Research Article

Existence Results for a Coupled System of Nonlinear Fractional Hybrid Differential Equations with Homogeneous Boundary Conditions

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We study an existence result for the following coupled system of nonlinear fractional hybrid differential equations with homogeneous boundary conditions

\[\begin{align*}
&D_{\alpha}0^+[x(t)/f(t, x(t), y(t))] = g(t, x(t), y(t)),
&D_{\alpha}0^+[y(t)/f(t, y(t), x(t))] = g(t, y(t), x(t)),
\end{align*}\]

where \(0 < t < 1\), and \(x(0) = y(0) = 0\), \(\alpha \in (0, 1)\) and \(D_{\alpha}0^+\) denotes the Riemann-Liouville fractional derivative. The main tools in our study are the techniques associated to measures of noncompactness in the Banach algebras and a fixed point theorem of Darbo type.

1. Introduction

Fractional differential equations arise in many engineering and scientific disciplines as the mathematical modelling of a great number of processes which appear in physics, chemistry, aerodynamics, and so forth and involve also derivatives of fractional order. For details, see [1–5] and the references therein.

On the other hand, about the theory of hybrid differential equations, we refer to the paper [6] where the authors studied the hybrid differential equation of first order:

\[\begin{align*}
\frac{d}{dt}\left[\frac{x(t)}{f(t, x(t))}\right] &= g\left(t, x(t)\right), \quad t \in [0, T),
\end{align*}\]

\[x(t_0) = x_0 \in \mathbb{R},\]

where \(f \in C(J \times \mathbb{R}, \mathbb{R} \setminus \{0\})\) and \(g \in C(J \times \mathbb{R}, \mathbb{R}).\)

In [7], the authors studied the fractional version of the abovementioned problem, that is,

\[\begin{align*}
&D_{\alpha}0^+\left[\frac{x(t)}{f(t, x(t))}\right] = g\left(t, x(t)\right), \quad t \in J, \quad 0 < \alpha < 1,
\end{align*}\]

\[x(0) = 0,\]

under the same assumptions on \(f\) and \(g\) in [6].

Recently, in [8], the authors studied the following fractional hybrid initial value problem with supremum:

\[\begin{align*}
&D_{\alpha}0^+\left[\frac{x(t)}{f(t, x(t), \max_{0 \leq \tau \leq t} |x(\tau)|)}\right] = g\left(t, x(t)\right),
\end{align*}\]

\[0 < t < 1,\]

\[x(0) = 0,\]

where \(0 < \alpha < 1, f \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R} \setminus \{0\})\), and \(g \in C([0, 1] \times \mathbb{R}, \mathbb{R}).\)
The coupled systems involving fractional differential equations are very important because they occur in numerous problems of applied nature; for instance, see [9–13]. In this paper, we consider the following coupled system:

\[ D^{\alpha}_{0^+} \left( \frac{x(t)}{f(t, x(t), y(t))} \right) = g(t, x(t), y(t)), \]

\[ D^{\alpha}_{0^+} \left( \frac{y(t)}{f(t, y(t), x(t))} \right) = g(t, y(t), x(t)), \]  

where \( \alpha \in (0, 1) \) and \( D^{\alpha}_{0^+} \) is the standard Riemann–Liouville fractional derivative.

The main tool in our study is a fixed point theorem of Darbo type associated to measures of noncompactness.

## 2. Preliminaries

We begin this section with some definitions and results about fractional calculus.

Let \( \alpha > 0 \) and \( n = \lfloor \alpha \rfloor + 1 \), where \( \lfloor \alpha \rfloor \) denotes the integer part of \( \alpha \). For a function \( f : (0, \infty) \to \mathbb{R} \), the Riemann–Liouville fractional integral of order \( \alpha > 0 \) of \( f \) is defined as

\[ I^{\alpha}_{0^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_{0}^{t} \frac{f(s)}{(t-s)^{1-\alpha}} ds, \]

provided that the right side is pointwise defined on \( (0, \infty) \).

The Riemann–Liouville fractional derivative of order \( \alpha \) of a continuous function \( f \) is defined by

\[ D^{\alpha}_{0^+} f(x) = \frac{d^{n}}{dx^n} I^{\alpha-n}_{0^+} f(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \int_{0}^{x} \frac{f(s)}{(x-s)^{n-\alpha}} ds, \]

provided that the right side is pointwise defined on \( (0, \infty) \).

The following lemma will be useful for our study, [14].

**Lemma 1.** Let \( h \in L^{1}(0, 1) \) and \( 0 < \alpha < 1 \). Then,

(a)

\[ D^{\alpha}_{0^+} I^{\alpha}_{0^+} h(x) = h(x); \]

(b)

\[ I^{\alpha}_{0^+} D^{\alpha}_{0^+} h(x) = h(x) - \frac{I^{1-\alpha}_{0^+} h(x)}{\Gamma(\alpha)} x^{\alpha-1}, \]

a.e. on \( (0, 1) \).

