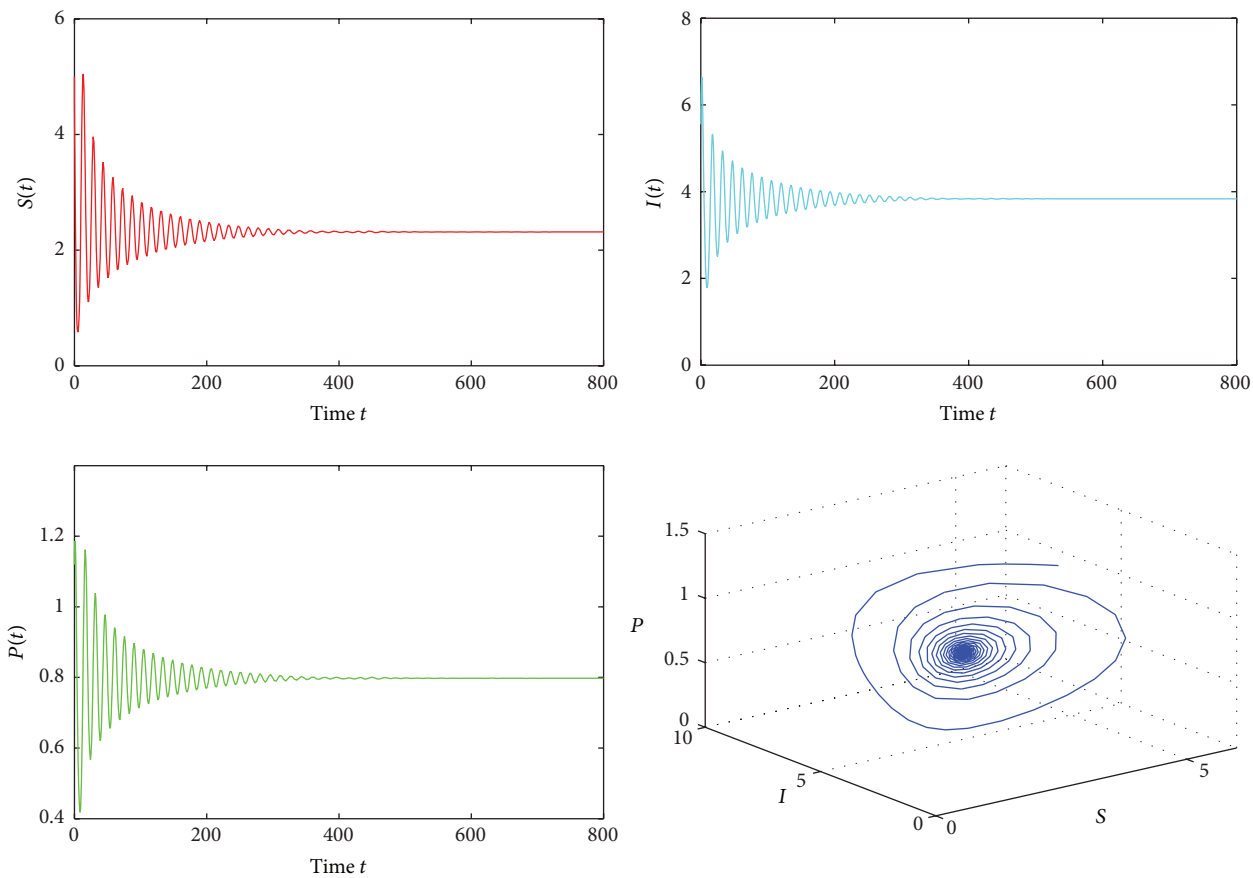

FIGURE 5: E^* is asymptotically stable when $\tau = 0.25 < 0.3652 = \tau_0$.

