Regularity of Functions on the Reduced Quaternion Field in Clifford Analysis

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We define a new hypercomplex structure of $\mathbb{R}^3$ and a regular function with values in that structure. From the properties of regular functions, we research the exponential function on the reduced quaternion field and represent the corresponding Cauchy-Riemann equations in hypercomplex structures of $\mathbb{R}^3$.

1. Introduction


We shall denote by $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{Z}$, respectively, the field of complex numbers, the field of real numbers, and the set of all integers. We [15, 16] showed that any complex-valued harmonic function $f_1$ in a pseudoconvex domain $D$ of $\mathbb{C}^2 \times \mathbb{C}^2$ has a hyperconjugate harmonic function $f_2$ in $D$ such that the quaternion-valued function $f_1 + f_2 j$ is hyperholomorphic in $D$ and gave a regeneration theorem in quaternion analysis in the view of complex and Clifford analysis. Further, we [17, 18] investigated the existence of the hyperconjugate harmonic functions of the octonion number system and some properties of dual quaternion functions.

In this paper, we introduce the Fueter variables on $\mathbb{R}^3$ and investigate a hypercomplex structure of $\mathbb{R}^3$. We define regular functions and obtain the representation of the corresponding Cauchy-Riemann equations for regular functions in the reduced quaternion field.

2. Preliminaries

A three-dimensional, noncommutative, and associative real field, called a ternary number system, is constructed by three base elements $e_0$, $e_1$, and $e_2$ which satisfy

\begin{equation}
\begin{aligned}
e_0^2 &= 1, & e_1^2 &= e_2^2 &= -1, \\
e_1 e_2 &= -e_2 e_1, \\
\bar{e}_r &= e_0, & e_r &= -e_r & (r &= 1, 2).
\end{aligned}
\end{equation}

In addition, let $e_0$ be the identity of a ternary number system and $e_1$ identifies the imaginary unit $\sqrt{-1}$ in the complex field, and

\begin{equation}
\mathbb{C}(T) := \{ z = e_1 z_1 + e_2 z_2 \mid z_1, z_2 \in \mathbb{C} \}.
\end{equation}
where \( z_r = x_r - (1/2)e_r x_0 \) (\( r = 1, 2 \)) and \( x_m \) (\( m = 0, 1, 2 \)) are real variables. They satisfy the equations

\[
\overline{z}_r \overline{w}_k = \overline{w}_k z_r \quad (r \neq k),
\]

where \( \overline{z}_r = x_r + (1/2)e_r x_0 \) (\( r = 1, 2 \)), \( \overline{w}_k = y_k - (1/2)e_k y_0 \), \( \overline{w}_k = y_k + (1/2)e_k y_0 \) (\( k = 1, 2 \)), and \( y_m \) (\( m = 0, 1, 2 \)) are real variables.

For any two elements \( z = e_1 z_1 + e_2 z_2 \) and \( w = e_1 w_1 + e_2 w_2 \) of \( \mathbb{C}(\mathbb{T}) \), their product is given by

\[
z w = z \cdot w + z \odot w,
\]

where the corresponding commutative inner product \( \cdot \) satisfies

\[
z \cdot w = \frac{1}{2} (zw + wz) = \sum_{r=1}^{2} z_r w_r + \frac{1}{2} e_1 e_2 \left( \overline{z}_1 w_2 - \overline{w}_2 z_1 + \overline{w}_1 z_2 - \overline{z}_2 w_1 \right)
\]

and the corresponding noncommutative outer product \( \odot \) satisfies

\[
z \odot w = \frac{1}{2} (zw - wz) = \frac{1}{2} e_1 e_2 \left( \overline{z}_1 w_2 + \overline{w}_2 z_1 - \overline{w}_1 z_2 - \overline{z}_2 w_1 \right)
\]

The conjugation \( z^* \), the corresponding norm \( |z| \), and the inverse \( z^{-1} \) of \( z \) in \( \mathbb{C}(\mathbb{T}) \) are given by

\[
z^* = e_1 \overline{z}_1 + e_2 \overline{z}_2, \quad |z|^2 = zz^* = z \cdot z^* = \sum_{r=1}^{2} z_r \overline{z}_r, \quad z^{-1} = \frac{z^*}{|z|^2} \quad (z \neq 0).
\]