**Lemma 2.** Let \( 0 < \alpha < 1 \) and suppose that \( f \in C([0, 1], \mathbb{R}\setminus\{0\}) \) and \( y \in C[0, 1] \). Then, the unique solution of the fractional hybrid initial value problem

\[ D^{\alpha}_{0^+} \left( \frac{x(t)}{f(t)} \right) = y(t), \quad 0 < t < 1 \]

\[ x(0) = 0, \]

is given by

\[ x(t) = f(t) \int_{0}^{t} \frac{y(s)}{(t-s)^{1-\alpha}} ds, \quad t \in [0, 1]. \]

**Proof.** Suppose that \( x(t) \) is a solution of problem (9). Using the operator \( I^{\alpha}_{0^+} \) and taking into account Lemma 1, we get

\[ I^{\alpha}_{0^+} D^{\alpha}_{0^+} \left( \frac{x(t)}{f(t)} \right) = I^{\alpha}_{0^+} y(t), \]

or, equivalently,

\[ \frac{x(t)}{f(t)} = \frac{I^{1-\alpha}_{0^+} (x(t)/f(t))}{\Gamma(\alpha)} = I^{\alpha}_{0^+} y(t). \]

Since \( x(t)/f(t)|_{t=0} = x(0)/f(0) = 0/0 = 0 \) (because \( f(0) \neq 0 \)), we have

\[ x(t) = f(t) I^{\alpha}_{0^+} y(t). \]

This means that

\[ x(t) = f(t) I^{\alpha}_{0^+} y(t), \quad t \in [0, 1]. \]

Conversely, suppose that \( x(t) \) is given by

\[ x(t) = f(t) I^{\alpha}_{0^+} y(t), \quad t \in [0, 1]. \]

This means that

\[ x(t) = f(t) I^{\alpha}_{0^+} y(t), \quad t \in [0, 1]. \]

Applying \( D^{\alpha}_{0^+} \) and taking into account Lemma 1 and that \( f(t) \neq 0 \) for \( t \in [0, 1] \), we obtain

\[ D^{\alpha}_{0^+} \left( \frac{x(t)}{f(t)} \right) = D^{\alpha}_{0^+} I^{\alpha}_{0^+} y(t) = y(t), \quad 0 < t < 1. \]

Moreover, for \( t = 0 \) in (16), we have \( x(0) = f(0) \cdot 0 = 0 \). This completes the proof. \( \Box \)

In the sequel, we recall some definitions and basic facts about measures of noncompactness.

Assume that \( E \) is a real Banach space with norm \( \| \cdot \| \) and zero element \( \theta \). By \( B(x, r) \) we denote the closed ball in \( E \) centered at \( x \) with radius \( r \). By \( B \), we denote the ball \( B(\theta, r) \). If \( X \) is a nonempty subset of \( E \), by the symbols \( \overline{X} \) and \( \text{Conv}X \) we denote the closure and the convex closure of \( X \), respectively. By \( \| X \| \) we denote the quantity \( \| X \| = \sup \{ \| x \| : x \in X \} \). Finally, by \( \mathcal{M}_{E} \) we will denote the family of all nonempty and bounded subsets of \( E \) and by \( \mathcal{M}_{E}^{*} \) we denote its subfamily consisting of all relatively compact subsets of \( E \).

**Definition 3.** A mapping \( \mu : \mathcal{M}_{E} \to \mathbb{R}_{+} = [0, \infty) \) is said to be a measure of noncompactness in \( E \) if it satisfies the following conditions.

(a) The family \( \ker \mu = \{ X \in \mathcal{M}_{E} : \mu(X) = 0 \} \) is nonempty and \( \ker \mu \subset \mathcal{M}_{E} \).
(b) $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

(c) $\mu(\text{Conv}X) = \mu(X)$.

(d) $\mu(\lambda X + (1 - \lambda)Y) \leq \lambda \mu(X) + (1 - \lambda)\mu(Y)$ for $\lambda \in [0, 1]$.

(e) If $(X_n)$ is a sequence of closed subsets from $\mathcal{M}_E$ such that $X_{n+1} \subset X_n$ for $n \geq 1$ and $\lim_{n \to \infty} \mu(X_n) = 0$, then $X_\infty = \cap_{n=1}^\infty X_n \neq \emptyset$.

The family $\ker \mu$ appearing in (a) is called the kernel of the measure of noncompactness $\mu$. Notice that the set $X_\infty$ appearing in (e) is an element of $\ker \mu$. Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for $n = 1, 2, \ldots$, we infer that $\mu(X_\infty) = 0$ and this says that $X_\infty \in \ker \mu$.

An important theorem about fixed point theorem in the context of measures of noncompactness is the following Darbo's fixed point theorem [15].

**Theorem 4.** Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$ and let $T : C \to C$ be a continuous mapping. Suppose that there exists a constant $k < 1$ such that

$$\mu(T(X)) \leq k\mu(X),$$

for any nonempty subset $X$ of $C$.

Then, $T$ has a fixed point.

A generalization of Theorem 4 which will be very useful in our study is the following, due to Sadovskii [16].

