Research Article

Quasiperiodic Solutions of Completely Resonant Wave Equations with Quasiperiodic Forced Terms

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This paper is concerned with the existence of quasiperiodic solutions with two frequencies of completely resonant, quasiperiodically forced nonlinear wave equations subject to periodic spatial boundary conditions. The solutions turn out to be, at the first order, the superposition of traveling waves, traveling in the opposite or the same directions. The proofs are based on the variational Lyapunov-Schmidt reduction and the linking theorem, while the bifurcation equations are solved by variational methods.

1. Introduction

This paper is devoted to the study of the existence of small-amplitude quasiperiodic solutions of completely resonant forced nonlinear wave equations like

\[ u_{tt} - u_{xx} + f (\omega_1 t, \omega_2 t, x, u) = 0, \]  

with periodic boundary conditions

\[ u (t, x) = u (t, x + 2\pi), \quad (t, x) \in \mathbb{R}^2, \]  

while the nonlinear forced term \( f \) is

\[ f (\omega_1 t, \omega_2 t, x, u) := \sum_{k \geq 3} a_k (\omega_1 t + x, \omega_2 t + x) u^k, \]  

when the traveling waves are in the same directions, and \( f \) is

\[ f (\omega_1 t, \omega_2 t, x, u) := \sum_{k \geq 3} a_k (\omega_1 t - x, \omega_2 t + x) u^k, \]  

when the traveling waves are in the opposite directions. Moreover, the nonlinear forced terms are all analytic in a neighborhood of \( u = 0 \).

Periodic or quasiperiodic solutions in nonresonant PDEs have been obtained, for instance, in [1–12] by the Lyapunov-Schmidt reduction together with Nash-Moser theory and KAM theory, while the completely resonant autonomous PDEs have been originally studied by variational methods starting from Rabinowitz [13–19]. They obtained the existence of periodic solutions with period being a rational multiple of \( \pi \), and such solutions correspond to a zero-measure set of values of the amplitudes. The case with period being irrational of \( \pi \), which in principle could provide a large measure of values, has been mostly studied under strong Diophantine conditions; see [20–25] and the references therein. In [26, 27], using the Lindstedt series method, Gentile and Procesi obtained the existence of quasiperiodic solutions for a large measure set of frequencies for the nonlinear wave equations and nonlinear Schrödinger equations with periodic boundary conditions. In [28], Yuan obtained the existence of quasiperiodic solution for a large measure set of at least three dimensional rotation vectors by the KAM method. In [29], under the periodic boundary condition and with the periodic forced nonlinearities \( f (\omega_1 t, u) = a (\omega_1 t) u^{2d-1} + O (u^{2d}), \quad d > 2 \), Berti and Procesi got the existence of quasiperiodic solution of nonlinear wave equation in the form of \( v (t, x) = u (\omega_1 t, \omega_2 t + x) \). In [30], Procesi firstly
obtained the quasiperiodic solutions with two frequencies in the form of \( v(t, x) = u(\omega_1 t + x, \omega_2 t - x) \) for the specific nonlinearities \( f = u^3 + O(u^4) \), where the forced terms do not depend on the time and the bifurcation equations are solved by ODE methods. In [31], Baldi proved the existence of small-amplitude quasiperiodic solutions in the form of \( v(t, x) = u(\omega_1 t + x, \omega_2 t + x) \) with the general nonlinearities \( f = u^3 + O(u^4) \), which also do not depend on the time. In [32], they considered the existence of quasiperiodic solution of the forced wave equation, in which the solutions are traveling in opposite directions. However, they did not give the regularity of the solutions, and the results are the special case of our results in Section 4. Moreover, we mention the work [33] of Bambusi, where a simple proof of an infinite-dimensional extension of the Lyapunov center theorem is given.

In this paper, for the completely resonant wave equation (1) subjecting to the quasiperiodic forced terms, we will prove the existence and regularity of quasiperiodic solutions with two frequencies, \( \omega_1, \omega_2 \), in both of the following two cases.

**Case (A1).** The first case considers the wave traveling in the same directions \( u(t, x) = v(\omega_1 t + x, \omega_2 t + x) \).

**Case (A2).** The second case considers the wave traveling in opposite directions \( u(t, x) = v(\omega_1 t - x, \omega_2 t + x) \).

### 2. Main Results

We look for quasiperiodic solutions \( u(t, x) \) of (1) of the following form:

\[
\begin{align*}
\text{(in the same directions)} & \\
\quad u(t, x) &= v(\omega_1 t + x, \omega_2 t + x) = v(\varphi_1, \varphi_2), \\
\quad v(\varphi_1, \varphi_2) &= v(\varphi_1 + 2k_1 \pi, \varphi_2 + 2k_2 \pi), \quad \forall k_1, k_2 \in \mathbb{Z}, \tag{5}
\end{align*}
\]

with frequencies \( \omega = (\omega_1, \omega_2) = (1 + \epsilon, 1 + a \epsilon^2), \epsilon \in \mathbb{R}_-, \) or

\[
\begin{align*}
\text{(in opposite directions)} & \\
\quad u(t, x) &= v(\omega_1 t - x, \omega_2 t + x) = v(\varphi_1, \varphi_2), \\
\quad v(\varphi_1, \varphi_2) &= v(\varphi_1 + 2k_1 \pi, \varphi_2 + 2k_2 \pi), \quad \forall k_1, k_2 \in \mathbb{Z}, \tag{6}
\end{align*}
\]

with frequencies \( \omega = (\omega_1, \omega_2) = (1 + \epsilon, 1 + a \epsilon^2), \epsilon \in \mathbb{R}_-, \) imposing the frequencies \( \omega = (\omega_1, \omega_2) \) to be close to linear frequency \( l \). Therefore, finding the quasiperiodic solutions of (1) with frequencies, respectively, \( (\omega_1, \omega_2) \) is equivalent to finding \( 2\pi \) periodic solutions with respect to \( (\varphi_1, \varphi_2) \) for the following equations:

\[
\begin{align*}
\text{(in the same directions)} & \\
\left( \hat{\sigma}^2_{tt} - \hat{\sigma}^2_{xx} \right) v &= (\partial_t - \partial_x) \circ (\partial_t + \partial_x) v \\
&= [\omega_1 \partial_{\varphi_1} + \omega_2 \partial_{\varphi_2} - \partial_{\varphi_1} - \partial_{\varphi_2}] v
\end{align*}
\]

\[
\begin{align*}
\text{(in opposite directions)} & \\
\left( \hat{\sigma}^2_{tt} - \hat{\sigma}^2_{xx} \right) v &= (\partial_t - \partial_x) \circ (\partial_t + \partial_x) v \\
&= [\omega_1 \partial_{\varphi_1} + \omega_2 \partial_{\varphi_2} - \partial_{\varphi_1} - \partial_{\varphi_2}] v
\end{align*}
\]

\[
\begin{align*}
\hat{\sigma}^2_{tt} - \hat{\sigma}^2_{xx} = (\omega_1^2 - 1) \partial^2_{\varphi_1} + (\omega_2^2 - 1) \partial^2_{\varphi_2} \\
+ 2 (\omega_1 \omega_2 - 1) \partial_{\varphi_1} \partial_{\varphi_2} v
\end{align*}
\]

\[
= - f (\varphi_1, \varphi_2, v), \tag{7}
\]

We assume that the quasiperiodic forced term \( f : \mathbb{T}^2 \times \mathbb{R} \to \mathbb{R} \),

\[
f (\varphi_1, \varphi_2) = a_3 (\varphi_1, \varphi_2) v^3 + O (v^4), \tag{9}
\]

is analytic in \( v \) but has only finite regularity in \( \varphi_1, \varphi_2 \). More precisely,

\[
(f (\varphi_1, \varphi_2)) = \sum_{k=3}^{\infty} a_k (\varphi_1, \varphi_2) v^k, \quad \text{and the coefficients} \quad a_k (\varphi_1, \varphi_2) \in H^r (\mathbb{T}^2), \quad \text{for some} \ r > 0,
\]

\[
\sum_{k=3}^{\infty} |a_k| \|v\|^k < \infty. \quad \text{The function} \ f (\varphi_1, \varphi_2) \ \text{is not} \ \text{identically constant in} \ (\varphi_1, \varphi_2).
\]

We look for solutions \( v \) of (7)-(8) in the Banach space

\[
\mathcal{H}_{\sigma, s} = \left\{ v (\varphi) \mid \sum_{(l_1, l_2) \in \mathbb{Z}^2} \hat{v}_{(l_1, l_2)} e^{i l_1 \varphi_1} e^{i l_2 \varphi_2} = \hat{v}_{(l_1, l_2)}, \quad |\hat{v}_{(l_1, l_2)}| \|v\|_{\mathcal{H}_{\sigma, s}} \right\},
\]

\[
|v|_{\mathcal{H}_{\sigma, s}} = \sum_{(l_1, l_2) \in \mathbb{Z}^2} |\hat{v}_{(l_1, l_2)}| \left( \max \{0, |l_1| \} \right)^s < + \infty, \tag{10}
\]

where \( l = (l_1, l_2) \in \mathbb{Z}^2, \hat{v}_{(l_1, l_2)} \) denotes its complex conjugate, and \( \sigma > 0, \quad s \geq 0. \)

The space \( \mathcal{H}_{\sigma, s} \) is a Banach algebra with respect to multiplications of functions, namely,

\[
v_1, v_2 \in \mathcal{H}_{\sigma, s} \implies v_1 v_2 \in \mathcal{H}_{\sigma, s}, \quad |v_1 v_2|_{\mathcal{H}_{\sigma, s}} \leq C |v_1|_{\mathcal{H}_{\sigma, s}} |v_2|_{\mathcal{H}_{\sigma, s}}. \tag{11}
\]

We will prove the following theorems.
Theorem 1. Assume that the nonlinearity $f$ satisfies (H) and $a_3(\varphi_1, \varphi_2) \neq 0, \forall (\varphi_1, \varphi_2) \in \mathbb{T}^2$. Let $\mathcal{B}_y$ be the uncountable zero-measure Cantor set

$$\mathcal{B}_y := \left\{ (a, e) \in \mathbb{R} \times \mathbb{R}, 1 + e \neq 0, 1 + ae^2 \neq 0, 2 + ae^2 \neq 0, \frac{2 + e}{2 + ae^2} \in \mathcal{C}_{\gamma_1} \cap \{1 - e_0, 1 + e_0\} \times \mathcal{C}_{\gamma_2} \cap \{-e_0, e_0\}, \frac{1 + e}{1 + ae^2} \in \mathcal{Q}, \text{ for } e_0 \in \left(0, \frac{1}{2}\right)\right\},$$

(12)

where $\mathcal{C}_{\gamma_i}, i = 1, 2,$ are sets of badly approximate numbers defined as

$$\mathcal{C}_{\gamma_i} := \left\{ \frac{2 + e}{2 + ae^2} : \left| l_2 + \frac{2 + e}{2 + ae^2} l_1 \right| > \frac{\gamma}{|l_1|} \right\},$$

