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Research Article

State-Feedback Stabilization for Stochastic High-Order Nonlinear Systems with Time-Varying Delays

Fangzheng Gao,[1] Zheng Yuan,[2] and Fushun Yuan[1]

[1] *School of Mathematics and Statistics, Anyang Normal University, Anyang 455000, China*
[2] *School of Civil and Architecture Engineering, Anyang Institute of Technology, Anyang 455000, China*

Correspondence should be addressed to Fangzheng Gao; gaofz@126.com

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This paper investigates the problem of state-feedback stabilization for a class of stochastic high-order nonlinear systems with time-varying delays. Under the weaker conditions on the power order and the nonlinear growth, by using the method of adding a power integrator, a state-feedback controller is successfully designed, and the global asymptotic stability in the probability of the resulting closed-loop system is proven with the help of an appropriate Lyapunov-Krasovskii functional. A simulation example is given to demonstrate the effectiveness of the proposed design procedure.

1. Introduction

It is well known that stochastic modeling has come to play an important role in the field of engineering where stochastic differential equations have been applied for the analysis and control of stochastic systems. As a consequence, the study of stochastic nonlinear systems has drawn an increasing attention in the past decades [1–9]. Especially, by applying the so-called method of adding a power integrator [10], which can be viewed as the latest achievement of the traditional backstepping method, a series of research results for stochastic high-order nonlinear systems have been achieved [11–18].

On the other hand, time delay extensively exists in various engineering systems, such as electrical networks, microwave oscillator, and hydraulic systems. It is well known that the existence of time delay often deteriorates the control performance of systems and even causes the instability of closed-loop systems [19]. Therefore, the control design and stability analysis of time-delay systems has been an active research area within the automation and control community. In recent years, by employing the Lyapunov-Krasovskii method or Lyapunov-Razumikhin method to deal with the time delay, control theory and techniques for time-delay linear systems were developed and many advanced methods such as time-delay partitioning method and input-output method have been established; see, for instance, [20–25] and reference therein. However, for the time-delay nonlinear systems, there exist many open problems which are so important and interesting at least from the theoretical point of view and have been paid careful attention; see, for example, the lastly published papers [26–29].

In this paper, we consider a class of stochastic high-order systems with time-varying delays described by

$$\begin{aligned} dx_i(t) &= h_i(t)\, x_{i+1}^{p_i}(t) + f_i\left(t, \overline{x}_i(t), \overline{x}_i(t-d(t))\right) dt \\ &\quad + g_i^T\left(t, \overline{x}_i(t), \overline{x}_i(t-d(t))\right) dw, \quad i = 1, \ldots, n-1, \\ dx_n(t) &= h_n(t)\, u^{p_n}(t) + f_n\left(t, \overline{x}_n(t), \overline{x}_n(t-d(t))\right) dt \\ &\quad + g_n^T\left(t, \overline{x}_n(t), \overline{x}_n(t-d(t))\right) dw, \end{aligned} \tag{1}$$

where $x = (x_1, \ldots, x_n)^T \in R^n$ is system state vector and $u \in R$ is control input, respectively; $\overline{x}_i = (x_1, \ldots, x_i)^T$, $\overline{x}_n = x$; $d(t) : R^+ \to [0, d]$ is the time-varying delay

satisfying $\dot{d}(t) \le \eta < 1$ for a known constant η; $p_i \in R_{\text{odd}}^{\ge 1} := \{p/q \mid p \text{ and } q \text{ are positive odd integers, and } p \ge q\}$ are said to be the high orders of the system; $h_i(t)$, $i = 1, \ldots, n$, are disturbed virtual control coefficients; w is an m-dimensional independent standard Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, P)$ with Ω being a sample space, $\mathscr{F}$ being a σ-field, $\{\mathscr{F}_t\}_{t\ge 0}$ being a filtration, and P being a probability measure. The functions $f_i : R^+ \times R^i \times R^i \to R$ and $g_i : R^+ \times R^i \times R^i \to R^m$ are assumed to be locally Lipschitz with $f_i(t, 0, 0) = 0$ and $g_i(t, 0, 0) = 0$.