**Theorem 5.** Let $C$ be a nonempty, bounded, closed, and convex subset of a Banach space $E$ and let $T : C \to C$ be a continuous operator satisfying

$$\mu(T(X)) < \mu(X),$$

for any nonempty subset $X$ of $C$ with $\mu(X) > 0$.

Then, $T$ has a fixed point.

Next, we will assume that the space $E$ has structure of Banach algebra. By $xy$ we will denote the product of two elements $x, y \in E$ and by $XY$ we will denote the set defined by $XY = \{xy : x \in X, y \in Y\}$.

**Definition 6.** Let $E$ be a Banach algebra. We will say that a measure of noncompactness $\mu$ defined on $E$ satisfies condition (m) if

$$\mu(XY) \leq \|X\| \mu(Y) + \|Y\| \mu(X),$$

for any $X, Y \in \mathcal{M}_E$.

This definition appears in [17].

In this paper, we will work in the space $C[0, 1]$ consisting of all real functions defined and continuous on $[0, 1]$ with the standard supremum norm

$$\|x\| = \sup \{|x(t)| : t \in [0, 1]\},$$

for $x \in C[0, 1]$. It is clear that $(C[0, 1], \|\cdot\|)$ is a Banach algebra, where the multiplication is defined as the usual product of real functions.

Next, we present the measure of noncompactness in $C[0, 1]$ which will be used later. Let us fix $X \in \mathcal{M}_{C[0,1]}$ and $\varepsilon > 0$. For $x \in X$, we denote by $\omega(x, \varepsilon)$ the modulus of continuity of $x$; that is,

$$\omega(x, \varepsilon) = \sup \{|x(t) - x(s)| : t, s \in [0, 1], |t - s| \leq \varepsilon\}. \quad (22)$$

Put

$$\omega(X, \varepsilon) = \sup \{\omega(x, \varepsilon) : x \in X\},$$

$$\omega_0(X) = \lim_{\varepsilon \to 0} \omega(X, \varepsilon). \quad (23)$$

In [15], it is proved that $\omega_0(X)$ is a measure of noncompactness in $C[0, 1]$.

**Proposition 7.** The measure of noncompactness $\omega_0$ on $C[0, 1]$ satisfies condition (m).

**Proof.** Fix $X, Y \in \mathcal{M}_{C[0,1]}$, $\varepsilon > 0$, and $t, s \in [0, 1]$ with $|t - s| \leq \varepsilon$. Then, we have

$$\|xy(t) - y(s) - (xy)(t) - x(s)y(s)\| \leq \|xy(t) - y(s)\| + \|y(s) - x(s)y(s)\|$$

and, therefore,

$$\omega(X, \varepsilon) \leq \|X\| \omega(Y, \varepsilon) + \|Y\| \omega(X, \varepsilon). \quad (26)$$

Taking $\varepsilon \to 0$, we get

$$\omega_0(XY) \leq \|X\| \omega_0(Y) + \|Y\| \omega_0(X). \quad (27)$$

This completes the proof. □

Proposition 7 appears in [17] and we have given the proof for the paper is self-contained.

### 3. Main Results

We begin this section introducing the following class $\mathcal{A}$ of functions:

$$\mathcal{A} = \{ \varphi : \mathbb{R}_+ \to \mathbb{R}_+ : \varphi \text{ is nondecreasing} \}$$

and

$$\lim_{n \to \infty} \varphi^n(t) = 0 \text{ for any } t > 0,$$

where $\varphi^n$ denotes the $n$-iteration of $\varphi$. 
Remark 8. Notice that if \( \varphi \in \mathcal{A} \), then \( \varphi(t) < t \) for any \( t > 0 \). Indeed, in contrary case, we can find \( t_0 > 0 \) and \( t_0 \leq \varphi(t_0) \). Since \( \varphi \) is nondecreasing, we have
\[
0 < t_0 \leq \varphi(t_0) \leq \varphi^2(t_0) \leq \cdots \leq \varphi^n(t_0) \leq \cdots ,
\]
and, therefore, \( 0 < t_0 \leq \lim_{n \to \infty} \varphi^n(t_0) \) and this contradicts the fact that \( \varphi \in \mathcal{A} \).
Moreover, the fact that \( \varphi(t) < t \) for any \( t > 0 \) proves that if \( \varphi \in \mathcal{A} \), then \( \varphi \) is continuous at \( t_0 = 0 \).

Using Remark 8 and Theorem 5, we have the following fixed point theorem.

**Theorem 9.** Let \( C \) be a nonempty, bounded, closed, and convex subset of a Banach space \( E \) and let \( T : C \to C \) be a continuous operator satisfying
\[
\mu(T(X)) \leq \varphi(\mu(X)),
\]
(30)
for any nonempty subset \( X \) of \( C \), where \( \varphi \in \mathcal{A} \). Then, \( T \) has a fixed point.

Theorem 9 appears in [18], where the authors present a proof without using Theorem 5.

The following result which appears in [19] will be interesting in our study.