(13)

$$\mathcal{C}_{\gamma_i} := \left\{ ae : |l_1 + ael_2| > \frac{\gamma}{|l_2|} \right\},$$

for $\forall l_1, l_2 \in \mathbb{Z} \setminus \{0\},$ and $0 < \gamma < 1/4$. There exist constants $\bar{\sigma}, \bar{\tau} > 2$, and $\bar{\sigma}(\mathbb{R}) > 0$, such that, for $(a, e) \in \mathcal{B}_y$, $|\gamma|/\bar{\tau}(\mathbb{R})$, there exists a solution $v(\epsilon, \varphi_1, \varphi_2) \in \mathcal{H}_{\pi y}$ of (7), having the form

$$v(\epsilon, \varphi_1, \varphi_2) = |\epsilon| \left(\tilde{q}_0 + \tilde{q}_1(\varphi_2) + \tilde{q}_2(\varphi_1 + \varphi_2) + \tilde{p}(\epsilon, \varphi_1, \varphi_2)\right),$$

(14)

with

$$|\tilde{p}(\epsilon, \varphi_1, \varphi_2)|_{\pi y} \leq C\left(\frac{|\epsilon|}{\gamma}\right),$$

(15)

where $C$ is a constant. As a consequence, (1) possesses the quasiperiodic solutions, traveling in the same directions, $u(\epsilon, t, x) = v(\epsilon, \omega_1 t + x, \omega_2 t + x)$, with two frequencies $(\omega_1, \omega_2) = (1 + e, 1 + ae^2)$.

Theorem 2. Assume that $f$ satisfies assumption (H) and $a_3(\varphi_1, \varphi_2) \neq 0, \forall (\varphi_1, \varphi_2) \in \mathbb{T}^2$. Let $\mathcal{B}_y \subset (-e_0, e_0) \times (-e_0, e_0)$ be the uncountable zero-measure Cantor set

$$\mathcal{B}_y := \left\{ (a, e) \in \mathbb{R} \times \mathbb{R}, \left(\frac{ae}{2 + e}, \frac{e}{2 + ae}\right) \in \mathcal{C}_y, \frac{1 + e}{1 + ae} \notin \mathcal{Q}, 1 + e \neq 0, 1 + ae \neq 0, 2 + ae \neq 0\right\},$$

(16)

where $\mathcal{C}_y$ is a set of badly approximate numbers defined as

$$\mathcal{C}_y := \left\{ \left(\frac{ae}{2 + e}, \frac{e}{2 + ae}\right) : (e_1, e_2) \in (-e_0, e_0) \times (-e_0, e_0), \right\},$$

$$\left| l_1 + e_1 l_2 \right| > \frac{\gamma}{|l_2|}, \left| l_2 + e_2 l_1 \right| > \frac{\gamma}{|l_1|} \right\},$$

(17)

for $\forall l_1, l_2 \in \mathbb{Z} \setminus \{0\},$ and $0 < \gamma < 1/4, e_0 \in (0, 1/2)$. There exist positive numbers $\bar{\sigma}, \bar{\tau}, \bar{C}, \bar{\mathcal{S}} > 2$, such that, $\forall (a, e) \in \mathcal{B}_y$, (8) admits solutions in the form of

$$v_{a,e}(t, x) = v(\epsilon, \varphi_1, \varphi_2)$$

$$= \sqrt{|\epsilon|} \left(\tilde{q}_0 + \tilde{q}_1(\varphi_2) + \tilde{q}_2(\varphi_1) + \tilde{p}(\epsilon, \varphi_1, \varphi_2)\right),$$

(18)

satisfying

$$|\tilde{p}(\epsilon, \varphi_1, \varphi_2)|_{\pi y} \leq C\left(\frac{|\epsilon|}{\gamma}\right).$$

(19)

As a consequence, (1) possesses the quasiperiodic solutions, $u_{a,e}(t, x) = v_{a,e}(1 + e)t + x, (1 + ae)t - x)$, traveling in opposite directions.

Remark 3. The quasiperiodic solutions of traveling waves we obtained are different from the ones got by KAM methods since the quasiperiodic solutions we get depending on $x$ and $t$ are coupled and in the form of the traveling waves.

Remark 4. We can get the similar result with more general nonlinearity, such as $f(\omega_1 t, \omega_2 t, x, u) = a_3(\varphi_1, \varphi_2) u^d + O(u^{d+1})$, for any $d \in \mathbb{N}, d \geq 3$.

This paper is organized as follows: we first prove the existence of quasiperiodic solutions, at the first order, to the superposition of traveling waves, traveling in the same directions. In Section 4, we prove the existence of quasiperiodic solutions traveling in opposite directions.

3. Waves Traveling in the Same Directions

Substituting $\omega_1 = 1 + e, \omega_2 = 1 + ae^2$ into (7), we can obtain the equations

$$\mathcal{L} v + f(\varphi_1, \varphi_2, v) = 0,$$

(20)

where, see (7), we have

$$\mathcal{L} = \left(\partial_{\varphi_1}\varphi_1, \partial_{\varphi_2}\varphi_2, 2 (\omega_1^2 - 1) \partial_{\varphi_1}^2 \varphi_1, 2 (\omega_2^2 - 1) \partial_{\varphi_2}^2 \varphi_2\right),$$

$$+ e \left(2 \partial_{\varphi_1}^2 + 2 \partial_{\varphi_1} \partial_{\varphi_2} \right) \partial_{\varphi_2}^2 + 2a (1 + e) \partial_{\varphi_1} \partial_{\varphi_2},$$

(21)

and $f(\varphi_1, \varphi_2, v) = a_3(\varphi_1, \varphi_2) v^3 + O(v^4)$. To prove Theorem 1, instead of looking for solutions of (7) in a shrinking neighborhood of zero, it is convenient to perform the rescaling $v(\varphi_1, \varphi_2) \rightarrow v(\varphi_1, \varphi_2)$, enhancing the relation between the amplitude and the frequencies. Without confusion, we define

$$\mathcal{L}_{a,e} = \left(2 \partial_{\varphi_1}^2 + 2 \partial_{\varphi_1} \partial_{\varphi_2} \right) \partial_{\varphi_2}^2 + 2a (1 + e) \partial_{\varphi_1} \partial_{\varphi_2},$$

(22)

$$= \mathcal{L}_0 + e \mathcal{L}_1,$$
so the problem becomes

$$L_{\alpha,\varepsilon} + \varepsilon f(\varphi_1, \varphi_2; \varepsilon, \epsilon) = 0.$$  \tag{23}$$

To find the solutions of (23), we will apply the Lyapunov-Schmidt reduction method which leads to solving separately a "range equation" and a "bifurcation equation." In order to solve the range equation (avoiding small divisor problems), we restrict \( \epsilon \) to the countable zero-measure set \( \mathcal{S}_\gamma \) for Theorem I, and we apply the Contraction Mapping Theorem; similar nonresonance conditions have been employed, for example, in [21–23, 25, 29, 30].

Equation (23) is the Euler–Lagrange equation of the action functional \( \Psi_\epsilon \in C^1(\mathcal{H}_{\alpha,\varepsilon}, \mathbb{R}) \) defined by

$$\Psi_\epsilon (\nu) := \int_{\mathbb{T}^2} \left( \frac{1}{2} \left( \partial_{\varphi_1} \nu \right)^2 + \frac{2a + \alpha^2 \varepsilon^2}{2} \left( \partial_{\varphi_2} \nu \right)^2 \right)$$

$$+ \alpha \left( 1 + \epsilon \right) \left( \partial_{\varphi_1} \nu \right) \left( \partial_{\varphi_2} \nu \right)$$

$$- \varepsilon F (\varphi_1, \varphi_2, \nu, \epsilon) = \Psi_0 (\nu) + \varepsilon \Psi_1 (\nu, \epsilon),$$

where

$$F (\varphi_1, \varphi_2, \nu, \epsilon) := \int_{\mathbb{T}^2} f (\varphi_1, \varphi_2, \xi, \epsilon) \, d\xi,$$

$$\Psi_0 (\nu) = \int_{\mathbb{T}^2} \left( \partial_{\varphi_1} \nu \right)^2 + \left( \partial_{\varphi_2} \nu \right)^2,$$

$$\Psi_1 (\nu, \epsilon) := \int_{\mathbb{T}^2} \left( \frac{1}{2} \left( \partial_{\varphi_1} \nu \right)^2 + \frac{2a + \alpha^2 \varepsilon^2}{2} \left( \partial_{\varphi_2} \nu \right)^2 \right) + \alpha \left( 1 + \epsilon \right) \left( \partial_{\varphi_1} \nu \right) \left( \partial_{\varphi_2} \nu \right)$$

$$- \varepsilon F (\varphi_1, \varphi_2, \nu, \epsilon).$$

To find critical points of \( \Psi_\epsilon (\nu) \), we perform a variational Lyapunov-Schmidt reduction inspired by Berti and Bolle [22, 23, 29]; see also Ambrosetti and Badiale [34].

3.1. The Variational Lyapunov-Schmidt Reduction. The operator \( L_{\alpha,\varepsilon} \) is diagonal defined on the Banach space \( \mathcal{H}_{\alpha,\varepsilon} \) under the Fourier basis \( \psi_1 = e^{i\varphi_1} e^{j\varphi_2} \) with eigenvalues

$$-D_1 = \left( 2l_1^2 + 2l_1 l_2 \right) + \epsilon \left( l_1^2 + \left( 2a + \alpha^2 \varepsilon^2 \right) l_2^2 + 2a \left( 1 + \epsilon \right) l_1 l_2 \right)$$

$$= \left( l_1 + \alpha l_2 \right) \left( 2 + \epsilon \right) l_1 + \left( 2 + \alpha^2 \varepsilon^2 \right) l_2.$$

So, we have

$$L_{\alpha,\varepsilon} \left[ \nu \right] = \sum_{(l_1, l_2) \in \mathbb{Z}^2} D_1 \tilde{p}_{l_1,l_2} e^{i\varphi_1} e^{j\varphi_2}, \quad \forall \nu \in \mathcal{H}_{\alpha,\varepsilon}.$$

The critical points of the unperturbed functional \( \Psi_0 : \mathcal{H}_{\alpha,\varepsilon} \to \mathbb{R} \) form an infinite-dimensional linear space \( Q \), and they are the solutions of the equation

$$L_0 q = \left( 2\partial_{\varphi_1}^2 + 2\partial_{\varphi_2} \partial_{\varphi_2} \right) q = 0.$$  \tag{28}$$

The space \( Q \) can be written as

$$Q = \left\{ q \in \sum_{(l_1, l_2) \in \mathbb{Z}^2} \tilde{p}_{l_1,l_2} e^{i\varphi_1} e^{j\varphi_2} \in \mathcal{H}_{\alpha,\varepsilon} \mid \tilde{p}_{l_1,l_2} = 0, \right\}.$$  \tag{29}$$