For any element \( z \) in \( \mathbb{C}(\mathbb{T}) \), we have the corresponding exponential function \( e^z \) denoted by

\[
\exp(z) = \exp(e_1 z_1 + e_2 z_2).
\]

**Theorem 1.** Let \( z \) be an arbitrary number in \( \mathbb{C}(\mathbb{T}) \). Then the corresponding exponential function \( \exp(z) \) of \( z \) in \( \mathbb{C}(\mathbb{T}) \) is given as

\[
\exp(z) = \begin{cases} (-1)^k \exp(x_0) \exp(e_2 x_2), & \text{if } x_1 = k \pi, \\ (-1)^t \exp(x_0) \exp(e_1 x_1), & \text{if } x_2 = t \pi, \end{cases}
\]

where \( k, t \in \mathbb{Z} \).

Furthermore, as hyperbolic functions, one has

\[
\exp(z) = \begin{cases} (-1)^k \exp(e_2 x_2) (\cosh(x_2) - \sinh(x_0)), & \text{if } x_1 = k \pi, \\ (-1)^t \exp(e_1 x_1) (\cosh(x_0) - \sinh(x_2)), & \text{if } x_2 = t \pi, \end{cases}
\]

where \( k, t \in \mathbb{Z} \).

**Proof.** For any element \( z = e_1 z_1 + e_2 z_2 \) of \( \mathbb{C}(\mathbb{T}) \),

\[
\exp(z) = \exp(e_1 z_1 + e_2 z_2) = \exp(e_1 z_1) \exp(e_2 z_2).
\]

Since a scalar part of \( e_1 z_1 \) is \( (1/2) x_0 \), a vector part of \( e_1 z_1 \) is \( e_1 x_1 \), and \( |e_1| = 1 \), by [19],

\[
\exp(e_1 z_1) = \exp \left( \frac{x_0}{2} \left\{ \cos(|e_1 x_1|) + \frac{e_1 x_1}{|e_1 x_1|} \sin(|e_1 x_1|) \right\} \right)
\]

\[
= \exp \left( \frac{x_0}{2} \{ \cos(x_1) + e_1 \sin(x_1) \} \right)
\]

and, similarly, we have

\[
\exp(e_2 z_2) = \exp \left( \frac{x_0}{2} \{ \cos(x_2) + e_2 \sin(x_2) \} \right)
\]

Then we have

\[
\exp(z) = \exp \left( \frac{x_0}{2} \{ \cos(x_1) + e_1 \sin(x_1) \} \right)
\]

\[
\times \exp \left( \frac{x_0}{2} \{ \cos(x_2) + e_2 \sin(x_2) \} \right)
\]

\[
= \exp(x_0) \{ \cos(x_1) + e_1 \sin(x_1) \}
\]

\[
\times \{ \cos(x_2) + e_2 \cos(x_2) \}
\]

\[
\times \sin(x_2) + e_1 \sin(x_1) \cos(x_2)
\]

\[
+ \exp(x_0) e_1 e_2 \sin(x_1) \sin(x_2).
\]

Also, we obtain

\[
\exp(z) = \exp(e_2 z_2 + e_1 z_1)
\]

\[
= \exp(e_2 z_2) \exp(e_1 z_1)
\]

\[
= \exp(x_0) \{ \cos(x_2) + e_2 \sin(x_2) \}
\]

\[
\times \{ \cos(x_1) + e_1 \sin(x_1) \}
\]

\[
= \exp(x_0) \{ \cos(x_1) \cos(x_2) + e_2 \cos(x_1) \}
\]

\[
\times \sin(x_2) + e_1 \sin(x_1) \cos(x_2)
\]

\[
+ \exp(x_0) e_1 e_2 \sin(x_1) \sin(x_2).
\]

Since (15) has to be equal to (14), \( \sin(x_1) \sin(x_2) = 0 \), that is, \( \sin(x_1) = 0 \) or \( \sin(x_2) = 0 \). Therefore, \( x_1 = k \pi \) or \( x_2 = t \pi \), and then \( \cos(x_1) = (-1)^k \) or \( \cos(x_2) = (-1)^t \), where \( k, t \in \mathbb{Z} \).