**Theorem 10.** Let \( \mu_1, \mu_2, \ldots, \mu_n \) be measures of noncompactness in the Banach spaces \( E_1, E_2, \ldots, E_n \), respectively. Suppose that \( F : [0, \infty)^n \to [0, \infty) \) is a convex function such that \( F(x_1, x_2, \ldots, x_n) = 0 \) if and only if \( x_i = 0 \) for \( i = 1, 2, \ldots, n \). Then,
\[
\tilde{\mu}(X) = F(\mu_1(X), \mu_2(X), \ldots, \mu_n(X))
\]
(31)
defines a measure of noncompactness in \( E_1 \times E_2 \times \cdots \times E_n \), where \( X \) denotes the natural projection of \( X \) into \( E_i \), for \( i = 1, 2, \ldots, n \).

Remark 11. As a consequence of Theorem 10, we have that if \( \mu \) is a measure of noncompactness on a Banach space \( E \) and we consider the function \( F : [0, \infty) \times [0, \infty) \to [0, \infty) \) defined by \( F(x, y) = \max(x, y) \), then, since \( F \) is convex and \( F(x, y) = 0 \) if and only if \( x = y = 0 \), \( \tilde{\mu}(X) = \max(\mu(X_1), \mu(X_2)) \) defines a measure of noncompactness in the space \( E \times E \).

Next, we present the definition of a coupled fixed point.

**Definition 12.** An element \( (x, y) \in X \times X \) is said to be a coupled fixed point of a mapping \( G : X \times X \to X \) if \( G(x, y) = x \) and \( G(y, x) = y \).

The following result is crucial for our study.

**Theorem 13.** Let \( \Omega \) be a nonempty, bounded, closed, and convex subset of a Banach space \( E \), and let \( \mu \) be a measure of noncompactness in \( E \). Suppose that \( G : \Omega \times \Omega \to \Omega \) is a continuous operator satisfying
\[
\mu(G(X_1 \times X_2)) \leq \varphi(\max(\mu(X_1), \mu(X_2)))
\]
(32)
for all nonempty subsets \( X_1 \) and \( X_2 \) of \( \Omega \), where \( \varphi \in \mathcal{A} \). Then, \( G \) has at least a coupled fixed point.
respectively, for any \( t \in [0, 1] \) and \( x_1, x_2, y_1, y_2 \in \mathbb{R} \), where \( \varphi_1, \varphi_2 \in \mathcal{A} \) and \( \varphi_1 \) is continuous.

Notice that assumption (\( H_2 \)) gives us the existence of two nonnegative constants \( k_1 \) and \( k_2 \) such that 
\[ |f(t, 0, 0)| \leq k_1 \quad \text{and} \quad |g(t, 0, 0)| \leq k_2, \]
for any \( t \in [0, 1] \).

(\( H_3 \)) There exists \( r_0 > 0 \) satisfying the inequalities
\[
\begin{align*}
(\varphi_1 (r) + k_1) \cdot (\varphi_2 (r) + k_2) & \leq r \Gamma (\alpha + 1), \\
\varphi_2 (r) + k_2 & \geq \Gamma (\alpha + 1).
\end{align*}
\] (36)

**Theorem 15.** Under assumptions (\( \text{H}_1 \))–(\( \text{H}_3 \)), problem (4) has at least one solution in \( C[0, 1] \times C[0, 1] \).

**Proof.** In virtue of Lemma 14, a solution \((x, y) \in C[0, 1] \times C[0, 1] \) of problem (4) satisfies
\[
\begin{align*}
x(t) &= \frac{f(t, x(t), y(t))}{\Gamma (\alpha)} \int_0^t \frac{g(s, x(s), y(s))}{(t-s)^{1-\alpha}} ds, \\
y(t) &= \frac{f(t, y(t), x(t))}{\Gamma (\alpha)} \int_0^t \frac{g(s, y(s), x(s))}{(t-s)^{1-\alpha}} ds,
\end{align*}
\] (37)

\( t \in [0, 1] \).

We consider the space \( C[0, 1] \times C[0, 1] \) equipped with the norm \( \|(x, y)\|_{C[0,1]\times C[0,1]} = \max \{|x|, |y|\} \), for any \((x, y) \in C[0, 1] \times C[0, 1] \).

In \( C[0, 1] \times C[0, 1] \), we define the operator
\[
G(x, y)(t) = \frac{f(t, x(t), y(t))}{\Gamma (\alpha)} \int_0^t \frac{g(s, x(s), y(s))}{(t-s)^{1-\alpha}} ds,
\] (38)

\( t \in [0, 1] \).

Let \( \mathcal{F} \) and \( \mathcal{G} \) be the operators given by
\[
\mathcal{F}(x, y)(t) = f(t, x(t), y(t)),
\]
\[
\mathcal{G}(x, y)(t) = \frac{1}{\Gamma (\alpha)} \int_0^t \frac{g(s, x(s), y(s))}{(t-s)^{1-\alpha}} ds
\] (39)

for any \((x, y) \in C[0, 1] \times C[0, 1] \) and any \( t \in [0, 1] \). Then,
\[
G(x, y) = \mathcal{F}(x, y) \cdot \mathcal{G}(x, y).
\] (40)

Firstly, we will prove that \( G \) applies \( C[0, 1] \times C[0, 1] \) into \( C[0, 1] \). To do this, it is sufficient to prove that \( \mathcal{F}(x, y) \), \( \mathcal{G}(x, y) \in C[0, 1] \) for any \((x, y) \in C[0, 1] \times C[0, 1] \) since the product of continuous functions is continuous.