In view of the variational argument that we will use to solve the bifurcation equation, we split \( Q \) as \( Q = Q_+ + Q_0 + Q_- \), where

$$Q_+ := \left\{ q \in Q : \tilde{p}_{l_1,l_2} = 0, \text{ for } (l_1, l_2) \notin \Lambda_+ \right\} = \left\{ q_+ := q_+ (\varphi) \in \mathcal{H}_{\alpha,\varepsilon} \right\},$$

$$Q_0 := \left\{ q \in Q : \tilde{p}_{0,0} = 0 \right\},$$

$$Q_- := \left\{ q \in Q : \tilde{p}_{l_1,l_2} = 0, \text{ for } (l_1, l_2) \notin \Lambda_- \right\} = \left\{ q_- := q_- (\varphi) \in \mathcal{H}_{\alpha,\varepsilon} \right\},$$

with

$$\Lambda_+ := \left\{ (l_1, l_2) \in \mathbb{Z}^2 : l_1 = 0, l_2 \neq 0 \right\},$$

$$\Lambda_- := \left\{ (l_1, l_2) \in \mathbb{Z}^2 : l_1 + l_2 = 0, (l_1, l_2) \neq (0,0) \right\},$$

$$\Lambda_0 := \left\{ (l_1, l_2) \in \mathbb{Z}^2 : (l_1, l_2) = (0,0) \right\}.$$  \tag{30}$$

We will also use in \( Q \) the norm

$$|q|_{\mathcal{H}}^2 := |q_{l_1,l_2}^0|^2 + 2a |q_{l_1,l_2}^0|^2 + 2a |q_{l_1,l_2}^0|^2 \sum_{(l_1, l_2) \in \Lambda_+ \cup \Lambda_-} |q_{l_1,l_2}|^2 + |q_{l_1,l_2}|^2 + 1.$$  \tag{32}$$

So, we can decompose the space \( \mathcal{H}_{\alpha,\varepsilon} = Q + P \), where

$$P := \left\{ p = \sum_{(l_1, l_2) \in \mathbb{Z}^2} \tilde{p}_{l_1,l_2} e^{i\varphi_1} e^{j\varphi_2} \in \mathcal{H}_{\alpha,\varepsilon} \mid \tilde{p}_{l_1,l_2} = 0, \right\}.$$  \tag{33}$$

for \( l_1 (2l_1 + 2l_2) = 0 \).$$

Projecting (23) onto the closed subspaces \( Q \) and \( P \), setting \( \nu = q + p \in \mathcal{H}_{\alpha,\varepsilon} \) with \( q \in Q \) and \( p \in P \), we obtain

\[ Q \mathcal{L}_{\alpha,\varepsilon} \left[ q \right] + \Pi_Q f (\varphi_1, \varphi_2, q + p, \epsilon) = 0, \]

\[ P \mathcal{L}_{\alpha,\varepsilon} \left[ p \right] + \epsilon \Pi_P f (\varphi_1, \varphi_2, q + p, \epsilon) = 0, \]  \tag{34}$$
where \( \Pi_Q : \mathcal{H}_{\sigma,s} \to Q \), \( \Pi_P : \mathcal{H}_{\sigma,s} \to P \) are the projectors, respectively, onto \( Q \) and \( P \).

In order to prove analyticity of the solutions and to highlight the compactness of the problem, we perform a finite-dimensional Lyapunov-Schmidt reduction, introducing the decomposition \( Q = Q_1 + Q_2 \), where

\[
Q_1 = \left\{ q = \sum_{|l_1| + |l_2| \leq N} \bar{q}_{l_1} \bar{q}_{l_2} e^{i\varphi_{l_1}} e^{i\varphi_{l_2}} | l_1, l_2 \in \mathbb{Z} \right\},
\]

\[
Q_2 = \left\{ q = \sum_{|l_1| + |l_2| \leq N} \tilde{q}_{l_1} \tilde{q}_{l_2} e^{i\varphi_{l_1}} e^{i\varphi_{l_2}} | l_1, l_2 \in \mathbb{Z} \right\}.
\] (35)

Setting \( q = q_1 + q_2 \) with \( q_1 \in Q_1 \) and \( q_2 \in Q_2 \), we finally get

\[
(1) \quad \mathcal{L}_1 [q_1] + \Pi_Q, \left[ f (\varphi_{l_1}, \varphi_{l_2}, q_1 + q_2 + p, e) \right] = 0 \\
\iff d\Psi_1 (v) [h] = 0, \quad \forall h \in Q_1;
\]

\[
(2) \quad \mathcal{L}_1 [q_2] + \Pi_Q, \left[ f (\varphi_{l_1}, \varphi_{l_2}, q_1 + q_2 + p, e) \right] = 0 \\
\iff d\Psi_1 (v) [h] = 0, \quad \forall h \in Q_2;
\]

\[
(P) \quad \mathcal{L}_{a,e} [p] + e\Pi_P, \left[ f (\varphi_{l_1}, \varphi_{l_2}, q_1 + q_2 + p, e) \right] = 0 \\
\iff d\Psi_1 (v) [h] = 0, \quad \forall h \in P,
\] (36) (37) (38)

where \( \Pi_Q : \mathcal{H}_{\sigma,s} \to Q_i \) are the projectors onto \( Q_i \) \( (i = 1, 2) \), and \( \Pi_P : \mathcal{H}_{\sigma,s} \to P \) is the projector onto \( P \). We will solve first the \((Q_2)-(P)\)-equations for all \( |q_2|_{L^1} \leq 2R \), provided \( e \) belongs to a suitable Cantor-like set, \( |e| \leq \varepsilon_0 (R) \) is sufficiently small, and \( N \geq N_0 (R) \) is large enough (see Lemma 7). Next, we will solve the \( Q_1 \)-equation by means of a variational linking argument; see Section 3.4.

3.2. The \((Q_2)-(P)\)-Equations. We first prove that \( \mathcal{L}_{a,e} \) restricted to \( P \) has a bounded inverse when \((a,e)\) belongs to the uncountable zero-measure set

\[
\mathcal{B}_y := \{ (a, e) \in \mathbb{R}^- \times \mathbb{R}, \ \ 1 + e \neq 0, \ \ 1 + ae^2 \neq 0, \ \ 2 + ae^2 \neq 0, \ \ \frac{2 + e}{2 + ae^2}, ae \} \in \mathcal{C}_{\gamma_1} \cap (1 - \varepsilon_0, 1 + \varepsilon_0) \times \mathcal{C}_{\gamma_2} \cap (-\varepsilon_0, \varepsilon_0), \ \ \frac{1 + e}{1 + ae^2} \notin Q, \ \ \varepsilon_0 \in \left(0, \frac{1}{2}\right) \},
\]

(39)

where \( \mathcal{C}_{\gamma_i}, i = 1, 2, \) is a set of badly approximate numbers defined by

\[
\mathcal{C}_{\gamma_1} := \left\{ \frac{2 + e}{2 + ae^2} : \left| l_2 + \frac{2 + e}{2 + ae^2} l_1 \right| > \frac{\gamma}{|l_1|} \right\},
\]

\[
\mathcal{C}_{\gamma_2} := \left\{ ae : |l_1 + ael_2| > \frac{\gamma}{|l_1|} \right\},
\] (40)

for \( \forall l_1, l_2 \in \mathbb{Z} \setminus \{0\} \), and \( 0 < \gamma < 1/4 \). \( \mathcal{C}_{\gamma_i}, i = 1, 2, \) accumulate at 1 and zero, respectively, from both the right and the left; see [21, 31, 35].

The operator \( \mathcal{L}_e \) is diagonal in the Fourier basis \( \{ e^{i\varphi_{l_1}}, e^{i\varphi_{l_2}}, (l_1, l_2) \in \mathbb{Z}^2 \} \) with eigenvalues

\[
D_{l_1,l_2} = -(l_1 + ael_2) \left( (2 + e) l_1 + (2 + ae^2) l_2 \right)
\]

\[
= -(2 + ae^2) \left( l_1 + ael_2 \right) \left( l_2 + \frac{2 + e}{2 + ae^2} l_1 \right).
\] (41)

Lemma 5. For \((a,e) \in \mathcal{B}_y\), the eigenvalues \( D_{l_1,l_2} \) of \( \mathcal{L}_{a,e} \) restricted to \( P \) satisfy

\[
|D_{l_1,l_2}| = \left| (l_1 + ael_2) \left( (2 + e) l_1 + (2 + ae^2) l_2 \right) \right| > \gamma,
\]

\[
\forall l_1, l_2 \neq 0.
\] (42)

As a consequence, the operator \( \mathcal{L}_{a,e}^{-1} : P \to P \) has a bounded inverse \( \mathcal{L}_{a,e}^{-1} \) and satisfies

\[
\left| \mathcal{L}_{a,e}^{-1} [h] \right|_{\sigma,s} \leq \frac{|h|_{\sigma,s}}{\gamma}, \quad \forall h \in P.
\] (43)

Proof. Denote by \([x]\) the nearest integer close to \( x \) and by \( \{x\} = x - [x] \) its fractional part. If both \( l_1 \neq -[ael_2] \) and \( l_2 \neq -((2 + e)/(2 + ae^2))l_1 \), we have

\[
|D_{l_1,l_2}| = \left| (l_1 + ael_2) \left( (2 + e) l_1 + (2 + ae^2) l_2 \right) \right| > 1.
\] (44)

If \( l_1 = -[ael_2] \), then \( |l_1| \leq (1/2)|l_2| \), so that \( |l_2 + ((2 + e)/(2 + ae^2))l_2| > (1/2)|l_2| \). This implies that

\[
|D_{l_1,l_2}|
\]

\[
= \left| l_1 + ael_2 \left( 2 + ae^2 \right) l_1 \right| \left| l_2 + ael_2 \left( 2 + ae^2 \right) l_2 \right|
\]

\[
\geq \frac{\gamma}{|l_1|} \cdot \left( 2 + e \right) \times \left( e \left( \frac{ae}{2 + ae^2} l_1 \right) \right) \left( 2 + ae^2 \right) \left( e \left( \frac{2 + e}{2 + ae^2} l_1 \right) \right) \geq \gamma.
\] (45)

In the same way, if \( l_2 = -((2 + e)/(2 + ae^2))l_1 \), we have

\[
|D_{l_1,l_2}| \geq \frac{\gamma}{|l_1|} \left( 2 + e \right) \times \left( e \left( \frac{ae}{2 + ae^2} l_1 \right) \right) \left( 2 + ae^2 \right) \left( e \left( \frac{2 + e}{2 + ae^2} l_1 \right) \right) \geq \gamma.
\] (46)

So, the operator \( \mathcal{L}_{a,e} \) restricted to \( P \) has a bounded inverse and satisfies

\[
\left| \mathcal{L}_{a,e}^{-1} [h] \right|_{\sigma,s} = \sum_{\{l_1,l_2\} \in \mathbb{Z}^2} \frac{|h|_{l_1} e^{i\varphi_{l_1}} (\max \{0,|l_1|\})^{\gamma}}{D_{l_1,l_2}} \geq \frac{|h|_{l_1} e^{i\varphi_{l_1}}}{\gamma}, \quad \forall h \in P.
\] (47)

\]
Lemma 6. The operator $\mathcal{L}_1 : Q_2 \rightarrow Q_2$ has a bounded inverse $\mathcal{L}_1^{-1}$, satisfying

$$|\mathcal{L}_1^{-1}[h]|_{\sigma,s} \leq \frac{|h|_{\sigma,s}}{N^2}. \tag{48}$$

Proof. $\mathcal{L}_1$ is diagonal in the Fourier basis of $Q : e^{i\varphi_x e^{i\varphi_t} x}$ with $(l_1, l_2) \in \Lambda_+ \cup \Lambda_0 \cup \Lambda_-$ with eigenvalues

$$d_{l_1, l_2} = \left\{ \begin{array}{ll}
\frac{1 + e^{-(-a + a^2 e)}}{a (2 + a^2 e)} l_2^2, & \text{if } l_1 + l_2 = 0,
0, & \text{if } l_1 = 0.
\end{array} \right. \tag{49}$$

The eigenvalues of $\mathcal{L}_1$ restricted to $Q_2(N)$ verify $|d_{l_1, l_2}| \geq N^2/C$, where the constant $C$ depends on $(\epsilon, a)$, and (48) holds.