$$
\begin{aligned}
g_{11} = \tau_{20}^{*}\bar{D}\Bigg[& 2a_{21} + a_{22}\left(\bar{q}^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right) + q^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right)\right) \\
& + a_{23}\left(q^{(3)}(0) + \bar{q}^{(3)}(0)\right) \\
& + \bar{q}_{2}^{*}\left(2b_{21} + b_{22}\left(\bar{q}^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right) + q^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right)\right)\right) \\
& + \bar{q}_{3}^{*}\left(2c_{21}q^{(3)}(0)\,\bar{q}^{(3)}(0) + c_{22}\left(q^{(3)}(0)\,\bar{q}^{(1)}(-1) + \bar{q}^{(3)}(0)\,q^{(1)}(-1)\right) + 2c_{23}q^{(1)}(-1)\,\bar{q}^{(1)}(-1)\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
g_{02} = 2\tau_{20}^{*}\bar{D}\Bigg[& a_{21} + a_{22}\bar{q}^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right) + a_{23}\bar{q}^{(3)}(0) \\
& + \bar{q}_{2}^{*}\left(b_{21} + b_{22}\bar{q}^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right)\right) \\
& + \bar{q}_{3}^{*}\left(c_{21}\left(\bar{q}^{(3)}(0)\right)^{2} + c_{22}\bar{q}^{(1)}(-1)\,\bar{q}^{(3)}(0) + c_{23}\left(\bar{q}^{(1)}(-1)\right)^{2}\right)\Bigg],
\end{aligned}
$$

$$
\begin{aligned}
g_{21} = 2\tau_{20}^{*}\bar{D}\Bigg[& a_{21}\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) \\
& + a_{22}\left(W_{11}^{(1)}(0)\,q^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right) + \frac{1}{2}W_{20}^{(1)}(0)\,\bar{q}^{(2)} \times \left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right) + W_{11}^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right) + \frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right)\right) \\
& + a_{23}\left(W_{11}^{(1)}(0)\,q^{(3)}(0) + \frac{1}{2}W_{20}^{(1)}(0)\,\bar{q}^{(3)}(0) + W_{11}^{(3)}(0) + \frac{1}{2}W_{20}^{(3)}(0)\right) \\
& + 3a_{31} + a_{32}\left(\bar{q}^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right) + 2q^{(2)}\left(-\frac{\tau_{10}^{*}}{\tau_{20}^{*}}\right)\right)
\end{aligned}
$$