It is worth pointing out that, when $p_i = 1$ and $h_i(t) = 1$, system (1) degenerates to the stochastic time-delay system in strict-feedback form, whose study on feedback control problem has achieved great development in recent years; see, for example, [30–33] and reference therein. However, when $p_i > 1$, the control design for system (1) is challenging because on one hand, the presence of the fractional power makes system (1) not only nonsmooth but also having the uncontrollable unstable linearization, and on the other hand, the existence of time-delay effect makes the common assumption on the high-order systems nonlinearities infeasible and which conditions should be placed to the nonlinearities remains unanswered. To the best of the authors' knowledge, up to now only [34, 35] two papers considered the state-feedback stabilization problem of system (1) under the strict power order restriction ($p_1 = p_2 = \cdots = p_n = p$) and some rigorous growth condition. However, from both practical and theoretical points of view, it is somewhat restrictive to require system (1) to satisfy such restrictions. Immediately, the following interesting questions are proposed. *Is it possible to relax the power order restriction and the nonlinear growth condition? Under these weaker conditions, how to design a state-feedback controller such that the closed-loop system is globally asymptotically stable in probability?*

Inspired by the recent works [15, 29], we will solve the aforementioned problems here. The contribution of this paper is highlighted as follows.

(i) By comparison with the existing results in [34, 35], the power order restriction is completely removed, the nonlinear growth condition is largely relaxed, and a much weaker sufficient condition is given.

(ii) Different from the result in the deterministic case [29], in which the controller need not satisfy the locally Lipschitz condition, for stochastic system, to guarantee the existence and uniqueness of the solution, the designed controller must guarantee that the closed-loop system satisfies the locally Lipschitz condition. Hence, how to construct the controller constitutes one of the main contributions.

(iii) Due to the appearance of high-order, time-varying delay and nonlinear assumption, how to construct an appropriate Lyapunov-Krasovskii functional for system (1), especially under the assumption that $f_i(t, \overline{x}_i(t), \overline{x}_i(t - d(t)))$ and $g_i(t, \overline{x}_i(t), \overline{x}_i(t - d(t)))$ are not bounded linearly, is a nontrivial work.

The remainder of this paper is organized as follows. Section 2 offers notations and some preliminary results. Section 3 presents the control design procedure and the main result, while Section 4 gives a simulation example to illustrate the theoretical finding of this paper. Finally, concluding remarks are proposed in Section 5.

2. Notations and Preliminary Results

Throughout this paper, the following notations are adopted. R^+ denotes the set of all nonnegative real numbers and R^n denotes the real n-dimensional space. $R_{\text{odd}}^{+} := \{p/q \mid p \text{ and } q \text{ positive integers}\}$, $R_{\text{odd}}^{\ge 1} := \{p/q \mid p \text{ and } q \text{ are positive odd integers, and } p \ge q\}$. For a given vector or matrix X, X^T denotes its transpose, $\mathrm{Tr}\{X\}$ denotes its trace when X is square, and $|X|$ is the Euclidean norm of a vector X. $C([-d, 0]; R^n)$ denotes the space of continuous R^n-value functions on $[-d, 0]$ endowed with the norm $\|\cdot\|$ defined by $\|f\| = \sup_{x\in[-d,0]} |f(x)|$ for $f \in C([-d, 0]; R^n)$. $C_{\mathscr{F}_0}^b([-d, 0]; R^n)$ denotes the family of all $\mathscr{F}_0$-measurable bounded $C([-d, 0]; R^n)$-valued random variables $\xi = \{\xi(\theta) : -d \le s \le 0\}$. C^i denotes the set of all functions with continuous ith partial derivatives; $C^{1,2}(R^+; R^n \times [-d, \infty))$ denotes the family of all nonnegative functions $V(t, x)$ on $[-d, \infty) \times R^n$ which are C^1 in t and C^2 in x; $C^{1,2}$ denotes the family of all functions which are C^1 in the first argument and C^2 in the second argument.

Consider the following stochastic time-delay system:

$$\begin{aligned} dx(t) = {} & f(t, x(t), x(t - d(t)))\, dt \\ & + g(t, x(t), x(t - d(t)))\, dw, \end{aligned} \tag{2}$$

with initial data $\{x(\theta) : -d \le s \le 0\} = \xi \in C_{\mathscr{F}_0}^b$, where $d(t) : R^+ \to [0, d]$ is a Borel measurable function; w is an m-dimensional independent standard Wiener process defined on a complete probability space $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\ge 0}, P)$. The functions $f : R^+ \times R^n \times R^n \to R^n$ and $g : R^+ \times R^n \times R^n \to R^{n\times m}$ are locally Lipschitz with $f(t, 0, 0) = 0$ and $g(t, 0, 0) = 0$.