In virtue of assumption (\( \text{H}_1 \)), it is clear that \( \mathcal{F}(x, y) \in C[0, 1] \) for \((x, y) \in C[0, 1] \times C[0, 1] \). In order to prove that \( \mathcal{G}(x, y) \in C[0, 1] \) for \((x, y) \in C[0, 1] \times C[0, 1] \), we fix \( t_0 \in [0, 1] \) and consider a sequence \( (t_n) \in [0, 1] \) such that \( t_n \to t_0 \), and we have to prove that \( \mathcal{G}(x, y)(t_n) \to \mathcal{G}(x, y)(t_0) \).

Without loss of generality, we can suppose that \( t_n > t_0 \). Then we have
\[
\begin{align*}
\mathcal{G}(x, y)(t_n) - \mathcal{G}(x, y)(t_0) &= \frac{1}{\Gamma (\alpha)} \int_0^{t_n} \left( \frac{g(s, x(s), y(s))}{(t_n-s)^{1-\alpha}} - \frac{g(s, x(s), y(s))}{(t_0-s)^{1-\alpha}} \right) ds \\
&+ \frac{1}{\Gamma (\alpha)} \int_{t_0}^{t_n} \frac{g(s, x(s), y(s))}{(t_0-s)^{1-\alpha}} ds \\
&\leq \frac{1}{\Gamma (\alpha)} \int_0^{t_n} \left| \frac{g(s, x(s), y(s))}{(t_n-s)^{1-\alpha}} - \frac{g(s, x(s), y(s))}{(t_0-s)^{1-\alpha}} \right| ds \\
&+ \frac{M}{\Gamma (\alpha)} \int_{t_0}^{t_n} \left| (t_0-s)^{1-\alpha} - (t_n-s)^{1-\alpha} \right| ds.
\end{align*}
\] (41)

By assumption (\( \text{H}_1 \)), since \( g \in C([0, 1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \), \( g \) is bounded on the compact set \([0, 1] \times [-\|x\|, \|x\|] \times [-\|y\|, \|y\|] \).

Denote by
\[
M = \sup \{ |g(s, x_1, y_1)| : s \in [0, 1], x_1 \in [-\|x\|, \|x\|], y_1 \in [-\|y\|, \|y\|] \}.
\] (42)

From the last estimate, we obtain
\[
\begin{align*}
\mathcal{G}(x, y)(t_n) - \mathcal{G}(x, y)(t_0) &\leq \frac{M}{\Gamma (\alpha)} \int_0^{t_n} \left| (t_0-s)^{1-\alpha} - (t_n-s)^{1-\alpha} \right| ds \\
&+ \frac{M}{\Gamma (\alpha)} \int_{t_0}^{t_n} \left| (t_0-s)^{1-\alpha} \right| ds.
\end{align*}
\] (43)
As \(0 < \alpha < 1\) and \(t_n > t_0\), we infer that
\[
|G(x, y)(t_n) - G(x, y)(t_0)| \leq \frac{M}{\Gamma(\alpha)} \left[ \int_0^{t_0} |(t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1}| \, ds + \int_{t_0}^{t_n} |(t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1}| \, ds \right]
\]
\[
+ \frac{M}{\Gamma(\alpha)} \int_{t_0}^{t_n} \frac{1}{(s - t_0)^{1-\alpha}} \, ds
\]
\[
= \frac{M}{\Gamma(\alpha)} \left[ \int_0^{t_0} |(t_n - s)^{\alpha-1} - (t_0 - s)^{\alpha-1}| \, ds + \int_{t_0}^{t_n} \frac{ds}{(t_n - s)^{1-\alpha}} + \int_{t_0}^{t_n} \frac{ds}{(s - t_0)^{1-\alpha}} \right]
\]
\[
\leq \frac{M}{\Gamma(\alpha + 1)} [(t_n - t_0)^\alpha + t_n^\alpha - t_0^\alpha + (t_n - t_0)^\alpha + (t_n - t_0)^{\alpha-1}]
\]
\[
+ \frac{M}{\Gamma(\alpha + 1)} (t_n - t_0)^\alpha
\]
\[
= \frac{4M}{\Gamma(\alpha + 1)} (t_n - t_0)^\alpha + \frac{M}{\Gamma(\alpha + 1)} (t_n^\alpha - t_0^\alpha)
\]
\[
\leq \frac{4M}{\Gamma(\alpha + 1)} (t_n - t_0)^\alpha,
\]
where the last inequality has been obtained by using the fact that \(t_n^\alpha - t_0^\alpha < 0\).