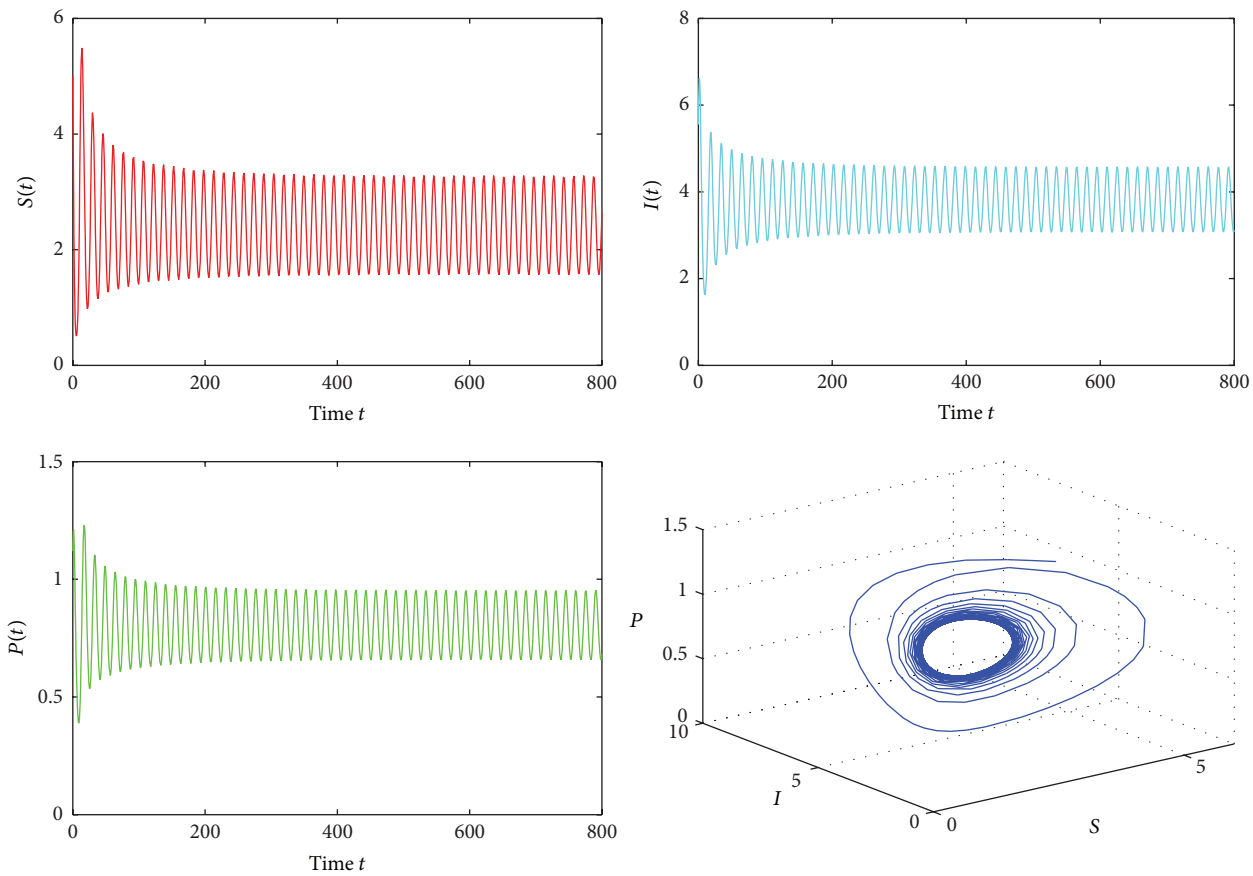

FIGURE 6: E^* is unstable when $\tau = 0.45 > 0.3652 = \tau_0$.

$$
\begin{aligned}
&+ a_{33}\left(\bar{q}^{(3)}(0) + 2q^{(3)}(0)\right) \\
&+ \bar{q}_2^*\Bigg(b_{21}\left(2W_{11}^{(1)}(0) + W_{20}^{(1)}(0)\right) \\
&\quad + b_{22}\Bigg(W_{11}^{(1)}(0)\, q^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + \frac{1}{2}W_{20}^{(1)}(0)\,\bar{q}^{(2)} \\
&\qquad \times\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + W_{11}^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) \\
&\qquad + \frac{1}{2}W_{20}^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right)\Bigg) \\
&\quad + 3b_{31} + b_{32} \\
&\qquad \times\left(\bar{q}^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + 2q^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right)\right)\Bigg) \\
&+ \bar{q}_3^*\Bigg(c_{21}\left(2W_{11}^{(3)}(0)\, q^{(3)}(0) + W_{20}^{(3)}(0)\,\bar{q}^{(3)}(0)\right) \\
&\quad + c_{22}\Bigg(W_{11}^{(1)}(-1)\, q^{(3)}(0) + \frac{1}{2}W_{20}^{(1)}(-1) \\
&\qquad \times \bar{q}^{(3)}(0) + W_{11}^{(3)}(0)\, q^{(1)}(-1) \\
&\qquad + \frac{1}{2}W_{20}^{(3)}(0)\,\bar{q}^{(1)}(-1)\Bigg) \\
&\quad + c_{23}\left(2W_{11}^{(1)}(-1) + W_{20}^{(1)}(-1)\right) \\
&\quad + c_{31}\Bigg(\left(q^{(1)}(-1)\right)^2\bar{q}^{(3)}(0) + 2q^{(1)}(-1) \\
&\qquad \times q^{(3)}(0)\,\bar{q}^{(1)}(-1)\Bigg) \\
&\quad + 3c_{32}\left(q^{(1)}(-1)\right)^2\bar{q}^{(1)}(-1)\Bigg)\Bigg],
\end{aligned}
\tag{70}
$$

with

$$
\begin{aligned}
W_{20}(\theta) &= \frac{ig_{20}q(0)}{\omega_{2*}\tau_{20}^*}e^{i\omega_{2*}\tau_{20}^*\theta} + \frac{i\bar{g}_{02}\bar{q}(0)}{3\omega_{2*}\tau_{20}^*}e^{-i\omega_{2*}\tau_{20}^*\theta} \\
&\quad + E_1 e^{2i\omega_{2*}\tau_{20}^*\theta}, \\
W_{11}(\theta) &= -\frac{ig_{11}q(0)}{\omega_{2*}\tau_{20}^*}e^{i\omega_{2*}\tau_{20}^*\theta} + \frac{i\bar{g}_{11}\bar{q}(0)}{\omega_{2*}\tau_{20}^*}e^{-i\omega_{2*}\tau_{20}^*\theta} + E_2,
\end{aligned}
\tag{71}
$$