If \( x_1 = k \pi \) (\( k \in \mathbb{Z} \)), then

\[
\exp(z) = \exp(x_0) \left\{ (-1)^k (\cos(x_2) + e_2 \sin(x_2)) \right\}
\]

\[
= (-1)^k \exp(x_0) \exp(e_2 x_2).
\]
Similarly, if \( x_2 = t\pi \ (t \in \mathbb{Z}) \), then
\[
\exp(z) = \exp(x_0) \left\{ (-1)^t (\cos(x_1) + e_1 \sin(x_1)) \right\} = (-1)^t \exp(x_0) \exp(e_1 x_1).
\]

Further, by the Euler formula and the addition rule of trigonometric functions,
\[
\exp(z) = \exp(e_1 z_1 + e_2 z_2) \exp(-e_1 z_1) \exp(e_2 z_2) = (\cos(z_1) + e_1 \sin(z_1)) (\cos(z_2) + e_2 \sin(z_2))
\]
\[
= \left\{ \cos(x_1) \cos(e_1 x_0) + \sin(x_1) \sin(e_1 x_0) \right\}
\]
\[
+ e_1 \left( \sin(x_1) \cos(e_1 x_0) - \cos(x_1) \sin(e_1 x_0) \right) \}
\]
\[
\cdot \left\{ \cos(x_2) \cos(e_2 x_0) + \sin(x_2) \sin(e_2 x_0) \right\}
\]
\[
+ e_2 \left( \sin(x_2) \cos(e_2 x_0) - \cos(x_2) \sin(e_2 x_0) \right) \}
\]
\[
\cdot \left\{ \cos(x_1) \cos(e_2 x_0) + \sin(x_1) \sin(e_2 x_0) \right\}.
\]

Since \( \cos(e_r(x_0/2)) = \cosh(x_0/2) \) and \( \sin(e_r(x_0/2)) = e_r \sinh(x_0/2) \) \((r = 1, 2)\), we have
\[
\exp(z) = \left\{ \cos(x_1) \cosh(x_0/2) + e_1 \sin(x_1) \sinh(x_0/2) \right\}
\]
\[
+ e_1 \left( \sin(x_1) \cosh(x_0/2) - e_1 \cos(x_1) \sinh(x_0/2) \right) \}
\]
\[
\cdot \left\{ \cos(x_2) \cosh(x_0/2) + e_2 \sin(x_2) \sinh(x_0/2) \right\}
\]
\[
+ e_2 \left( \sin(x_2) \cosh(x_0/2) - e_2 \cos(x_2) \sinh(x_0/2) \right) \}
\]
\[
\cdot \left\{ \cos(x_1) \cosh(x_0/2) + e_1 \sin(x_1) \sinh(x_0/2) \right\}.
\]

Since \((22)\) has to be equal to \((21)\), \( \sin(x_1) \sin(z_1) = 0 \), that is, \( \sin(x_1) = 0 \) or \( \sin(z_1) = 0 \). Therefore, \( x_1 = k\pi \) or \( x_2 = t\pi \), and then \( \cos(x_1) = (-1)^k \) or \( \cos(x_2) = (-1)^t \), where \( k, t \in \mathbb{Z} \). If \( x_1 = k\pi \ (k \in \mathbb{Z}) \), then
\[
\exp(z) = (\cos(x_1) + e_1 \sin(x_1)) (\cosh(x_0/2) - \sinh(x_0/2))
\]
\[
\times (-1)^k (\cosh(x_0) - \sinh(x_0)) \]
\[
= (-1)^k \exp(e_2 x_2) (\cosh(x_0) - \sinh(x_0)).
\]

Similarly, if \( x_2 = t\pi \ (t \in \mathbb{Z}) \), then
\[
\exp(z) = (\cos(x_1) + e_1 \sin(x_1)) (\cosh(x_0/2) - \sinh(x_0/2))
\]
\[
\times (-1)^t (\cosh(x_0) - \sinh(x_0)) \]
\[
= (-1)^t \exp(e_1 x_1) (\cosh(x_0) - \sinh(x_0)).
\]

Remark 2. By Theorem 1 and the properties of the Euler formula, if \( x_1 = k\pi \), then we can write
\[
\exp(z) = (-1)^k \exp(e_2 x_2) (\cosh(x_0) - \sinh(x_0))
\]
\[
= (-1)^k \exp(e_2 x_2 - x_0) = (-1)^k \exp(e_2 F_2).
\]
also, if \( x_2 = t \pi \), then
\[
\exp(z) = (-1)^k \exp(e_1 x_1)(\cosh(x_0) - \sinh(x_0)) = (-1)^k \exp(e_1 \bar{F}_1), \tag{26}
\]
where \( k, t \in \mathbb{Z} \) and \( \bar{F}_1 = x_r + e_2 x_0 \) are the conjugate Fueter variables of \( F_r = x_r - e_2 x_0 \) (see [20]).