Fixed points of the nonlinear operator $\mathfrak{P} : Q_2 \oplus P \rightarrow Q_2 \oplus P$ defined by

$$\mathfrak{P}(q_2, p, q_1) := \left( -\mathcal{L}_1^{-1} P \Pi f (\varphi_1, \varphi_2, q_1 + q_2 + p, \epsilon), -\epsilon \mathcal{L}_1^{-1} P \Pi f (\varphi_1, \varphi_2, q_1 + q_2 + p, \epsilon) \right) \tag{50}$$

are solutions of the $(Q_2)$-$(P)$-equations. Using the Contraction Mapping Theorem, we can prove the following lemma.

Lemma 7. For any $R > 0$, there exist an integer $N_0(R) \in \mathbb{N}^*$ and positive constants $\epsilon_0(R) > 0$, $C_0(R) > 0$ such that

$$\forall |q_1|_{l_1} \leq 2R, \quad \forall \epsilon \in \mathfrak{B}, \quad |\epsilon|_{l_1}^{-1} \leq \epsilon_0(R), \quad \forall N \geq N_0(R) : 0 \leq N \sigma \leq 1,$n

and there exists a unique solution $(q_2(q_1), p(q_1)) := (q_2(e, N, q_1), p(e, N, q_1)) \in Q_2 \oplus P$ of the $(Q_2)$-$(P)$-equations satisfying

$$|q_2(e, N, q_1)|_{\sigma,s} \leq \frac{C_0(R)}{N^2}, \tag{52}$$

$$|p(e, N, q_1)|_{\sigma,s} \leq C_0(R) |\epsilon|_{l_1}^{-1}. \tag{53}$$

Moreover, the map $q_1 \rightarrow (q_2(q_1), p(q_1))$ is $C^1(B_{2R} Q_2 \oplus P)$ and

$$|q_2'(q_1) [h]|_{\sigma,s} \leq \frac{C_0(R)}{N^2} |h|_{l_1}^{-1}, \tag{55}$$

$$|p'(q_1) [h]|_{\sigma,s} \leq C_0(R) |\epsilon|_{l_1}^{-1} |h|_{l_1} \quad \forall h \in Q_1.$$

Proof. Let us consider the ball

$$B := \{ (q_2, p) \in Q_2 \oplus P : |q_2|_{\sigma,s} \leq \rho_1, |p|_{\sigma,s} \leq \rho_2 \} \tag{54}$$

with norm $|(q_2, p)|_{\sigma,s} := |q_2|_{\sigma,s} + |p|_{\sigma,s}$. We can claim that, under assumptions (51), there exist $\rho_1, \rho_2 \in (0, 1)$ such that the map $(q_2, p) \rightarrow \mathfrak{P}(q_2, p; q_1)$ is a contraction mapping in $B$, that is, we have to prove

(i) $(q_2, p) \in B \Rightarrow \mathfrak{P}(q_2, p; q_1) \in B$;
(ii) $|\mathfrak{P}_1(q_2, p; q_1) - \mathfrak{P}_1(q_2, p; q_1)|_{\sigma,s} \leq \eta |(q_2, p) - (\tilde{q}_2, \tilde{p})|_{\sigma,s},$

$$\forall (q_2, p), (\tilde{q}_2, \tilde{p}) \in B,$$

where the constant $\eta \in (0, 1)$. In the following, $\kappa_i (i = 1, 2, \ldots , 5)$ denote different constants. By (48) and the Banach property of $\mathcal{H}_{\sigma,s}$,

$$|\mathfrak{P}_1(q_2, p; q_1)|_{\sigma,s} \leq \frac{\kappa_1}{N^2} \left( |q_1^3 + |q_2^3|_s + |p|_s^3 \right). \tag{55}$$

Similarly, for $(a, c) \in \mathfrak{B}$, by (43), we have

$$|\mathfrak{P}_2(q_2, p; q_1)|_{\sigma,s} \leq \kappa_2 \left( |q_1^3 + |q_2^3|_s + |p|_s^3 \right). \tag{56}$$

For all $q_1 \in Q_1(N)$, setting $|x| = \max\{0, |x|\}$, we can get

$$|q_1|_{l_1} \leq \sum_{|l_1| \leq NJ_1} \sum_{l_1 \neq l_1} \frac{1}{|l_1|} \left( |\tilde{q}_l|_{l_1} \left( |l_2| - |l_2| \right)^{s} \right)^{1/2} + \left( \frac{1}{|l_1|} \left( |l_2| \right)^{2(1-s)} \right)^{1/2} \leq \kappa_3 \left( \sum_{|l_1| \leq NJ_1} \sum_{l_1 \neq l_1} \frac{1}{|l_1|} \left( |l_2| \right)^{2(1-s)} \right)^{1/2} \leq \kappa_4 |q_1|_{H_1}.$$

whenever $0 \leq N \sigma \leq 1$, $0 \leq s < 1/2$. Thus, $|q_1|_{H_1} \leq 2R, |q_2|_{\sigma,s} \leq \rho_1, |p|_{\sigma,s} \leq \rho_2$, we get

$$|\mathfrak{P}_1(q_2, p; q_1)|_{\sigma,s} \leq \frac{\kappa_5}{N^2} (R^3 + \rho_3 + \rho_3), \tag{57}$$

$$|\mathfrak{P}_2(q_2, p; q_1)|_{\sigma,s} \leq \kappa_3 \left( R^3 + \rho_3 + \rho_3 \right). \tag{58}$$

Now, set $C_0(R) := \kappa_3 R^3$, and we define $\rho_1 := 2C_0(R)/N^{3/2}$, $\rho_2 := 2C_0(R)(|e|/\gamma)$. By the inequality above, there exist
\[
N_0(R) \in \mathbb{N}^+ \text{ and } \epsilon_0(R) > 0 \text{ such that } \forall N \geq N_0(R) \text{ and } \forall \epsilon \gamma^{-1} \leq \epsilon_0(R),
\]
\[
|P_1(q_2, p; q_1)|_{\sigma, s} \leq \rho_1, \quad |P_2(q_2, p; q_1)|_{\sigma, s} \leq \rho_2. \quad (59)
\]
So, we get the proof of (i). Item (ii) can be obtained with the same estimates.

By the Contraction Mapping Theorem, there exists a unique fixed point \((q_2(q_1), p(q_1)) := (q_2(\epsilon, N, q_1), p(\epsilon, N, q_1))\) of \(\mathfrak{P}\) in \(B\). The bounds of (52) follow by the definition of \(\rho_1\) and \(\rho_2\).

Since the map \(\mathfrak{P} \in C^1(Q_2 \oplus P \times Q_1; Q_2 \oplus P \times Q_1)\), the Implicit Function Theorem implies that the map \(q_1 \rightarrow (q_2(\epsilon, N, q_1), p(\epsilon, N, q_1))\) is \(C^1\). Differentiating both sides of \((q_2(q_1), p(q_1)) = \mathfrak{P}(q_2(q_1), p(q_1), q_1)\), we can get
\[
q_2'(q_1)[h] = -\mathcal{L}_1^{-1} \Pi_{Q_2}(\partial_{q_2} f)(q_1, q_2, q_1 + q_2(q_1) + p(q_1), \epsilon)
\times (h + q_2'(q_1)[h] + p'(q_1)[h]),
\]
\[
p'(q_1)[h] = \epsilon \mathcal{L}_1^{-1} \Pi_{P}(\partial_{p} f)(q_1, q_2, q_1 + q_2(q_1) + p(q_1), \epsilon)
\times (h + q_2'(q_1)[h] + p'(q_1)[h]).
\]

Using (43)–(48) and the Banach property of \(\mathcal{F}_{\sigma, s}\), we get
\[
|q_2'(q_1)[h]|_{\sigma, s} \leq \frac{C(R)}{N^2} \left( |q_2'(q_1)[h]|_{\sigma, s} + |p'(q_1)[h]|_{\sigma, s} \right),
\]
\[
|p'(q_1)[h]|_{\sigma, s} \leq \frac{C(R)}{\gamma} \left( |h|_{\sigma, s} + |q_2'(q_1)[h]|_{\sigma, s} + |p'(q_1)[h]|_{\sigma, s} \right),
\]
which implies the bounds (53), since when \(C(R)|\epsilon|/\gamma + 1/N^2\) is small enough, we can get
\[
\det \begin{pmatrix}
1 - \frac{C(R)}{N^2} & \frac{C(R)}{N^2} \\
\frac{C(R)|\epsilon|}{\gamma} & 1 - \frac{C(R)|\epsilon|}{\gamma}
\end{pmatrix} \geq \frac{1}{2}. \quad (62)
\]

3.3. The \((Q_2)\)-Equation. Once the \((Q_1)\)-(P)-equations have been solved by \((q_2(q_1), p(q_1)) \in Q_2 \oplus P\), there remains the finite-dimensional \(Q_2\)-equation
\[
\mathcal{L}_1[q_1] + \Pi_{Q_2}(\partial_{q_2} f)(q_1, q_2, q_1 + q_2(q_1) + p(q_1), \epsilon) = 0. \quad (63)
\]
The geometric interpretation of the construction of \((q_2(q_1), p(q_1))\) is that, on the finite-dimensional submanifold \(Z := \{q_1 + q_2(q_1) + p(q_1) : |q_1| < 2R\}\), diffeomorphic to the ball \(B_{2R} := \{q \in Q_1 : |q_1|_{H^1} < 2R\}\).