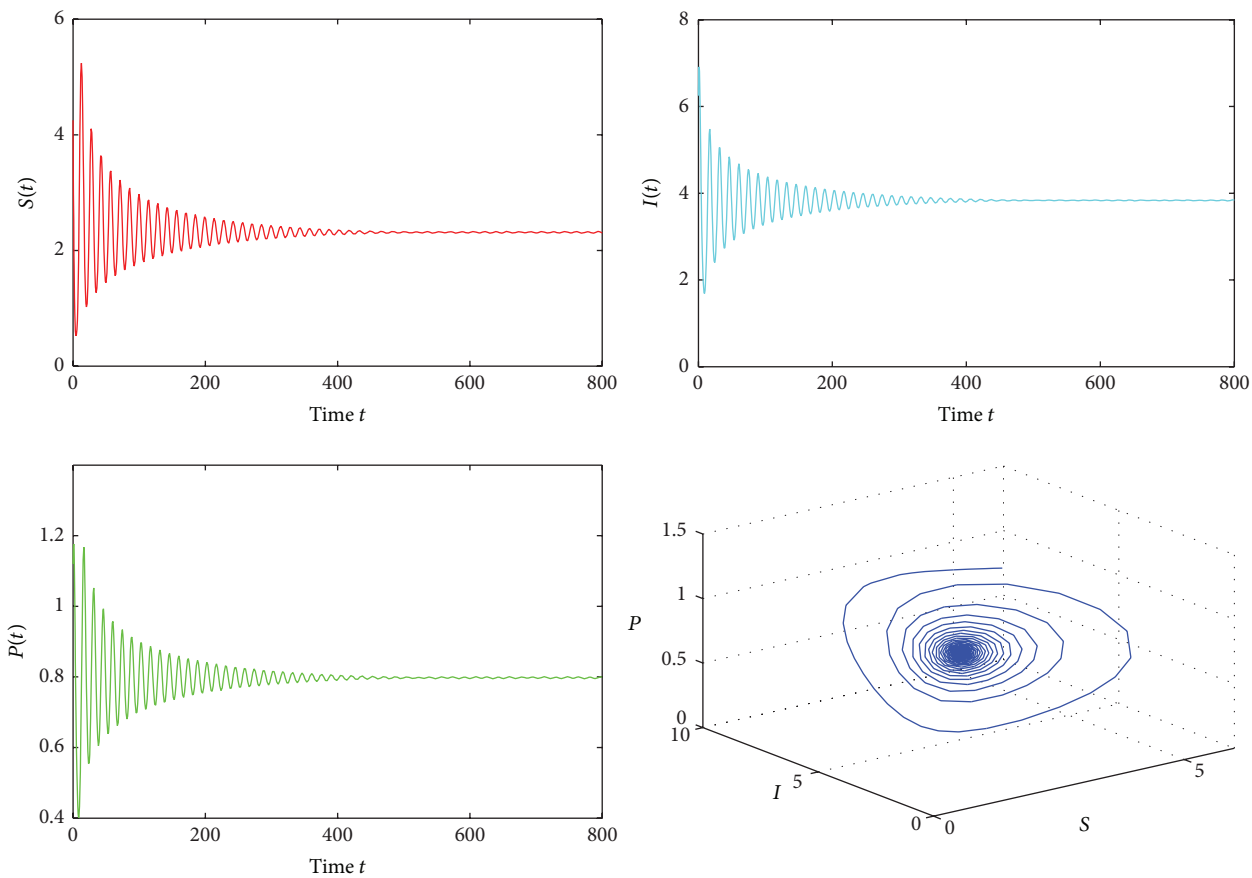

FIGURE 7: E^* is asymptotically stable when $\tau_2 = 0.75 < 1.0782 = \tau_{20}^*$ and $\tau_1 = 0.15 \in (0, \tau_{10})$.