Let \( \Omega \) be an open subset of \( \mathbb{R}^3 \) and let a function \( f(a) \) be defined by the following form on \( \Omega \) with values in \( C(T) \):
\[
f : \Omega \rightarrow C(T), \tag{27}
\]
satisfying
\[
a = (x_0, x_1, x_2) \in \Omega \mapsto f(a) = e_1 f_1(x_0, x_1, x_2) + e_2 f_2(x_0, x_1, x_2) \in C(T), \tag{28}
\]
where \( f_r = u_r - (1/2) e_r \mu_0, \bar{f}_r = u_r + (1/2) e_r \mu_0 \) (\( r = 1, 2 \)) and \( u_m \) (\( m = 0, 1, 2 \)) are real-valued functions.

From the chain rule, we use the following differential operators:
\[
\frac{\partial}{\partial A} := 2 \frac{\partial}{\partial x_0} - \frac{1}{2} e_1 \frac{\partial}{\partial x_1} - \frac{1}{2} e_2 \frac{\partial}{\partial x_2} = -e_1 \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial z_2},
\]
\[
\frac{\partial}{\partial A^*} = 2 \frac{\partial}{\partial x_0} + \frac{1}{2} e_1 \frac{\partial}{\partial x_1} + \frac{1}{2} e_2 \frac{\partial}{\partial x_2} = e_1 \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial z_2},
\]
\[
\frac{\partial}{\partial \bar{A}} = 2 \frac{\partial}{\partial x_0} - e_1 \frac{\partial}{\partial x_1} - e_2 \frac{\partial}{\partial x_2},
\]
in \( C(T) \). We have the following equations:
\[
f_r \frac{\partial}{\partial z_r} = \frac{\partial f_r}{\partial z_r}, \quad f_r \frac{\partial}{\partial \bar{z}_r} = \frac{\partial f_r}{\partial \bar{z}_r} \quad (r = 1, 2), \tag{31}
\]
and then, the operator \( \partial / \partial A \) operates to \( f \) as follows:
\[
\frac{\partial f}{\partial A} \ = \ \left( -e_1 \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial z_2} \right) (e_1 f_1 + e_2 f_2) = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + e_1 e_2 \left( \frac{\partial f_1}{\partial z_2} - \frac{\partial f_2}{\partial z_1} \right),
\]
\[
\frac{\partial f}{\partial A^*} = \left( e_1 \frac{\partial}{\partial z_1} + e_2 \frac{\partial}{\partial z_2} \right) (e_1 f_1 + e_2 f_2) = -\frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2} + e_1 e_2 \left( \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2} \right),
\]
\[
f \frac{\partial}{\partial A} = (e_1 f_1 + e_2 f_2) \left( -e_1 \frac{\partial}{\partial z_1} - e_2 \frac{\partial}{\partial z_2} \right) = f_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2} + e_1 e_2 \left( f_2 \frac{\partial}{\partial z_1} - f_1 \frac{\partial}{\partial z_2} \right) = \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + e_1 e_2 \left( \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2} \right), \tag{32}
\]
Thus, we have a corresponding Laplacian in the reduced quaternion \( C(T) \):
\[
\Delta_n = \frac{\partial^2}{\partial A \partial A^*} = \frac{\partial^2}{\partial A \partial A^*} = 4 \frac{\partial^2}{\partial x_0^2} + 1 \frac{\partial^2}{\partial x_1^2} + 1 \frac{\partial^2}{\partial x_2^2}. \tag{33}
\]