By (37) and (38), we have that, at \(v := q_1 + q_2(q_1) + p(q_1)\),
\[
d\Psi_{\epsilon}(v)[h] = 0, \quad \forall h \in Q_2,
\]
\[
d\Psi_{\epsilon}(v)[h] = 0, \quad \forall h \in P.
\]
Since \(q_2'(q_1)[h] \in Q_2\) and \(p'(q_1)[h] \in P, \forall h \in Q_1\), we deduce that
\[
d\Phi_{\epsilon, N}(q_1)[h] = d\Psi_{\epsilon}(v)[h] + q_2'(q_1)[h] + p'(q_1)[h] = d\Psi_{\epsilon}(v)[h], \quad \forall h \in Q_1,
\]
and therefore $v := q_1 + q_2(q_1) + p(q_1)$ solves also the $(Q_1)$-equation (36). Write $\Psi_\varepsilon(v) := \Psi_\varepsilon^{(2)}(v) - \varepsilon \int_{T^2} F(q_1, v, \varepsilon)$, where

$$
\Psi_\varepsilon^{(2)}(v) = \int_{T^2} \left( 1 + \frac{1}{2\varepsilon} \right) (\partial_{q_1} v)^2 + \varepsilon \frac{2a + a^2 \varepsilon^2}{2} (\partial_{q_2} v)^2 + (1 + \varepsilon a (1 + \varepsilon)) (\partial_{q_1} v) (\partial_{q_2} v)
$$

(71)
is a homogeneous functional of degree two. By homogeneity,

$$
\Psi_\varepsilon(v) = \frac{1}{2} d\Psi_\varepsilon^{(2)}(v)[v] - \varepsilon \int_{T^2} F(q_1, q_2, \varepsilon),
$$

(72)
and, according to (69), we have

$$
d\Psi_\varepsilon^{(2)}(q_1 + q_2(q_1) + p(q_1)) [q_2(q_1) + p(q_1)] = \varepsilon \int_{T^2} f(q_1, q_2, \varepsilon) (q_2(q_1) + p(q_1)).
$$

(73)
Substituting the above equality into (72), we obtain, at $v := q_1 + q_2(q_1) + p(q_1)$,

$$
\Phi_{\varepsilon,N}(q_1) = \Psi_\varepsilon(q_1 + q_2(q_1) + p(q_1))
$$

$$
= \frac{1}{2} d\Psi_\varepsilon^{(2)}(v)[q_1 + q_2(q_1) + p(q_1)]
$$

$$
- \varepsilon \int_{T^2} F(q_1, q_2, \varepsilon)
$$

$$
= \frac{1}{2} d\Psi_\varepsilon^{(2)}(q_1)[q_1]
$$

$$
+ \frac{1}{2} \varepsilon \int_{T^2} f(q_1, q_2, \varepsilon) (q_2(q_1) + p(q_1))
$$

$$
- \varepsilon \int_{T^2} F(q_1, q_2, \varepsilon)
$$

$$
= \int_{T^2} (\partial_{q_1} q_1)^2 + (\partial_{q_1} q_1)(\partial_{q_2} q_1)
$$

$$
+ \varepsilon \left( \int_{T^2} \frac{1}{2} (\partial_{q_1} q_1)^2 + \frac{2a + a^2 \varepsilon^2}{2} (\partial_{q_2} q_1)^2 + a (1 + \varepsilon) (\partial_{q_1} q_1)(\partial_{q_2} q_1)
$$

$$
+ \frac{1}{2} f(q_1, q_2, \varepsilon) (q_2(q_1) + p(q_1))
$$

$$
- F(q_1, q_2, \varepsilon) \right)
$$

$$
= \Psi_0(q_1)
$$

$$
+ \varepsilon \left( \int_{T^2} \frac{1}{2} (\partial_{q_1} q_1)^2 + \frac{2a + a^2 \varepsilon^2}{2} (\partial_{q_2} q_1)^2 + a (1 + \varepsilon) (\partial_{q_1} q_1)(\partial_{q_2} q_1)
$$

$$
+ \frac{1}{2} f(q_1, q_2, \varepsilon) (q_2(q_1) + p(q_1))
$$

$$
- F(q_1, q_2, \varepsilon) \right).
$$

(74)
Because $\Psi_0(q_1) \equiv \text{const.}$ by (52), we can get the bounds of (68).

The problem of finding nontrivial solutions of the $Q_1$-equation is reduced to finding nontrivial critical points of the reduced action functional $\Phi_{\varepsilon,N}$ in $B_{2R}$. By (66), this is equivalent to finding critical points of the rescaled functional

$$
\tilde{\Phi}_{\varepsilon,N} = \Gamma(q_1) + R_{\varepsilon,N}(q_1)
$$

(75)
where the quadratic form

$$
\mathcal{Q}(q) := \int_{T^2} \frac{1}{2} (\partial_{q_1} q_1)^2 + \frac{2a + a^2 \varepsilon^2}{2} (\partial_{q_2} q_1)^2
$$

$$
+ a (1 + \varepsilon) (\partial_{q_1} q_1)(\partial_{q_2} q_1)
$$

$$
= \frac{1}{2} \sum_{l, j \in Z^2} t_l^2 (q_{l,j}^1)^2 e^{i q_{l,j} \varphi_1} e^{i q_{l,j} \varphi_2}
$$

$$
= \frac{\alpha^*}{2} |q_1|_{H^1}^2,
$$

(76)
is positive definite on $Q_+$, negative definite on $Q_-$, and zero definite on $Q_0$. For $q_1 = q_+ + q_- + q_0 \in Q_1$ and $a < 0$, we have

$$
\mathcal{Q}(q_+) = \int_{T^2} \frac{1}{2} (\partial_{q_1} q_1)^2 + \frac{2a + a^2 \varepsilon^2}{2} (\partial_{q_2} q_1)^2
$$

$$
+ a (1 + \varepsilon) (\partial_{q_1} q_1)(\partial_{q_2} q_1)
$$

$$
= \frac{1}{2} \sum_{l, j \in Z^2} t_l^2 (1 + \varepsilon a) q_{l,j}^1 e^{i q_{l,j} \varphi_1} e^{i q_{l,j} \varphi_2}
$$

$$
= - \frac{\alpha}{2} |q_-|_{H^1}^2,
$$

(77)
where the positive constants $\alpha_0(a), \alpha_1(a)$ are bounded away from zero and independent of $\varepsilon$. We will prove the existence of critical point of $\tilde{\Phi}_{\varepsilon,N}$ in $B_{2R}$ of linking type.

3.4. Linking Critical Points of the Reduced Action Functional $\tilde{\Phi}_{\varepsilon,N}$. We cannot directly apply the linking theorem because $\tilde{\Phi}_{\varepsilon,N}$ is defined only in the ball $B_{2R}$. Therefore, we first extend
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̂Φ,𝑁 to the whole space 𝑄1. We define the extended action

function ̂Φ,𝑁 ∈ 𝐶(1)(𝑄, 𝑅) as

where ̃Φ,𝑁(𝑞1) : 𝑄1 → 𝑅 is

and λ : [0, +∞) → [0, 1] is a smooth, nonincreasing, cut-off function such that

By definition, ̂Φ,𝑁(𝑞1) ≡ ̃Φ,𝑁(𝑞1) on 𝐵̂ := {𝑞1 ∈ 𝑄1 : |𝑞1|₁ ≤ 𝑅} and ̂Φ,𝑁(𝑞1) = Φ(𝑞1) outside 𝐵̂. Moreover, by (88), there is a constant 𝐶̃(𝑅) ≥ 𝐶(𝑅) > 0 such that

Now, we fix ρ > 0 small such that (α(α)/2)ρ² − 𝜅1ρ⁴ ≥ (α(α)/4)ρ². Since 𝐶(𝑅)(e + e/γ + 1/N²) ≤ (α(α)/8)ρ², by (84), we can get

(ii) Let

with |𝑞|₁ ≤ 𝑅.

Second, we will verify that ̂Φ,𝑁(𝑞1) satisfies the geometrical hypotheses of the linking theorem.

Lemma 9. There exist positive constants ρ, β, 𝑟, 𝑟₁, 𝑟₂ > ρ, and

0 < 𝜀(𝑅) ≤ 𝜀(𝑅), 𝑁(𝑅) ≥ 𝑁(𝑅), which are independent of (e, N, 𝑦), such that, ∀(e, y) ≤ 𝜀(e, y) and ∀𝑉 ≤ 𝑁(𝑉),

(i) inf, ∈ 𝑆̂ ̂Φ,𝑁(𝑞1) ≥ β > 0, ∀𝑞1 ∈ 𝑆̂ := {𝑞1 ∈ 𝑄1 ∩ 𝑄₀ : |𝑞1|₁ = 𝑅},

(ii) ∑₁,∈ 𝑊̄ ̂Φ,𝑁(𝑞1) ≤ β/2, ∀𝑞1 ∈ ∂𝑊̄,

where 𝑊̄ is the rectangle in 𝑄₁ ⊆ 𝑄₀,

𝑊̄ := {𝑞1 = 𝑞₀ + 𝑞₁ + 𝑟 + 𝑞, |𝑞₀ + 𝑞₁| ≤ 𝑟₁, 𝑞₁ ∈ 𝑄₁ ∩ 𝑄₀,

and 𝑒̂ is the unit vector in 𝑄₁ ⊆ 𝑄₀.

Proof. (i) ∀𝑞₁ ∈ 𝑄₁ ∩ 𝑄₀ with |𝑞₁|₁ = 𝑅 > ρ, we have

̂Φ,𝑁(𝑞₁) = ̂Φ,𝑁(𝑞₁)

= Ω(𝑞₁) − ∫₁(𝑞₁, 𝑞₂) 1 4 + ̃Φ,𝑁(𝑞₁)

≥ α(α)/2 + 𝜅1ρ⁴ − 𝐶(𝑅)(e + e/γ + 1/N²).

Next, setting 𝑀 := max(0, 𝑟) 𝜅² − 𝜅⁴, we fix 𝑟₁ large such that

α(α)/2 + 𝜅² + 𝜅₀⁴ ≥ 𝑀, ∀ |𝑞₀ + 𝑞₁| = 𝑟₁,
and therefore
\[ \Phi_{e,N}(q_1) \leq M - \left( \frac{\alpha_+(a)}{2} |q_1|_{H^1}^2 + \kappa q_0^4 \right) \leq 0, \quad \forall q_1 \in B_2. \]
(93)

Finally, if \( q_1 = q_- + q_0 \),
\[ \Phi_{e,N}(q_1) = \Phi(q_-) - \int_{\mathbb{T}^1} a_3(q_1, q_2) \frac{q_1^4}{4} + \Phi_{e,N}(q_1) \]
\[ \leq |\Phi_{e,N}(q_1)| \leq C_3(R) \left( \epsilon + |q| \gamma^{-1} + \frac{1}{N^2} \right). \]
(94)
So, if \( C_3(R)(\sqrt{\epsilon} + |q|) \gamma^{-1} + (1/N^2) \leq \beta/2 \), we can get \( \Phi_{e,N}(q_1) \leq \beta/2, \forall q_1 \in \partial W^- \).