Therefore, since \(t_n \to t_0\), from the last estimate, we deduce that \(G(x, y)(t_n) \to G(x, y)(t_0)\). This proves that \(G(x, y) \in C[0, 1]\). Consequently, \(G : C[0, 1] \times C[0, 1] \to C[0, 1]\). On the other hand, for \((x, y) \in C[0, 1] \times C[0, 1]\) and \(t \in C[0, 1]\), we have
\[
\|G(x, y)(t)\| \leq \frac{1}{\Gamma(\alpha)} [\varphi_1 (\max(\|x\|, \|y\|)) + k_1]
\]
\[
\times \left[ \int_0^t \frac{g(s, x(s), y(s) - g(s, 0, 0))}{(t-s)^{1-\alpha}} \, ds \right]
\]
\[
+ \left[ \int_0^t \frac{g(s, 0, 0)}{(t-s)^{1-\alpha}} \, ds \right]
\]
\[
\leq \frac{1}{\Gamma(\alpha)} [\varphi_1 (\max(\|x\|, \|y\|)) + k_1]
\]
\[
\times \left[ \int_0^t \varphi_2 (\max(\|x\|, \|y\|)) + k_2 \right]
\]
\[
\cdot \left[ \varphi_2 (\max(\|x\|, \|y\|)) + k_2 \right).
\]

Now, taking into account assumption (H_3), we infer that the operator \(G\) applies \(B_{r_2} \times B_{r_2}\) into \(B_{r_2}\). Moreover, from the last estimates, it follows that
\[
\|G(B_{r_2} \times B_{r_2})\| \leq \varphi_2 (r_2) + k_2
\]
\[
\leq \frac{\varphi_2 (r_2) + k_2}{\Gamma(\alpha + 1)}.
\]

Next, we will prove that the operators \(F\) and \(G\) are continuous on the ball \(B_{r_2} \times B_{r_2}\) and, consequently, \(G\) will be also continuous.

In fact, we fix \(\varepsilon > 0\) and we take \((x_0, y_0), (x, y) \in B_{r_2} \times B_{r_2}\) with \(\|(x, y) - (x_0, y_0)\| = \|(x - x_0, y - y_0)\| = \max\{\|x - x_0\|, \|y - y_0\|\} \leq \varepsilon\). Then, for \(t \in [0, 1]\), we have
\[
|F(x, y)(t) - F(x_0, y_0)(t)|
\]
\[
= |f(t, x(t), y(t))| \left[ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, x(s), y(s))}{(t-s)^{1-\alpha}} \, ds \right]
\]
\[
- \left[ \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s, x(s), y(s) - g(s, 0, 0))}{(t-s)^{1-\alpha}} \, ds \right]
\]
\[
\leq \varphi_1 (\max(\|x\|, \|y\|)) < \varepsilon,
\]
where we have used Remark 8. This proves the continuity of \(F\) on \(B_{r_2} \times B_{r_2}\).
In order to prove the continuity of $G$ on $B_r \times B_r$, we have
\[
|G(x, y)(t) - G(x_0, y_0)(t)| = \frac{1}{\Gamma(\alpha)} \left| \int_0^t \frac{g(s, x(s), y(s)) - g(s, x_0(s), y_0(s))}{(t-s)^{1-\alpha}} ds \right|
\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{|g(s, x(s), y(s)) - g(s, x_0(s), y_0(s))|}{(t-s)^{1-\alpha}} ds
\leq \frac{1}{\Gamma(\alpha)} \int_0^t \frac{\varphi_2(\max(\|x(s) - x_0(s)\|, \|y(s) - y_0(s)\|))}{(t-s)^{1-\alpha}} ds
\leq \frac{1}{\Gamma(\alpha + 1)} \varphi_2(\varepsilon)
\leq \varepsilon / \Gamma(\alpha + 1).
\]

Therefore,
\[
\|G(x, y) - G(x_0, y_0)\| < \frac{\varepsilon}{\Gamma(\alpha + 1)}
\]
and, consequently, $G$ is a continuous operator on $B_r \times B_r$.

In order to prove that $G$ satisfies assumptions of Theorem 13, only we have to check the condition
\[
\mu(G(X_1 \times X_2)) \leq \varphi(\max(\mu(X_1), \mu(X_2)))
\]
for any subsets $X_1$ and $X_2$ of $B_r$.

To do this, we fix $\varepsilon > 0, t_1, t_2 \in [0, 1]$ with $|t_1 - t_2| \leq \varepsilon$ and $(x, y) \in X_1 \times X_2$; then, we have
\[
|G(x, y)(t_1) - G(x, y)(t_2)| = |f(t_1, x(t_1), y(t_1)) - f(t_2, x(t_2), y(t_2))|
\leq |f(t_1, x(t_1), y(t_1)) - f(t_1, x(t_2), y(t_2))| + |f(t_1, x(t_2), y(t_2)) - f(t_2, x(t_2), y(t_2))|
\leq \varphi_1(\max(\|x(t_1) - x(t_2)\|, \|y(t_1) - y(t_2)\|)) + \omega(f, \varepsilon)
\leq \varphi_1(\max(\mu(x, \varepsilon), \mu(y, \varepsilon))) + \omega(f, \varepsilon),
\]
where $\omega(f, \varepsilon)$ denotes the quantity
\[
\omega(f, \varepsilon) = \sup \{ \|f(t, x, y) - f(s, x, y)\| : t, s \in [0, 1], \ |t - s| \leq \varepsilon, x, y \in [-r_0, r_0] \}.
\]