**Remark 3.** Let \( \Omega \) be an open set of \( \mathbb{R}^3 \). From the definition of the differential operators in \( C(T) \), we have
\[
\frac{\partial}{\partial A} \cdot f = \frac{1}{2} \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + f_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2} \right) + \frac{1}{2} e_1 e_2 \left( f_2 \frac{\partial}{\partial z_1} - f_1 \frac{\partial}{\partial z_2} \right) = f_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2} + e_1 e_2 \left( f_2 \frac{\partial}{\partial z_1} - f_1 \frac{\partial}{\partial z_2} \right) \times \left( \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2} + \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2} \right),
\]
\[
\frac{\partial}{\partial \bar{A}} \circ f = \frac{1}{2} \left( f_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2} - f_1 \frac{\partial}{\partial z_1} - f_2 \frac{\partial}{\partial z_2} \right) + \frac{1}{2} e_1 e_2 \left( f_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2} - f_1 \frac{\partial}{\partial z_1} - f_2 \frac{\partial}{\partial z_2} \right) = \frac{1}{2} e_1 e_2 \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} \right) \times \left( \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_2} + \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2} \right),
\]
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\[
\frac{\partial}{\partial A^*} \circ f = \frac{1}{2} \left( \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + f_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2} \right) \\
+ \frac{1}{2} e_1 e_2 \left( \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_1} + f_2 \frac{\partial}{\partial z_2} - \frac{\partial f_2}{\partial z_2} \right) \\
= - \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2} + \frac{1}{2} e_1 e_2 \\
\times \left\{ \frac{\partial f_2}{\partial z_1} \left( \frac{\partial f_1}{\partial z_2} + \frac{\partial f_1}{\partial z_1} - \frac{\partial f_1}{\partial z_2} \right) \right\}, \\
\frac{\partial}{\partial A} \circ f = \frac{1}{2} \left( - \frac{\partial f_1}{\partial z_1} - \frac{\partial f_2}{\partial z_2} + f_1 \frac{\partial}{\partial z_1} - f_2 \frac{\partial}{\partial z_2} \right) \\
+ \frac{1}{2} e_1 e_2 \left( \frac{\partial f_2}{\partial z_1} - \frac{\partial f_1}{\partial z_1} + f_2 \frac{\partial}{\partial z_2} - \frac{\partial f_2}{\partial z_2} \right) \\
= \frac{1}{2} e_1 e_2 \left\{ \frac{\partial f_2}{\partial z_1} \left( \frac{\partial f_1}{\partial z_2} + \frac{\partial f_1}{\partial z_1} - \frac{\partial f_1}{\partial z_2} \right) \right\}
\]

and, therefore,

\[
\frac{\partial f}{\partial A} = \frac{\partial}{\partial A^*} \circ f + \frac{\partial}{\partial A^*} \circ f, \quad \frac{\partial f}{\partial A} = \frac{\partial}{\partial A^*} \circ f \circ f.
\]

Similarly, we have

\[
f \circ \frac{\partial}{\partial A} = \frac{1}{2} \left( f_1 \frac{\partial}{\partial z_1} + f_2 \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right) \\
+ e_1 e_2 \left( f_1 \frac{\partial}{\partial z_2} - f_2 \frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_1} \right) \\
= \frac{\partial f_1}{\partial z_1} + \frac{\partial f_2}{\partial z_2} + \frac{1}{2} e_1 e_2 \\
\times \left\{ \frac{\partial f_2}{\partial z_1} \left( \frac{\partial f_1}{\partial z_2} + \frac{\partial f_1}{\partial z_1} - \frac{\partial f_1}{\partial z_2} \right) \right\}, \\
f \circ \frac{\partial}{\partial A^*} = \frac{1}{2} \left( - f_1 \frac{\partial}{\partial z_1} - f_2 \frac{\partial}{\partial z_2} - \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \right) \\
+ \frac{1}{2} e_1 e_2 \left( f_1 \frac{\partial}{\partial z_2} - f_2 \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} + \frac{\partial}{\partial z_1} \right) \\
= \frac{1}{2} e_1 e_2 \left\{ \frac{\partial f_2}{\partial z_1} \left( \frac{\partial f_1}{\partial z_2} + \frac{\partial f_1}{\partial z_1} - \frac{\partial f_1}{\partial z_2} \right) \right\}, \quad (34)
\]

Definition 4. Let \( \Omega \) be an open set in \( \mathbb{R}^3 \) and for any element \( a \) in \( \Omega \). A function \( f(a) \) is said to be \( L(R) \)-regular on \( \Omega \) if the following conditions are satisfied:

(i) \( f(r = 1, 2) \) are continuously differential functions on \( \Omega \), and 
(ii) \( df(a)/\partial A^* = 0 \) (\( f(a)(\partial/\partial A^*) = 0 \)) on \( \Omega \).