We introduce the minimax class \( S := \{ \psi \in C(\mathbb{W}, Q) \mid \psi = \text{Id} \text{ on } \partial \mathbb{W} \} \). According to Proposition 5.9 of [19], the maps of \( S \) have an important intersection property as follows.

**Proposition 10.** (\( S^* \) and \( W^- \) link with respect to \( \psi \)). Consider
\[ \psi \in S \implies \psi(W^-) \cap S^* \neq \emptyset. \]
(95)

One defines the minimax level as follows:
\[ \zeta_\epsilon := \inf_{q \in \partial W^-} \sup_{q \in \partial W^-} \Psi(q_1). \]
Obviously, by Proposition 10 and Lemma 9,
\[ \max_{q \in W^-} \Phi_{e,N}(\psi(q_1)) \geq \min_{q \in S^*} \Phi_{e,N}(q_1) \geq \beta > 0, \quad \forall \psi \in S', \]
and, therefore, \( \zeta_\epsilon > \beta > 0 \).

Since \( \text{Id} \in S' \), so
\[ \zeta_\epsilon \leq \max_{q \in W^-} \Phi_{e,N}(q_1) \leq \max_{q \in W^-} \left( \Gamma(q_1) + \Phi_{e,N}(q_1) \right) \]
\[ \leq \max_{q \in W^-} \left( \frac{\alpha_-(a)}{2} |q_1|_{H^1}^2 + \frac{\alpha_+(a)}{2} |q_1|_{H^1}^2 + \int_{\mathbb{T}^1} \kappa q_0^4 \right) \]
\[ + 1 \leq \zeta_{\text{co}} \leq +\infty, \]
where \( \zeta_{\text{co}} \) is independent of \( N, \epsilon, \gamma \). By the linking theorem, we deduce the existence of a Palais-Smale sequence \( q_n \in Q_1 \) at the level \( \zeta_\epsilon \), namely,
\[ \Phi_{e,N}(q_n) \to \zeta_\epsilon, \quad \Phi'_{e,N}(q_n) \to 0 \quad \text{when} \ n \to \infty. \]
(99)

Third, we will prove that the Palais-Smale sequence \( q_n \) (up to subsequence) converges to some nontrivial critical point \( q_1 \neq 0 \) in some open ball of \( Q_1 \), where \( \Phi_{e,N} \) and \( \Phi_{e,N} \) coincide. Because the space \( Q_1 \) is finite dimensional, we only need to prove that the sequence \( q_n \) is bounded.

**Lemma 11.** There is a constant \( M > 0 \) independent of \( R, \epsilon, N, \gamma, \) such that, for all \( |\gamma|/y \) small enough and \( N \) large enough, the Palais-Smale sequence \( q_n \) is bounded, that is, \( |q_n|_{H^1} < M \). So, there exists a subsequence of the P-S sequence that converges to some critical point \( q_1 \neq 0 \), and the functional \( \Phi_{e,N} \) possesses a nontrivial critical point \( q_1 \in Q_1 \) with critical value \( \Phi_{e,N}(q_1) = \zeta_\epsilon. \)

**Proof.** In the sequel, we will always assume that \( (\epsilon + |q| \gamma^{-1} + 1/N^2) < 1 \). Writing \( \Phi_{e,N}(q) = \Gamma(q) + \Phi_{e,N}(q) \), by (81)-(82), we can get
\[ \Phi_{e,N}(q_n) - \frac{1}{2} \Phi_{e,N}'(q_n)[q_n] \]
\[ = \Gamma(q_n) - \frac{1}{2} \Gamma'(q_n)[q_n] + \left( \Phi_{e,N}(q_n) - \frac{1}{2} \Phi_{e,N}'(q_n)[q_n] \right) \]
\[ = \left( \frac{1}{2} - \frac{1}{4} \right) \int_{\mathbb{T}^1} a_3(q_1) q_j^4 \]
\[ + \left( \Phi_{e,N}(q_n) - \frac{1}{2} \Phi_{e,N}'(q_n)[q_n] \right) \]
\[ \geq \alpha \left( \frac{1}{2} - \frac{1}{4} \right) \int_{\mathbb{T}^1} q_j^4 \epsilon + |q| \gamma^{-1} + \frac{1}{N^2}. \]
(100)
Then, by \( \zeta_\epsilon < \zeta_{\text{co}} < +\infty \) and \( \Phi_{e,N}(q_n) \to 0 \ (n \to \infty), \)
\[ \zeta_{\text{co}} + 1 + |q_n|_{H^1} \geq \kappa_\alpha \int_{\mathbb{T}^1} q_j^4 := \kappa_\alpha |q_n|_{H^1}^4. \]
(101)
By Hölder inequality and orthogonality,
\[ \zeta_{\text{co}} + 1 + |q_n|_{H^1} \geq \kappa_\alpha \left( \int_{\mathbb{T}^1} (q_n + q_- q_n + q_+ q_n)^2 \right) \]
\[ = \kappa_\alpha \left( \int_{\mathbb{T}^1} q_n^2 + q_-^2 + q_+^2 \right) \geq \kappa_\alpha |q_n|_{H^1}^4. \]
(102)
and therefore \( |q_n|_{H^1} \leq (1 + |q_n|_{H^1})^{1/4} \). In the same way, by Hölder inequality and (82),
\[ \Phi_{e,N}'(q_n)[q_{n+}] = \alpha_+(a) |q_{n+}|_{H^1}^2 \]
\[ - \int_{\mathbb{T}^1} a_3(q_1) q_3 q_{n+} + \Phi_{e,N}'(q_n)[q_{n+}] \]
\[ \\
\geq \alpha_+(a) |q_{n+}|_{H^1}^2 - \kappa_\alpha |q_{n+}|_{H^1} \int_{\mathbb{T}^1} q_{n+}^4 \]
\[ - C_3(R)(\sqrt{\epsilon} + |q| \gamma^{-1} + \frac{1}{N^3/2}) |q_{n+}|_{H^1} \]
\[ \geq \kappa_\alpha |q_{n+}|_{H^1} \left( |q_{n+}|_{H^1} - |q_{n+}|_{H^1} ^3 - 1 \right). \]
(103)

By (101) and the above inequalities, using \( \Phi_{e,N}(q_n) \to 0 \ (n \to \infty), \) we conclude that \( |q_{n+}|_{H^1} \leq \kappa_{11} (1 + |q_{n+}|_{H^1}/4) \).

Estimating analogously, we derive \( |q_{n-}|_{H^1} \leq \kappa_{12} (1 + |q_{n-}|_{H^1}/4) \).

Finally, we deduce that
\[ |q_n|_{H^1} = |q_{n+}| + |q_{n-}|_{H^1} + |q_{n-}|_{H^1} \]
\[ \leq \kappa_{13} \left( 1 + |q_{n+}|_{H^1}^3/4 + |q_{n-}|_{H^1}^3/4 \right). \]
(104)
So, we can conclude that $|q_n|_{H^1} \leq M$ for a suitable constant $M > 0$, which is independent of $R, \varepsilon, N, \gamma$. Since $Q_1$ is finite dimensional, $|q_n|$ converges, up to subsequence, to some critical point $\tilde{q}$ of $\Phi_{e,N}$ with $|\tilde{q}|_{H^1} < M$. Since $\Phi_{e,N}(\tilde{q}) = \xi_{\varepsilon} \geq \beta > 0$, we conclude that $\tilde{q} \neq 0$.

**Proof of Theorem 1.** Let us fix $R := M + 1$ and take $|e|\gamma^{-1} < e_0(R) := \bar{\gamma}$. According to Lemma 7, we can get, for $0 < \sigma N(R) \leq 1$, $0 \leq s < 1/2$, a solution $(q_2(q_1), p(q_1)) \in (Q_2(N) \cap P) \cap H_{\sigma,\gamma}$ of the $(Q_2)-(P)$-equations with $|q_1|_{H^1} \leq 2R$. By Lemma 11, the extended functional $\Phi_{e,N}(q_1)$ possesses a nontrivial critical point $\tilde{q}$ with $|q_1|_{H^1} \leq M < \bar{R}$ since $\Phi_{e,N}$ coincides with $\Phi_{e,N}$ on the ball $B_{\bar{R}}$ by Lemma 8, there exists a nontrivial weak solution $\tilde{q}_1 + q_2(\tilde{q}_1) + p(\tilde{q}_1) \in H_{\sigma,\gamma}$ of (23).

Finally, $u = u(\tilde{q}_1 + q_2(\tilde{q}_1) + p(\tilde{q}_1)) = [\tilde{q}_1 + q_2(\tilde{q}_1)](\tilde{q}_1)$ solves (7).

According to Lemma 6, by the regularizing property of the operator $\mathcal{L}_1$, the solution $\tilde{q}_1 := \tilde{q}_1 + q_2(\tilde{q}_1)$ of the $(Q)$-equation belongs to $H_{\sigma,\gamma} \cap H^{s+2}$. By the P-equation

$$(2a + e^2)^2 q_{y_1}^2 + 2(1 + a(\varepsilon + e^2)) \mathcal{L}_{y_1, q_2} \tilde{p} = -e \left((2a + e^2) \mathcal{L}_{y_1, q_2} + \Pi P f(\varphi_1, \varphi_2, \tilde{q}_1 + p, \varepsilon) \right) \tilde{p},$$

where $\tilde{p} = p(\tilde{q}_1)$, we can get that

$$-e \left((2a + e^2) \mathcal{L}_{y_1, q_2} + \Pi P f(\varphi_1, \varphi_2, \tilde{q}_1 + p, \varepsilon) \right) \tilde{p} \in H_{\sigma,\gamma},$$

for $0 < \sigma' < \sigma$, satisfying $|e(2a + a^2e^2)|_{L_2}^2 |\tilde{p}|_{L_2}^2 < \varepsilon^{2/3}$. For $(a, \varepsilon) \in B_{\varepsilon}$, the eigenvalues of operator $2(a + e^2) \mathcal{L}_{y_1, q_2} + (2(1 + a + e^2)) \mathcal{L}_{y_1, q_2}$, restricted to $P$ satisfy $|2(a + e^2)|_{L_2}^2 + 2(1 + a + a^2)|_{L_2}^2 \geq 2 + e)|_{L_2}^2 \geq 0$, $l_1 + l_2 = 0$, and, thus, we deduce that $\tilde{p} \in H_{\sigma,\gamma}$, for all $0 < \sigma' < \sigma$ (satisfying $C_1 |e|^{1/3} \leq \varepsilon^{2/3}$), and $|\mathcal{L}_{y_1, \tilde{p}}(\varphi_1, \varphi_2, \tilde{q}_1, p, \varepsilon)\tilde{p} = O(e/\gamma)$. By (105),

$$(2a + e^2)^2 q_{y_1}^2 \tilde{p} = -2 \left((1 + a(\varepsilon + e^2)) \mathcal{L}_{y_1, q_2} + \Pi P f(\varphi_1, \varphi_2, \tilde{q}_1 + p, \varepsilon) \right) \tilde{p},$$

we get $\tilde{p} \in H_{\sigma,\gamma}$, and $|\tilde{p}|_{\sigma,\gamma} = O(e\gamma^{-1})$, with $0 < \sigma < \sigma''$. Thus, (15) follows with $s = s + 2, 0 < \sigma < \sigma''$. By (5), $u(t, x) = e\psi((1 + \varepsilon)t + x, (1 + a^2\varepsilon)^r + x)$ is the solution of (1), for all $(a, \varepsilon) \in B_{\varepsilon}$. To show that $u(t, x)$ is quasiperiodic, it remains to prove that $v$ depends on both variables $(\varphi_1, \varphi_2)$ independently.