where E_1 and E_2 can be computed as the following equations, respectively:

$$\begin{pmatrix} 2i\omega_{2*} - a_1 & -b_1 e^{-2i\omega_{2*}\tau_{20}^*} & -a_2 \\ -a_3 & 2i\omega_{2*} - a_4 - b_2 e^{-2i\omega_{2*}\tau_{20}^*} & 0 \\ -c_1 e^{-2i\omega_{2*}\tau_{20}^*} & 0 & 2i\omega_{2*} - a_5 \end{pmatrix} E_1 = 2\begin{pmatrix} E_1^{(1)} \\ E_1^{(2)} \\ E_1^{(3)} \end{pmatrix},$$

$$\begin{pmatrix} a_1 & b_1 & a_2 \\ a_3 & a_4 + b_2 & 0 \\ c_1 & 0 & a_5 \end{pmatrix} E_2 = -\begin{pmatrix} E_2^{(1)} \\ E_2^{(2)} \\ E_2^{(3)} \end{pmatrix}, \tag{72}$$

with

$$E_1^{(1)} = a_{21} + a_{22} q^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + a_{23} q^{(3)}(0),$$

$$E_1^{(2)} = b_{21} + b_{22} q^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right),$$

$$E_1^{(3)} = c_{21}\left(q^{(3)}(0)\right)^2 + c_{22} q^{(1)}(-1)\, q^{(3)}(0) + c_{23}\left(q^{(1)}(-1)\right)^2,$$

$$E_2^{(1)} = 2a_{21} + a_{22}\left(\bar{q}^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + q^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right)\right) + a_{23}\left(q^{(3)}(0) + \bar{q}^{(3)}(0)\right),$$

$$E_2^{(2)} = 2b_{21} + b_{22}\left(\bar{q}^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right) + q^{(2)}\left(-\frac{\tau_{10}^*}{\tau_{20}^*}\right)\right),$$

$$E_2^{(3)} = 2c_{21} q^{(3)}(0)\, \bar{q}^{(3)}(0) + c_{22}\left(q^{(3)}(0)\, \bar{q}^{(1)}(-1) + \bar{q}^{(3)}(0)\, q^{(1)}(-1)\right) + 2c_{23} q^{(1)}(-1)\, \bar{q}^{(1)}(-1). \tag{73}$$

Therefore, we can calculate the following values:

$$C_1(0) = \frac{i}{2\omega_{2*}\tau_{20}^*}\left(g_{11}g_{20} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}\right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{\operatorname{Re}\{C_1(0)\}}{\operatorname{Re}\{\lambda'(\tau_{20}^*)\}},$$