From the last estimate, we infer that
\[
\omega(G(X_1 \times X_2), \varepsilon) \leq \varphi_1(\max(\mu(X_1, \varepsilon), \mu(X_2, \varepsilon))) + \omega(f, \varepsilon).
\]
where we have used the fact that $t_1^\alpha - t_2^\alpha \leq 0$. Therefore,
\[
\omega (\mathcal{G} (X_1 \times X_2), \epsilon) \leq \frac{2L}{\Gamma (\alpha + 1)} \epsilon^\alpha.
\] (58)

From this, it follows that
\[
\omega_0 (\mathcal{G} (X_1 \times X_2)) = 0.
\] (59)

Next, by Proposition 7, (46), (55), and (59), we have
\[
\omega_0 (G (X_1 \times X_2)) = \omega_0 (\mathcal{F} (X_1 \times X_2) \cdot \mathcal{F} (X_1 \times X_2)) \\
\leq \| \mathcal{F} (X_1 \times X_2) \| \omega_0 (\mathcal{F} (X_1 \times X_2)) \\
+ \| \mathcal{F} (X_1 \times X_2) \| \omega_0_0 (\mathcal{F} (X_1 \times X_2)) \\
\leq \| \mathcal{F} (B_{r_2} \times B_{r_3}) \| \omega_0 (\mathcal{F} (X_1 \times X_2)) \\
+ \| \mathcal{F} (B_{r_2} \times B_{r_3}) \| \omega_0 (\mathcal{F} (X_1 \times X_2)) \\
\leq \frac{q_2 (r_2) + k_2}{\Gamma (\alpha + 1)} \varphi_1 (\max (\omega_0 (X_1), \omega_0 (X_2))).
\] (60)

By assumption (H_2), since $q_2 (r_2) + k_2 \leq \Gamma (\alpha + 1)$ and since it is easily proved that if $\alpha \in [0, 1]$ and $\varphi \in \mathcal{A}$, then $\alpha \varphi \in \mathcal{A}$, we deduce
\[
\omega_0 (G (X_1 \times X_2)) \leq \varphi (\max (\omega_0 (X_1), \omega_0 (X_2))),
\] (61)

where $\varphi \in \mathcal{A}$.

Finally, by Theorem 13, the operator $\mathcal{G}$ has at least a coupled fixed point and this is the desired result. This completes the proof.

On the other hand, if we perturb the data function in problem (4) of the following manner:
\[
D_x^{\alpha_1} \left[ \frac{x (t)}{f (t, x (t), y (t))} \right] = g (t, x (t), y (t)) + \eta (t),
\]
\[
D_y^{\alpha_2} \left[ \frac{y (t)}{f (t, y (t), x (t))} \right] = g (t, y (t), x (t)) + \eta (t),
\] (63)

where $0 < \alpha < 1$, $f \in C([0, 1] \times \mathbb{R} \times \mathbb{R})$, $g \in C([0, 1] \times \mathbb{R} \times \mathbb{R})$, and $\eta \in C[0, 1]$, then under assumptions of Theorem 15, problem (63) can be studied by using Theorem 15 where assumptions (H_1) and (H_2) are automatically satisfied and we only have to check assumption (H_3). This fact gives a great applicability to Theorem 15.

Before presenting an example illustrating our results, we need some facts about the functions involving this example. The following lemma appears in [18].

**Lemma 16.** Let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a nondecreasing and upper semicontinuous function. Then, the following conditions are equivalent:

(i) $\lim_{n \to \infty} \varphi^n (t) = 0$ for any $t \geq 0$;
(ii) $\varphi (t) < t$ for any $t > 0$.

Particularly, the functions $\alpha_1, \alpha_2 : \mathbb{R}_+ \to \mathbb{R}_+$ given by $\alpha_1 (t) = \arctan t$ and $\alpha_2 (t) = t/(1 + t)$ belong to the class $\mathcal{A}$ since they are nondecreasing and continuous, and, as it is easily seen, they satisfy (ii) of Lemma 16.

On the other hand, since the function $\alpha_1 (t) = \arctan t$ is concave (because $\alpha'' (t) \leq 0$) and $\alpha_1 (0) = 0$, we infer that $\alpha_1$ is subadditive and, therefore, for any $t, t' \in \mathbb{R}_+$, we have

\[
|\alpha_1 (t) - \alpha_1 (t')| = \left| \arctan t - \arctan t' \right| \leq \arctan |t - t'|.
\] (64)

Moreover, it is easily seen that $\max (\alpha_1, \alpha_2)$ is a nondecreasing and continuous function because $\alpha_1$ and $\alpha_2$ are nondecreasing and continuous and $\max (\alpha_1, \alpha_2)$ satisfies (ii) of Lemma 16. Therefore, $\max (\alpha_1, \alpha_2) \in \mathcal{A}$.