In particular, the equation \( df/\partial A^* = 0 \) of Definition 4 is equivalent to

\[
\frac{\partial}{\partial A^*} \circ f = - \frac{\partial}{\partial A^*} \circ f. \quad (38)
\]

Moreover, (38) is equivalent to the following system:

\[
\frac{\partial f_1}{\partial z_1} = - \frac{\partial f_2}{\partial z_2}. \quad (39)
\]

The above system is a corresponding Cauchy-Riemann system in \( C(T) \).

Remark 5. From the multiplications of \( C(T) \), the equation \( f(\partial/\partial A^*) = 0 \) of Definition 4 is equivalent to

\[
\frac{\partial}{\partial A^*} \circ f = \frac{\partial}{\partial A^*} \circ f. \quad (40)
\]

Also, the above equation (40) is equivalent to the following system:

\[
\frac{\partial f_1}{\partial z_1} = - \frac{\partial f_2}{\partial z_2}. \quad (41)
\]

Further, the above system (41) is also a corresponding Cauchy-Riemann system in \( C(T) \). Since the system (39) is equivalent to the system (41), we say that \( f(a) \) of Definition 4 is a regular function on \( \Omega \subset \mathbb{R}^3 \). When the function \( f(a) \) is either an \( L \)-regular function or an \( R \)-regular function on \( \Omega \subset \mathbb{R}^3 \), we simply say that \( f(a) \) is a regular function on \( \Omega \subset \mathbb{R}^3 \).
3. Properties of Regular Functions with Values in $\mathbb{C}(\mathbb{T})$

We define the derivative $f'(a)$ of $f(a)$ by the following:

$$f'(a) := \frac{\partial f}{\partial A}(a).$$  \ (42)

**Proposition 6.** Let $\Omega$ be an open set in $\mathbb{R}^3$ and let a function $f(a)$ be a regular function defined on $\Omega$. Then

$$f'(a) = -2e_r\left(\frac{\partial f}{\partial z_r} - \frac{\partial f}{\partial x_r}\right) = 4\frac{\partial f}{\partial x_0}$$

$$= -e_1\frac{\partial f}{\partial x_1} - e_2\frac{\partial f}{\partial x_2} \quad (r = 1, 2).$$  \ (43)

**Proof.** From the definition of a regular function $\vdots (\partial f/\partial A^*) = 0$, we have

$$\frac{\partial f}{\partial A} = \frac{\partial f}{\partial A^*} = \frac{\partial f}{\partial z_r} - \frac{\partial f}{\partial x_r}.$$  \ (44)

Therefore,

$$\frac{\partial}{\partial z} \cdot f = \frac{\partial f}{\partial z_1} + 2e_1\frac{\partial u_1}{\partial x_0} + 2e_2\frac{\partial u_2}{\partial x_0} + \frac{\partial f}{\partial z_2} + 2e_1\frac{\partial u_1}{\partial x_0} + 2e_2\frac{\partial u_2}{\partial x_0} + \frac{\partial f}{\partial z_1} + 2e_1\frac{\partial u_1}{\partial x_0} + 2e_2\frac{\partial u_2}{\partial x_0}$$

$$+ \frac{1}{2}e_1e_2\left(\frac{\partial f}{\partial z_1} + e_1\frac{\partial u_1}{\partial x_0} + e_2\frac{\partial u_2}{\partial x_0} - \frac{\partial f}{\partial z_2}\right)$$

$$= 4\frac{\partial f}{\partial x_0} + \frac{1}{2}e_1e_2\left(\frac{\partial f}{\partial z_1} - \frac{1}{2}e_1e_2\left(\frac{\partial f}{\partial z_2}\right), \right.$$  \ (45)

$$\frac{\partial}{\partial z} \circ f = \frac{1}{2}e_1e_2$$

$$\times \left(\frac{\partial}{\partial z_1} + 2e_1\frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_0} - 2e_2\frac{\partial f}{\partial x_0} - \frac{\partial f}{\partial z_1} + 2e_1\frac{\partial f}{\partial x_0} - 2e_2\frac{\partial f}{\partial x_0} - \frac{\partial f}{\partial z_1}\right)$$