According to Lemma 9, $|q_1|_{H^1} \leq R$, $\Phi_{e,N}(q_1) \geq \beta$. On the other hand, $\Phi_{e,N}(\tilde{q}_1 + \tilde{q}_2) \leq \beta/2, |\tilde{q}_1|_{H^1} \leq R$, so that $\tilde{q}_1 \neq Q_0$. Then, therefore, $v$ depends on $\varphi_2$. In fact, any solution $v$ of (23) depending only on $\varphi_2$, that is, the solutions of

$$(2a + e^2)^2 \frac{d^2 v(\varphi_2)}{d\varphi_2^2} + f(\varphi_1, \varphi_2, \varphi_2, \varepsilon) = 0,$$

is $v(\varphi_2) \equiv 0$. Indeed, by the homogeneity of $f(\varphi_1, \varphi_2, \varepsilon, \varepsilon)$, we have

$$e^3 f(\varphi_1, \varphi_2, \varepsilon, \varepsilon) = f(\varphi_1, \varphi_2, \varepsilon, \varepsilon) = \sum_{k=3}^{\infty} a_k(\varphi_1, \varphi_2) (\varepsilon \varepsilon)^k.$$  

(109)

Now consider a smooth function $h(\varphi_2)$ with zero mean, ant it satisfies $\int a_k(\varphi_1, \varphi_2) h(\varphi_2) d\varphi_2 \neq 0$ for some $k$. Multiplying (108) by $h(\varphi_2)$ and integrating over $[0, \pi]$, we have

$$\int_0^{2\pi} 2a + a^2 e^2 \frac{d^2 v(\varphi_2)}{d\varphi_2^2} h(\varphi_2) d\varphi_2$$

$$+ \int_0^{2\pi} f(\varphi_1, \varphi_2, \varphi_2, \varepsilon) h(\varphi_2) d\varphi_2 = 0.$$  

(110)

According to $h(\varphi_2)$ that has zero mean and multiplying above equation by $e^3$, we get

$$\int_0^{2\pi} f(\varphi_1, \varphi_2, \varepsilon v(\varphi_2)) h(\varphi_2) d\varphi_2$$

$$= \sum_{k=3}^{\infty} a_k(\varphi_1, \varphi_2) h(\varphi_2) d\varphi_2 = 0.$$  

(111)

The function $G(z(\varphi_2)) = \sum_{k=3}^{\infty} b_k(\varphi_1, \varphi_2) z(\varphi_2)^k$, with $b_k = \int_0^{2\pi} a_k(\varphi_1, \varphi_2) h(\varphi_2) d\varphi_2$, is a nontrivial analytic function. Thus, the equation $G(v(\varphi_2)) = 0$ cannot have a sequence of zeros accumulating to zero. So, for $\varepsilon$ small enough, $v(\varphi_2) \equiv 0$.

**4. Waves Traveling in Opposite Directions**

Substituting $\omega_1 = 1 + \varepsilon, \omega_2 = 1 + a\varepsilon$ into (8), we get

$$\mathcal{L}_{a,\varepsilon} v + f(\varphi_1, \varphi_2, \varepsilon) = 0,$$

where (see (8))

$$\mathcal{L}_{a,\varepsilon} := \left[ (1 + a + 1) \mathcal{L}_{a_1} + (a) \mathcal{L}_{a_2} \right] e^{\left[ e \mathcal{L}_{a_1} + (2 + a) \mathcal{L}_{a_2} \right]}$$

$$= 4a \mathcal{L}_{a_1} + e \left[ 2(a + 1 + a) \mathcal{L}_{a_1} \mathcal{L}_{a_2} \right]$$

$$+ 2(a + 1 + a) \mathcal{L}_{a_1} \mathcal{L}_{a_2}.$$  

(112)

We rescale (112) in order to highlight the relationship between the amplitude and the variation in frequency: $v(\varphi_1, \varphi_2) \rightarrow \sqrt{\varepsilon} v(\varphi_1, \varphi_2)$, and, for convenience, we assume $\varepsilon > 0$. In the following, we consider the scaled equation

$$\mathcal{L}_{a,\varepsilon} v + e f(\varphi_1, \varphi_2, \varepsilon, \varepsilon) = 0,$$

where $f(\varphi_1, \varphi_2, \varepsilon, \varepsilon) = a_3(\varphi_1, \varphi_2) \varepsilon^3 + \sqrt{\varepsilon} O(\varepsilon^4).$
Equation (114) is the Euler-Lagrange equation of the Lagrange action functional $\Psi_\epsilon \in C^1(\mathcal{H}_{\sigma,s}, \mathbb{R})$ defined by

$$
\Psi_\epsilon (v) := \int_{\tau^2} \left( 2(\partial_\phi v)(\partial_\phi v) + \frac{\epsilon(2 + \epsilon)}{2}(\partial_\phi v)^2 
+ \frac{\epsilon(2 + \epsilon)}{2}(\partial_\phi v)^2 
+ \epsilon(2 + \epsilon)(\partial_\phi v)(\partial_\phi v) - 2F(\phi_1, v, \epsilon) \right),
$$

where $F(\phi_1, \phi_2, v, \epsilon) := \int_0^v f(\phi_1, \phi_2, \xi, \epsilon)d\xi$ and $\Psi_0(v) := \int_{\tau^2} 2(\partial_\phi v)(\partial_\phi v)$.

\[ (115) \]

In order to find critical points of $\Psi_\epsilon(v)$, we use the same method as in Section 3. The operator $\mathcal{L}_{a,\epsilon}$ is diagonal defined on the Banach space $\mathcal{H}_{\sigma,s}$ under the Fourier basis $e^{i[l_1,j_1]}$ with eigenvalue $D_{l_1,j_1} = -[(2 + \epsilon)l_1 + a\epsilon l_2][e^{l_1 + (2 + \epsilon)l_1}].$ So, we have

$$
\mathcal{L}_{a,\epsilon} [v] = \sum_{(l_1,j_1) \in \mathbb{Z}^2} D_{l_1,j_1} \tilde{v}_{l_1,j_1} e^{i[l_1,j_1]}v, \quad \forall v \in \mathcal{H}_{\sigma,s}. 
$$

The unperturbed functional $\Psi_0 : \mathcal{H}_{\sigma,s} \to \mathbb{R}$ possesses an infinite-dimensional linear space $Q$ of critical points, which are the solutions of the equation

$$
\mathcal{L}_0 q = 4\partial_{\phi_1} \partial_{\phi_2} q = 0.
$$

The space $Q$ can be written as

$$
Q = \left\{ q = \sum_{(l_1,j_1) \in \mathbb{Z}^2} \tilde{q}_{l_1,j_1} e^{i[l_1,j_1]} \in \mathcal{H}_{\sigma,s} \mid \tilde{q}_{l_1,j_1} = 0, \right\}
$$

for $l_1,j_1 \neq 0$.

\[ (119) \]

We split $Q$ as

$$
Q = Q_+ + Q_0 + Q_-, 
$$

where

\[ (120) \]

We decompose the space $\mathcal{H}_{\sigma,s} = Q + P$, where

$$
P := \left\{ p = \sum_{(l_1,j_1) \in \mathbb{Z}^2} \tilde{p}_{l_1,j_1} e^{i[l_1,j_1]} \in \mathcal{H}_{\sigma,s} \mid \tilde{p}_{l_1,j_1} = 0, \right\}
$$

Projecting (114) onto the closed subspaces $Q$ and $P$, setting $v = q + p \in \mathcal{H}_{\sigma,s}$ with $q \in Q$ and $p \in P$, we obtain

\[ (122) \]

In the same way, we decompose the space $Q = Q_1 + Q_2$. Setting $q = q_1 + q_2$, with $q_1 \in Q_1$ and $q_2 \in Q_2$, we finally get

\[ (123) \]

The operator $\mathcal{L}_{a,\epsilon}$ is diagonal in the Fourier basis $\{e^{i[l_1,j_1]} \in \mathbb{Z}^2\}$ with eigenvalues $D_{l_1,j_1} = -[(2 + \epsilon)l_1 + a\epsilon l_2][e^{l_1 + (2 + \epsilon)l_1}].$ We first prove that $\mathcal{L}_{a,\epsilon}$ restricted to $P$ has a bounded inverse when $(a,\epsilon)$ belongs to the uncountable zero-measure set

$$
\mathcal{D}_\gamma := \left\{ (a,\epsilon) \in \mathbb{R} \times \mathbb{R} \mid \frac{ae}{2 + \epsilon} < 1, 1 + \epsilon \neq 0, 1 + ae \neq 0, 2 + ae \neq 0 \right\},
$$

\[ (127) \]
where \( \mathcal{B}_y \) is a set of badly approximate numbers defined as
\[
\mathcal{B}_y := \left\{ \left( \frac{ae}{2 + \epsilon}, \frac{\epsilon}{2 + ae} \right) : (e_1, e_2) \in (-\epsilon, 0) \times (-\epsilon, 0) \right\}
\]
for all \( l_1, l_2 \in \mathbb{Z} \setminus \{0\} \), and \( 0 < y < 1/4 \), \( e_0 \in (0, 1/2) \).

**Lemma 12.** For \((a, e) \in \mathcal{B}_y \), the eigenvalues \( D_{l_1, l_2} \) of \( \mathcal{L}_{a, e} \) restricted to \( P \) satisfy
\[
|D_{l_1, l_2}| = \left| \left( 2 + e \right) \left( l_1 + \frac{ae}{2 + e} l_2 \right) \left( l_2 + \frac{\epsilon}{2 + ae} l_1 \right) \right| > y,
\]
for all \( l_1, l_2 \neq 0 \).