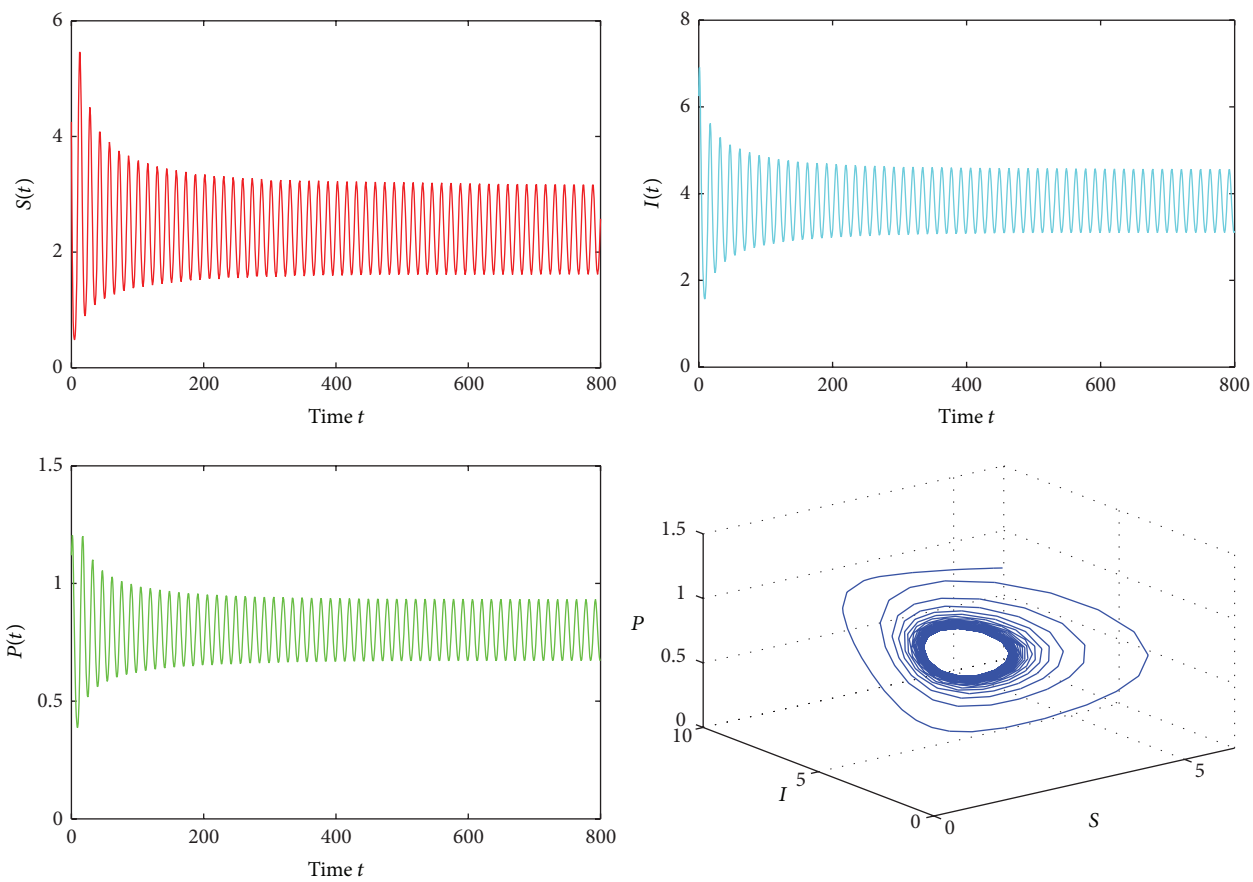

FIGURE 8: E^* is unstable when $\tau_2 = 1.35 > 1.0782 = \tau_{20}^*$ and $\tau_1 = 0.15 \in (0, \tau_{10})$.

$$\beta_2 = 2\,\mathrm{Re}\left\{C_1(0)\right\},$$
$$T_2 = -\frac{\mathrm{Im}\left\{C_1(0)\right\} + \mu_2\,\mathrm{Im}\left\{\lambda'\left(\tau_{20}^*\right)\right\}}{\omega_{2*}\tau_{20}^*}. \tag{74}$$

Based on the discussion above, we can obtain the following results.

Theorem 6. *For system* (2),

(i) *μ_2 determines the direction of the Hopf bifurcation. If $\mu_2 > 0$ ($\mu_2 < 0$), then the Hopf bifurcation is supercritical (subcritical).*

(ii) *β_2 determines the stability of the bifurcating periodic solutions. If $\beta_2 < 0$ ($\beta_2 > 0$), then the bifurcating periodic solutions are stable (unstable).*

(iii) *T_2 determines the period of the bifurcating periodic solutions. If $T_2 > 0$ ($T_2 < 0$), then the period of the bifurcating periodic solutions increases (decreases).*

4. Numerical Example

In this section, we give a numerical example to support the theoretical results in Sections 2 and 3. We use the same coefficients which are used by Jana and Kar in [6]. They are as follows: $a = 5.1$, $b = 3.995$, $m = 0.8$, $r = 1.1$, $K = 8.8$, $\alpha = 1.25$, $\beta = 1.3$, $\gamma = 0.39$, $\delta = 0.29$, and $\varepsilon = 0.15$. Thus, we get the following particular case of system (2):

$$\frac{dS(t)}{dt} = 1.1S\left(1 - \frac{S}{8.8}\right) - \frac{1.25SI(t-\tau_1)}{5.1+S} - \frac{1.3SP}{3.995+S},$$
$$\frac{dI(t)}{dt} = \frac{1.25SI(t-\tau_1)}{5.1+S} - 0.39I,$$
$$\frac{dP(t)}{dt} = P\left[\frac{1.04S(t-\tau_2)}{3.995+S(t-\tau_2)} - 0.29P - 0.15\right], \tag{75}$$

which has a positive equilibrium $E^*(2.3128, 3.8339, 0.7977)$.