$$= 2e_1e_2\left(e_1\frac{\partial f}{\partial x_0} - e_1\frac{\partial f}{\partial x_0} - e_2\frac{\partial f}{\partial x_0} + e_2\frac{\partial f}{\partial x_0}\right)$$

$$= -\frac{1}{2}e_1e_2\left(\frac{\partial f}{\partial z_1} + \frac{1}{2}e_1e_2\frac{\partial f}{\partial z_2} = 0. \right.$$  \ (46)

Similarly, by calculating the derivative $f'(z)$ of $f(z)$,

$$\frac{\partial}{\partial z} \cdot f = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}$$

$$+ \frac{1}{2}e_1e_2\left(\frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2} - \frac{\partial f}{\partial x_1} - \frac{\partial f}{\partial x_2}\right),$$

$$\frac{\partial}{\partial z} \circ f = \left(e_1\frac{\partial f}{\partial x_1} - e_1\frac{\partial f}{\partial x_2} - e_2\frac{\partial f}{\partial x_2} + e_2\frac{\partial f}{\partial x_2}\right)$$

$$- \frac{1}{2}e_1e_2\left(\frac{\partial f}{\partial x_1} - \frac{1}{2}e_1e_2\frac{\partial f}{\partial x_2}.\right.$$  \ (47)

Therefore, we have the equation

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} \cdot f + \frac{\partial}{\partial z} \circ f = -e_1\left(e_1\frac{\partial f}{\partial x_1} + e_2\frac{\partial f}{\partial x_2}\right)$$

$$- e_2\left(e_2\frac{\partial f}{\partial x_2} + e_1\frac{\partial f}{\partial x_1}\right) = -e_1\frac{\partial f}{\partial x_1} - e_2\frac{\partial f}{\partial x_2}. \quad \square \right.$$  \ (48)

**Proposition 7.** Let $\Omega$ be an open set in $\mathbb{R}^3$. If $f(a)$ is a regular function on $\Omega$, then we have

$$\frac{\partial^n f}{\partial A^n} = 4^n \frac{\partial^n f}{\partial x_0^n},$$

where $n$ is a positive integer.

**Proof.** Since $f$ is a regular function on $\Omega$ with values in $\mathbb{C}(\mathbb{T})$, by Definition 4,

$$\frac{\partial}{\partial A^*} \left(4 \frac{\partial f}{\partial x_0}\right) = 4 \frac{\partial f}{\partial A} \left(\frac{\partial f}{\partial A^*}\right) = 0.$$  \ (51)

Hence, $\partial f/\partial x_0$ is a regular function with values in $\mathbb{C}(\mathbb{T})$. From Proposition 6, we have

$$\frac{\partial^2 f}{\partial A^2} = \frac{\partial}{\partial A} \left(4 \frac{\partial f}{\partial x_0}\right) = 4^2 \frac{\partial^2 f}{\partial x_0^2}. \quad \square \right.$$  \ (52)

By repeating the above process, we can obtain the equation

$$\frac{\partial^n f}{\partial A^n} = 4^n \frac{\partial^n f}{\partial x_0^n}. \quad \square \right.$$  \ (53)

We let

$$\square_a = \sum_{r=1}^2 \frac{\partial^2}{\partial z_r \partial z_r} = 2 \frac{\partial^2}{\partial x_0^2} + \frac{1}{4}e_1 \frac{\partial^2}{\partial x_1^2} + \frac{1}{4}e_2 \frac{\partial^2}{\partial x_2^2} \right.$$  \ (54)

on an open set $\Omega$ in $\mathbb{R}^3$.
Theorem 8. Let $\Omega$ be an open set in $\mathbb{R}^3$. If $f$ is a regular function on $\Omega$, then the following equation holds true:

$$\Box_\omega f(a) = -\frac{1}{8} \frac{\partial^2 f(a)}{\partial A^2}. \quad (55)$$