Definition 1 (see [32]). For any given $V(t, x(t)) \in C^{1,2}$ associated with system (2), the differential operator $\mathscr{L}$ is defined as

$$\mathscr{L}V = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f + \frac{1}{2} \mathrm{Tr}\left\{ g^T \frac{\partial^2 V}{\partial x^2} g \right\}. \tag{3}$$

Lemma 2 (see [32]). *For system (2), if there is a function $V(t, x(t)) \in C^{1,2}(R^+; R^n \times [-d, \infty))$ and two class K_∞ functions $\beta_1(\cdot)$ and $\beta_2(\cdot)$, such that*

$$\begin{gathered} \beta_1(|x(t)|) \le V(t, x(t)) \le \beta_2\left(\sup_{-d\le s\le 0} |x(t + s)| \right), \\ \mathscr{L}V \le -W(x(t)), \end{gathered} \tag{4}$$

where $W(x(t))$ is a nonnegative continuous function, then

(i) *there exists a unique solution on $[-d, \infty)$;*

(ii) *the equilibrium $x = 0$ is globally stable in probability and the solution $x(t)$ satisfies*

$$P\left\{ \lim_{t\to\infty} W(x(t)) = 0 \right\} = 1. \tag{5}$$

Especially, if $W(x(t))$ is a positive definite function, then the equilibrium $x = 0$ is globally asymptotically stable (GAS) in probability.

In the remainder of this section, we list several lemmas that serve as the basis for the development of a state-feedback controller for system (1). They are the key tools for adding a power integrator technique.

Lemma 3 (see [36]). *For $x \in R$, $y \in R$, and $p \geq 1$ being a constant, the following inequalities hold:*

$$\begin{aligned} |x+y|^p &\leq 2^{p-1}|x^p+y^p|, \\ (|x|+|y|)^{1/p} &\leq |x|^{1/p}+|y|^{1/p} \\ &\leq 2^{(p-1)/p}(|x|+|y|)^{1/p}. \end{aligned} \tag{6}$$

If $p \geq 1$ is odd, then

$$\begin{aligned} |x-y|^p &\leq 2^{p-1}|x^p-y^p|, \\ |x^{1/p}-y^{1/p}| &\leq 2^{(p-1)/p}(|x-y|)^{1/p}. \end{aligned} \tag{7}$$

Lemma 4 (see [37]). *For $x, y \in R$ and positive real number p, the following inequality holds:*

$$\begin{aligned} |x^p-y^p| &\leq p|x-y||x^{p-1}+y^{p-1}| \\ &\leq c|x-y||(x-y)^{p-1}+y^{p-1}|, \end{aligned} \tag{8}$$

where $c = p$ for $1 < p \leq 2$ and $c = p2^{p-1}$ for $p > 2$.

Lemma 5 (see [38]). *Let x and y be real variables; then for any positive real numbers a, m, and n, one has*

$$\begin{aligned} a|x|^m|y|^n &\leq b|x|^{m+n} \\ &\quad + \frac{n}{m+n}\left(\frac{m+n}{m}\right)^{-m/n} a^{(m+n)/n} b^{-m/n}|y|^{m+n}, \end{aligned} \tag{9}$$

where $b > 0$ is any real number.

3. Controller Design and Main Result

The objective of this paper is to design a state-feedback controller of the form

$$u = u(x), \tag{10}$$

such that system (1) is GAS in probability at the origin.

To this end, the following assumptions regarding system (1) are imposed.

Assumption 6. For $i = 1, \ldots, n$, there are positive constants c_{i1} and c_{i2} such that

$$c_{i1} \leq h_i(t) \leq c_{i2}. \tag{11}$$

Assumption 7. For $i = 1, \ldots, n$, there are nonnegative smooth functions $a_{i1}(\overline{x}_i)$, $a_{i2}(\overline{x}_i)$ and constants $\tau_1 \geq \tau_2 \geq \cdots \geq \tau_n \geq 0$ such that

$$\begin{aligned} &|f_i(t, \overline{x}_i(t), \overline{x}_i(t-d(t)))| \\ &\quad \leq a_{i1}(\overline{x}_i)\sum_{j=1}^{i}\Big(|x_j(t)|^{(r_i+\tau_i)/r_j} \\ &\qquad + |x_j(t-d(t))|^{(r_i+\tau_i)/r_j}\Big), \\ &|g_i(t, \overline{x}_i(t), \overline{x}_i(t-d(t)))| \\ &\quad \leq a_{i2}(\overline{x}_i)\sum_{j=1}^{i}\Big(|x_j(t)|^{(2r_i+\tau_i)/2r_j} \\ &\qquad + |x_j(t-d(t))|^{(2r_i+\tau_i)/2r_j}\Big), \end{aligned} \tag{12}$$

where $r_1 = 1$ and $r_{i+1} = (r_i + \tau_i)/p_i > 0$. Let $r_0 = \max\{r_i\}$ and $\alpha_i = r_0/r_i$, $i = 1, \ldots, n$. Meanwhile, one of the following conditions should be satisfied:

(i) $r_n + \tau_n \geq r_i$ is required if $\alpha_i = 1$ or $\alpha_i \geq 2$, $i = 1, \ldots, n$;

(ii) $r_n + \tau_n \geq 2r_i$ is required if condition (i) does not hold.

Remark 8. Assumption 7 is a generalization of the homogeneous growth condition introduced in [15], where $d(t) = 0$. The assumption plays an essential role in ensuring the locally Lipschitz condition of the closed-loop system, which guarantees the existence and uniqueness of the solution. Moreover, it is worth pointing out that Assumption 7 encompasses the assumption in existing results [34, 35]. Specifically, when $p_i = p \in R_{\text{odd}}^{>2}$, $\tau_i = \tau$, and $a_{i1}(\overline{x}_i) = a_{i2}(\overline{x}_i) = a$, Assumption 7 can be seen to reduce to the condition used in [35]. Moreover, when $p_i = p$ and $a_{i1}(\overline{x}_i) = a_{i2}(\overline{x}_i) = a$ by choosing $\tau_i = p - 1$, Assumption 7 reduces to the following condition:

$$\begin{aligned} &|f_i(t, \overline{x}_i(t), \overline{x}_i(t-d(t)))| \\ &\quad \leq a\sum_{j=1}^{i}\Big(|x_j(t)|^p + |x_j(t-d(t))|^p\Big), \\ &|g_i(t, \overline{x}_i(t), \overline{x}_i(t-d(t)))| \\ &\quad \leq a\sum_{j=1}^{i}\Big(|x_j(t)|^{(p+1)/2} + |x_j(t-d(t))|^{(p+1)/2}\Big), \end{aligned} \tag{13}$$

which is equivalent to that in [34].

Remark 9. It should be emphasized that the available state-feedback results for high-order stochastic time-delay system (1) in [34, 35] are all based on the strict power order restriction ($p_1 = p_2 = \cdots = p_n = p$). In this paper, for p_i, we only need a much more general condition: $p_i \in R_{\text{odd}}^{\geq 1}$, $i = 1, \ldots, n$, which means that the power order restriction is completely removed in this paper.

Without loss of generality, we assume that $\tau_i = p_i/d_i$ with p_i being any even integer and d_i being any odd integer, under

which and with the definition of r_i in Assumption 7, we know that $r_i \in R^+_{\text{odd}}$. For general r_i, a similar technique in [39] can be used.

Choose $l \geq 1$ to satisfy $r_n + \tau_n \geq (r_i + \tau_i)/l$ and $r_0 \geq (r_i + \tau_i)/l$, $i = 1, \ldots, n$. With Assumption 7, σ is chosen in the following manner.

(a) Choose $\sigma = r_0$ if condition (i) of Assumption 7 is satisfied.

(b) σ can be chosen as any $\sigma \in R^+_{\text{odd}}$ satisfying $r_n + \tau_n \geq \sigma \geq \max_{1\leq i\leq n}\{(r_i + \tau_i)/l, 2r_i\}$ if condition (ii) of Assumption 7 holds.

Now, we are in the position to give the controller design procedure by applying the method of adding a power integrator. To simplify the deduction procedure, we sometimes denote $\chi(t)$ by χ.

Step 1. Define $\xi_1 = x_1^\sigma$ and choose the Lyapunov functional $V_1 = W_1 + (n/(1-\eta)) \int_{t-d(t)}^{t} x_1^{4l\sigma}(s)ds$, where $W_1 = (1/(4l\sigma - \tau_1))x_1^{4l\sigma-\tau_1}$. With the help of Assumptions 6 and 7 and Lemma 5, we have