Now, we are ready to present an example where our results can be applied.

**Example 17.** Consider the following coupled system of fractional hybrid differential equations:
\[
D^{1/2}_{\alpha} \left[ x (t) \times \left( \frac{1}{4} + \left( \frac{1}{10} \right) \arctan |x (t)| \right. \right.
\]
\[
\left. \left. + \left( \frac{1}{20} \right) \left( \frac{|y (t)|}{(1 + |y (t)|)} \right)^{-1} \right) \right] = \frac{1}{7} + \frac{1}{9} x (t) + \frac{1}{10} y (t),
\]
\[
D^{1/2}_{\beta} \left( \frac{y (t)}{f (t, y (t), x (t))} \right) = g (t, y (t), x (t)) + \eta (t),
\] (65)
Notice that problem (17) is a particular case of problem (4), where \( \alpha = 1/2 \), \( f(t,x,y) = \frac{1}{4} + \left( \frac{1}{10} \right) \arctan |y| \), and \( g(t,x,y) = \frac{1}{7} + \left( \frac{1}{9} \right) x + \left( \frac{1}{10} \right) y \). It is clear that \( f \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and \( g \in C([0,1] \times \mathbb{R} \times \mathbb{R}, \mathbb{R}) \) and, moreover, \( k_1 = \sup \{ |f(t,0,0)| : t \in [0,1] \} = 1/4 \) and \( k_2 = \sup \{ |g|: t \in [0,1] \} = 1/7 \). Therefore, assumption (H2) of Theorem 15 is satisfied.

Moreover, for \( t \in [0,1] \) and \( x_1, x_2, y_1, y_2 \in \mathbb{R} \), we have

\[
|f(t,x_1,y_1) - f(t,x_2,y_2)| \\
\leq \frac{1}{10} \left| \arctan |x_1| - \arctan |x_2| \right| \\
+ \frac{1}{20} \left| \frac{|y_1|}{1 + |y_1|} - \frac{|y_2|}{1 + |y_2|} \right| \\
\leq \frac{1}{10} \arctan |x_1 - x_2| \\
+ \frac{1}{20} \frac{|y_1 - y_2|}{(1 + |y_1|)(1 + |y_2|)} \\
\leq \frac{1}{10} \arctan (|x_1 - x_2|) \\
+ \frac{1}{20} \frac{|y_1 - y_2|}{1 + |y_1|} \\
= \frac{1}{10} \alpha_1 (|x_1 - x_2|) \\
+ \frac{1}{20} \alpha_2 (|y_1 - y_2|) \\
\leq \frac{1}{10} \max (\alpha_1, \alpha_2) (|x_1 - x_2|) \\
+ \frac{1}{10} \max (\alpha_1, \alpha_2) (|y_1 - y_2|) \\
\leq \frac{1}{10} \left[ 2 \max (\alpha_1, \alpha_2) \right] \\
\times \max (|x_1 - x_2|, |y_1 - y_2|) \\
= \frac{1}{5} \max (\alpha_1, \alpha_2) \left( \max (|x_1 - x_2|, |y_1 - y_2|) \right).
\]

Therefore, \( \varphi(t) = (1/5) \max (\alpha_1(t), \alpha_2(t)) \) and \( \varphi \in \mathcal{A} \).

On the other hand,

\[
|g(t,x_1,y_1) - g(t,x_2,y_2)| \\
\leq \frac{1}{9} |x_1 - x_2| + \frac{1}{10} |x_2 - y_2| \\
\leq \frac{1}{9} (|x_1 - y_1| + |y_2 - y_2|) \\
\leq \frac{1}{9} (2 \max (|x_1 - y_1|, |y_2 - y_2|)) \\
= \frac{2}{9} \max (|x_1 - y_1|, |y_2 - y_2|),
\]

and \( \varphi(t) = (2/9)t \). It is clear that \( \varphi(t) < \mathcal{A} \). Therefore, assumption (H3) of Theorem 15 is satisfied.

In our case, the inequality appearing in assumption (H3) of Theorem 15 has the expression

\[
\left[ \frac{1}{5} \max \left( \arctan r, \frac{r}{1 + r} \right) + \frac{1}{4} \right] \left( \frac{2}{9} r + \frac{1}{7} \right) \leq r \Gamma \left( \frac{3}{2} \right).
\]

(67)

It is easily seen that \( r_0 = 1 \) satisfies the last inequality. Moreover,

\[
\frac{2}{9} r_0 + \frac{1}{7} = \frac{2}{9} + \frac{1}{7} \leq \Gamma \left( \frac{3}{2} \right) = 1.88623.
\]

(69)

Finally, Theorem 15 says that problem (17) has at least one solution \((x,y) \in C[0,1] \) such that \( \max (\|x\|, \|y\|) \leq 1 \).

**Conflict of Interests**

The authors declare that there is no conflict of interests in the submitted paper.

**References**


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