As a consequence, the operator \( \mathcal{L}_{a, e} : P \to P \) has a bounded inverse \( \mathcal{L}_{a, e}^{-1} \) and satisfies
\[
\left| \mathcal{L}_{a, e}^{-1} \right|_{\sigma,s} \leq \frac{|h|_{\sigma,s}}{y}, \quad \forall h \in P.
\]

**Proof.** Denote by \( [x] \) the nearest integer close to \( x \) and \( [x] = x - \lfloor x \rfloor \). If both \( l_1 \neq -\left\lfloor (ae)/(2 + \epsilon) \right\rfloor \) and \( l_2 \neq -\left\lfloor (e/2 + ae) \right\rfloor \), then we have
\[
|D_{l_1, l_2}| = \left| \left( 2 + e \right) \left( l_1 + \frac{ae}{2 + e} l_2 \right) \left( l_2 + \frac{e}{2 + ae} l_1 \right) \right| > 1.
\]
If \( l_1 = -\left\lfloor (ae)/(2 + \epsilon) \right\rfloor \), then
\[
|D_{l_1, l_2}| = \left| \left( 2 + e \right) \left( l_1 + \frac{ae}{2 + e} l_2 \right) \right| \cdot \left| (2 + ae) l_2 - \frac{ae}{2 + e} e l_2 \right| \geq \frac{y}{|l_2|} \cdot \left| (2 + e) \left( \frac{ae}{2 + e} l_2 \right) + 2l_2 \right|
\]
\[
+ ae \left( 1 - \frac{e}{2 + e} \right) l_1 \geq y.
\]
In the same way, if \( l_2 = -\left\lfloor (e/2 + ae) \right\rfloor \), then we have
\[
|D_{l_1, l_2}| \geq \frac{y}{|l_1|} \cdot \left| (2 + ae) \left( \frac{\epsilon}{2 + ae} l_1 \right) + 2l_1 \right|
\]
\[
+ e \left( 1 - \frac{\epsilon^2}{2 + ae} \right) l_1 \geq y.
\]

**Lemma 13.** The operator \( \mathcal{L}_1 : Q_2 \to Q_2 \) has a bounded inverse \( \mathcal{L}_1^{-1} \) which satisfies
\[
|\mathcal{L}_1^{-1} [h]|_{\sigma,s} \leq \frac{|h|_{\sigma,s}}{N^2}.
\]

**Proof.** \( \mathcal{L}_1 \) is diagonal in the Fourier basis of \( Q : e^{i\varphi} e^{i\varphi^2} \) with \((l_1, l_2) \in \Lambda_+ \cup \{(0,0)\} \cup \Lambda_- \) with eigenvalues
\[
d_{l_1, l_2} = \begin{cases} \left( 2 + e \right) l_1^2, & \text{if } l_2 = 0, \\ -(2 + ae) l_2^2, & \text{if } l_1 = 0. \end{cases}
\]

The eigenvalues of \( \mathcal{L}_1 \) restricted to \( Q_2(N) \) verify \( |d_{l_1, l_2}| \geq N^2/C \), where the constant \( C \) depends on \((e, a)\), and \( (134) \) holds.

Similarly, the solution of \( Q_1 \)-equation (124) is the Euler-Lagrange equation of the reduced Lagrangian action functional:
\[
\Phi_{e, N} : B_{2R} \subset Q_1 \to \mathbb{R},
\]

\[
\Phi_{e, N}(q_1) := \Psi_e (q_1 + q_2 (q_1 + p (q_1))),
\]

where \( \Gamma(q_1) := \int_{\mathbb{T}^2} \frac{(2 + e)}{2} \left( \partial_{\varphi_1} q_1 \right)^2 \]
\[
+ (1 + a + ae) \left( \partial_{\varphi_1} q_1 \right) \left( \partial_{\varphi_2} q_1 \right)
\]
\[
+ \frac{2a + a^2 e}{2} \left( \partial_{\varphi_1} q_1 \right)^2 - a_3 (q_1, q_2) \frac{q_1^4}{4},
\]

\[
R_{e, N}(q_1) := \int_{\mathbb{T}^2} \alpha_4 (q_1, q_2) \frac{q_1^4}{4}
\]
\[
- F (q_1, q_2, q_1 + q_2 + p, e)
\]
\[
+ \frac{1}{2} f (q_1, q_2, v, e) (q_2 + p),
\]
and, for some positive constant \( C_2(R) \geq C_1(R) \), we can get
\[
|R_{e, N}(q_1)| \leq C_2 (R) \left( \sqrt{e} + e^y + \frac{1}{N^2} \right),
\]

\[
|R_{e, N}(q_1)| \leq C_2 (R) \left( \sqrt{e} + e^y + \frac{1}{N^2} \right)|h|_{H^1},
\]
\[
\forall h \in Q_1.
\]

The problem of finding nontrivial solutions of the \( Q_1 \)-equation is reduced to finding nontrivial critical points of the reduced action functional \( \Phi_{e, N} \) in \( B_{2R} \). By (137), this is
equivalent to finding critical points of the rescaled functional denoted by \( \Phi_{\epsilon,N} \) and called the reduced action functional
\[
\tilde{\Phi}_{\epsilon,N}(q_1) = \Gamma(q_1) + R_{\epsilon,N}(q_1)
\]
\[
= \left( \Omega(q_1) - \int_{\mathbb{T}^2} a_3(q_1) \frac{q_1^4}{4} + R_{\epsilon,N}(q_1) \right),
\]
where the quadratic form
\[
\Omega(q_1) = \int_{\mathbb{T}^2} \frac{(2 + \epsilon)}{2} (\partial_{q_1} q_1)^2 + (1 + a + a\epsilon) (\partial_{q_1} q_1) (\partial_{q_1} q_1)
\]
\[
+ 2a + a^2 \epsilon \left( \frac{a_1}{2} \right)^2 \partial_{q_1} q_1^2
\]
is positive definite on \( Q_{+} \), negative definite on \( Q_{-} \), and zero definite on \( Q_{0} \). For \( q_1 = q_1 + q_2 + q_0 \in Q_1 \), we have
\[
\Omega(q_+)= \int_{\mathbb{T}^2} \frac{(2 + \epsilon)}{2} (\partial_{q_1} q_1)^2 = \frac{\alpha_+}{2} |q_1|^2_{H^1},
\]
\[
\Omega(q_0) = 0,
\]
\[
\Omega(q_-)= \int_{\mathbb{T}^2} \frac{(a + a\epsilon)}{2} (\partial_{q_1} q_1)^2 = -\frac{\alpha_-}{2} |q_1|^2_{H^1},
\]
where the positive constants \( \alpha_+, \alpha_- \) are bounded away from zero and independent of \( \epsilon \). The following steps of finding the nontrivial solutions of the \( Q_{-} \)-equation are similar to Lemmas 9 and 11 in Section 3.4.

**Proof of Theorem 2.** We can get the solution of (8) as follows:
\[
\nu = \sqrt{|e|} \left[ \tilde{q}_1 + q_1(\tilde{q}_1) + p(\tilde{q}_1) \right] = \sqrt{|e|} \left[ \tilde{q}_1 + p(\tilde{q}_1) \right].
\]
(143)
According to Lemma 13, by the regularizing property of the operator \( S_{\epsilon} \), the solution \( \tilde{q}_1 := \tilde{q}_1 + q_1(\tilde{q}_1) \) of the \( (Q) \)-equation belongs to \( \mathscr{H}_{\epsilon,+2} \cap Q \). By the \( P \)-equation
\[
(\epsilon + a + a\epsilon) \partial_{\tilde{q}_1} q_1 + (4 + 2\epsilon a + 2\epsilon + 2a^2) \partial_{\tilde{q}_1} \tilde{q}_1 \tilde{q}_1 = -e \left( 2a + a^2 \epsilon \right) \partial_{\tilde{q}_1} q_1 + \Pi_{P} f (q_1, \varphi_1, \varphi_2, p, e) \tilde{p}
\]
(144)
where \( \tilde{p} = p(\tilde{q}_1) \), we can get that
\[
-e \left( 2a + a^2 \epsilon \right) \partial_{\tilde{q}_1} q_1 + \Pi_{P} f (q_1, \varphi_1, \varphi_2, p, e) \tilde{p} \in \mathscr{H}_{\epsilon,+2},
\]
(145)
for \( 0 < \sigma' < \sigma \), satisfying \( |e(2a + a^2 \epsilon) \partial_{\tilde{q}_1} q_1| < e^{1/2} |e| \). For \( (a, e) \in \mathcal{D}_{\epsilon} \), the eigenvalues of operator
\[
(\epsilon + a + a\epsilon) \partial_{\tilde{q}_1} q_1 + (4 + 2\epsilon a + 2\epsilon + 2a^2) \partial_{\tilde{q}_1} \tilde{q}_1
\]
(146)
restricted to \( P \) satisfy
\[
\left| (\epsilon + a + a\epsilon) \partial_{\tilde{q}_1} q_1 + (4 + 2\epsilon a + 2\epsilon + 2a^2) \partial_{\tilde{q}_1} \tilde{q}_1 \right| \geq 2 |e| \frac{|\partial_{\tilde{q}_1}|}{|\tilde{q}_1|}.
\]
(147)
\[
\forall l_1 \neq 0, l_2 \neq 0,
\]
and, thus, we deduce that \( \tilde{p} \in \mathscr{H}_{\epsilon,+2} \), for all \( 0 < \sigma' < \sigma \) (satisfying \( C l_2 e^{1/2} |e| < e^{1/2} |e| \)), and \( |\partial_{\tilde{q}_1} \tilde{p}|_{\epsilon,+2} = O(|e|/\gamma) \). By (144),
\[
(\epsilon + a + a\epsilon) \partial_{\tilde{q}_1} q_1 + (4 + 2\epsilon a + 2\epsilon + 2a^2) \partial_{\tilde{q}_1} \tilde{q}_1 \tilde{q}_1
\]
(148)
we get \( \tilde{p} \in \mathscr{H}_{\epsilon,+2} \), and \( |\tilde{p}|_{\epsilon,+2} = O(|e|/\gamma) \), with \( 0 < \sigma < \sigma' \).

5. Conclusion

In this paper, for the completely resonant nonlinear wave equations, under periodic boundary conditions, we obtain the existence and regularity of quasiperiodic solutions. The forced terms we consider are quasiperiodic, and, according to the linking theorem, the bifurcation equations are solved by variational method. Moreover, the solutions depending on the spatial and time variables are coupled and in the form of traveling waves. In [28], Yuan got the existence of quasiperiodic solutions with \( n \in \mathbb{N} \) \((n \geq 3)\) frequencies by KAM theory, in which the form of the solutions is \( u(t,x) = v(\omega_1 t, \omega_2 t, \ldots, \omega_n t, x) \). In the future work, we will investigate the existence of quasiperiodic solutions with the traveling wave form as \( u(t,x) = v(\omega_1 t + x, \omega_2 t + x, \ldots, \omega_n t + x) \).

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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