Proof. Since $f$ is a regular function on $\Omega$, we have the following system:

$$\begin{align*}
4 \frac{\partial u_0}{\partial x_0} &= \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}, \\
4 \frac{\partial u_0}{\partial x_0} &= -\frac{\partial u_0}{\partial x_r} \quad (r = 1, 2). \quad (56)
\end{align*}$$

By the definition of $\Box_\omega$, we have

$$\Box_\omega f = \left( \frac{\partial^2}{\partial x_0^2} + \frac{1}{4} e_1 \frac{\partial^2}{\partial x_1^2} + \frac{1}{4} e_2 \frac{\partial^2}{\partial x_2^2} \right) (u_0 + e_1 u_1 + e_2 u_2)$$

$$= 2 \frac{\partial^2 u_0}{\partial x_0^2} + 2e_1 \frac{\partial^2 u_1}{\partial x_0^2} + 2e_2 \frac{\partial^2 u_2}{\partial x_0^2} - \frac{\partial^2 u_1}{\partial x_0 \partial x_1}$$

$$+ e_1 \frac{\partial^2 u_0}{\partial x_0 \partial x_1} - \frac{\partial^2 u_2}{\partial x_0 \partial x_2} + e_2 \frac{\partial^2 u_0}{\partial x_0 \partial x_2}$$

$$= -2 \frac{\partial^2 u_0}{\partial x_0^2} - 2e_1 \frac{\partial^2 u_1}{\partial x_0^2} - 2e_2 \frac{\partial^2 u_2}{\partial x_0^2} = -\frac{\partial^2 f}{\partial A^2}. \quad (57)$$

From Proposition 7, we have $\partial^3 f / \partial A^2 = 4^2 (\partial^2 f / \partial x_0^2)$. Hence, by calculating and comparing the above polynomials, we obtain that $\Box_\omega f$ is equal to $-(1/8) (\partial^3 f / \partial A^2) f$. \qed

Next, we consider a differential form

$$\omega = 4 dx_1 \wedge dx_2 - e_1 dx_0 \wedge dx_2 + e_2 dx_0 \wedge dx_1. \quad (58)$$

Theorem 9. Let $\Omega$ be an open set in $\mathbb{R}^3$ and let $U$ be any domain on $\Omega$ with a smooth distinguished boundary $bU$ such that $U \subset \Omega$. If $f$ is a regular function on $\Omega$, then one has

$$\int_{bU} \omega f = 0, \quad (59)$$

where $\omega f$ is the reduced quaternionic product of the form $\omega$ on the function $f(a)$. \hspace{1cm} \Box

Proof. Since $\omega f = 4 dx_1 \wedge dx_2 - e_1 dx_0 \wedge dx_2 + e_2 dx_0 \wedge dx_1$, we have

$$d (\omega f) = 4 \frac{\partial f}{\partial x_0} dx_0 \wedge dx_1 \wedge dx_2 + e_1 \frac{\partial f}{\partial x_1} dx_0 \wedge dx_1 \wedge dx_2$$

$$+ e_2 \frac{\partial f}{\partial x_2} dx_0 \wedge dx_1 \wedge dx_2$$

$$= 4 \frac{\partial (e_1 f_1 + e_2 f_2)}{\partial x_0} dI + e_1 \frac{\partial (e_1 f_1 + e_2 f_2)}{\partial x_1} dI$$

$$+ e_2 \frac{\partial (e_1 f_1 + e_2 f_2)}{\partial x_2} dI$$

$$= \left\{ \left( \frac{4 \partial u_0}{\partial x_0} - \frac{\partial u_0}{\partial x_1} - \frac{\partial u_0}{\partial x_2} \right) + e_1 \left( \frac{4 \partial u_1}{\partial x_0} + \frac{\partial u_1}{\partial x_1} \right) \right. \right.$$}

$$+ e_2 \left( \frac{4 \partial u_2}{\partial x_0} + \frac{\partial u_2}{\partial x_1} \right) + e_1 e_2 \left( \frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \right\} dI, \quad (60)$$

where $dI = dx_0 \wedge dx_1 \wedge dx_2$ in $U$. From the corresponding Cauchy-Riemann system (39) for $f(a)$ in $C(T)$, we have the system (56). Hence, $d(\omega f) = 0$ and, therefore, by Stokes theorem, we obtain the following result:

$$\int_{bU} \omega f = \int_U d (\omega f) = 0. \quad (61) \ Box$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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