For $\tau_1 > 0, \tau_2 = 0$. We get $\omega_{10} = 0.4149 < 0$, $\tau_{10} = 0.5148$. Further, we have $f_1'(v_{1*}) = 0.1022 > 0$. Thus, conditions (H_{21}) and (H_{22}) hold. From Theorem 2, the positive equilibrium E^* is asymptotically stable when $\tau_1 < \tau_{10}$ as illustrated by Figure 1. When τ_1 passes through the critical value τ_{10}, the positive equilibrium E^* loses its stability and a Hopf bifurcation occurs and a family of periodic solutions bifurcate from the positive equilibrium E^*, which can be shown as in Figure 2. Similarly, we have $\omega_{20} = 0.6010$, $\tau_{20} = 1.8829$ for $\tau_1 = 0$, $\tau_2 > 0$. The corresponding waveforms and the phase plots are shown in Figures 3 and 4.

For $\tau_1 = \tau_2 = \tau > 0$, we can obtain $\omega_0 = 2.0030$ and then we get $\tau_0 = 0.3652$. From Theorem 4, we know that when τ increases from zero to the critical value τ_0 the positive equilibrium E^* is asymptotically stable; then it will lose its stability and a Hopf bifurcation occurs once $\tau > \tau_0$. These properties can be shown as in Figures 5 and 6.

For $\tau_2 > 0$ and $\tau_1 = 0.15 \in (0, \tau_{10})$, we can obtain $\omega_{2*} = 2.1105$, $\tau_{20}^* = 1.0782$. By Theorem 5, the positive equilibrium E^* is asymptotically stable when $\tau_2 \in [0, \tau_{20}^*)$ and E^* is unstable when $\tau_2 > \tau_{20}^*$ and a Hopf bifurcation occurs, which can be illustrated by Figures 7 and 8. In addition, by complex computations, we obtain $C_1(0) = -2.2071 + 1.6159i$, and further we have $\mu_2 = 6.8522 > 0$, $\beta_2 = -4.4142 < 0$, $T_2 = -3.7367 < 0$. By Theorem 6, we know that the Hopf bifurcation with respect to τ_2 with $\tau_1 = 0.15 \in (0, \tau_{10})$ is supercritical; the bifurcating periodic solutions are stable and decrease. From the viewpoint of ecology, if the periodic solutions bifurcating from the Hopf bifurcation are stable, the species in a prey-predator system may coexist in an oscillatory mode. Therefore, we can conclude that the three species in system (75) can coexist in an oscillatory mode, since the bifurcating periodic solutions are stable.

5. Conclusion

In this present paper a prey-predator system with disease in the prey and two delays is considered. Based on the system proposed in [6], we further incorporate the time delay due to the gestation of the predator. The main purpose of this paper is to investigate the effects of the two delays on the system. We have shown that the two delays play a complicated role in the system. By choosing the possible combinations of the two delays as bifurcation parameters, sufficient conditions for local stability and existence of local Hopf bifurcation are obtained. When the time delay is below the corresponding critical value, we get that the system is local stable. Otherwise, a local Hopf bifurcation occurs at the positive equilibrium. We also find that the delay due to the susceptible prey becoming the infected prey is more marked compared with the delay due to the gestation of the predator, because the critical value of τ_1 is much smaller than that of τ_2 when we only consider one of the two delays, which can be seen from the numerical simulations. Further, the properties of the bifurcated periodic solutions such as the direction and the stability are determined. And a numerical example is also given to support the theoretical results. From the numerical simulations we can see that the species in the system considered in this paper can coexist under some certain conditions.

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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