$$\begin{aligned}
\mathscr{L}V_1 \leq{}& h_1 x_1^{4l\sigma-\tau_1-1} x_2^{p_1} \\
&+ a_{11} x_1^{4l\sigma-\tau_1-1}\left(|x_1|^{1+\tau_1} + |x_1(t-d(t))|^{1+\tau_1}\right) \\
&+ (4\sigma - \tau_1 - 1) a_{12}^2 x_1^{4l\sigma-\tau_1-2} \\
&\times \left(|x_1|^{2+\tau_1} + |x_1(t-d(t))|^{2+\tau_1}\right) \\
&+ \frac{n}{1-\eta} x_1^{4l\sigma} - \frac{n\left(1-\dot{d}(t)\right)}{1-\eta} x_1^{4l\sigma}(t-d(t)) \\
\leq{}& h_1 x_1^{4l\sigma-\tau_1-1}\left(x_2^{p_1} - x_2^{*p_1}\right) \\
&- (n-1) x_1^{4l\sigma}(t-d(t)) + h_1 x_1^{4l\sigma-\tau_1-1} x_2^{*p_1} \\
&+ x_1^{4l\sigma}\left(\frac{n}{1-\eta} + a_{11} + (4\sigma - \tau_1 - 1) a_{12}^2 + l_{11} + l_{12}\right),
\end{aligned} \tag{14}$$

where $l_{11}(x_1) \geq ((4l\sigma - \tau_1 - 1)/4l\sigma) \times ((1+\tau_1)/2l\sigma)^{(1+\tau_1)/(4l\sigma-\tau_1-1)} \times a_{11}^{4l\sigma/(4l\sigma-\tau_1-1)}$ and $l_{12}(x_1) \geq ((4l\sigma - \tau_1 - 2)/4l\sigma) \times ((2+\tau_1)/2l\sigma)^{(2+\tau_1)/(4l\sigma-\tau_1-2)} \times [(4\sigma - \tau_1 - 1)a_{12}^2]^{4l\sigma/(4l\sigma-\tau_1-2)}$ are nonnegative smooth functions.

Obviously, the first virtual controller

$$\begin{aligned}
x_2^* ={}& -\frac{1}{c_{11}}\left(n + \frac{n}{1-\eta} + a_{11} + (4\sigma - \tau_1 - 1) a_{12}^2 \right. \\
&\left. + l_{11} + l_{12}\right) x_1^{(1+\tau_1)/p_1} \\
:={}& -\beta_1^{r_2/\sigma}\xi_1^{r_2/\sigma},
\end{aligned} \tag{15}$$

with $\beta_1 > 0$ being smooth results in

$$\begin{aligned}
\mathscr{L}V_1 \leq{}& -n\xi_1^{4l} - (n-1)\xi_1^{4l}(t-d(t)) \\
&+ h_1 \xi_1^{(4l\sigma-\tau_1-1)/\sigma}\left(x_2^{p_1} - x_2^{*p_1}\right).
\end{aligned} \tag{16}$$

Remark 10. It is necessary to mention that in the first design step, the functions $l_{11}(x_1)$ and $l_{12}(x_1)$ have been provided with explicit expression in order to deduce the completely explicit virtual controller. However, in the later design steps, sometimes for the sake of briefness, we will not explicitly write out the functions which are easily defined.

Inductive Step. Suppose at step $k-1$ that there are a C^2, proper and positive definite Lyapunov function V_{k-1}, and a set of virtual controllers $x_1^*, \ldots, x_{k-1}^*$ defined by

$$\begin{aligned}
x_1^* &= 0, & \xi_1 &= x_1^{\sigma/r_1} - x_1^{*\sigma/r_1}, \\
x_2^* &= -\beta_1^{r_2/\sigma}\xi_1^{r_2/\sigma}, & \xi_2 &= x_2^{\sigma/r_2} - x_2^{*\sigma/r_2}, \\
&\vdots & &\vdots \\
x_k^* &= -\beta_{k-1}^{r_k/\sigma}\xi_{k-1}^{r_k/\sigma}, & \xi_k &= x_k^{\sigma/r_k} - x_k^{*\sigma/r_k},
\end{aligned} \tag{17}$$

with $\beta_1 > 0, \ldots, \beta_{k-1} > 0$ being smooth, such that

$$\begin{aligned}
&\mathscr{L}V_{k-1} \\
&\quad\leq -(n-k+2)\sum_{i=1}^{k-1}\xi_i^{4l} - (n-k+1)\sum_{i=1}^{k-1}\xi_i^{4l}(t-d(t)) \\
&\quad + h_{k-1}\xi_{k-1}^{(4l\sigma-\tau_{k-1}-r_{k-1})/\sigma}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right).
\end{aligned} \tag{18}$$

To complete the induction, at the kth step, we choose the following Lyapunov functional candidate

$$\begin{aligned}
V_k(\overline{x}_k) ={}& V_{k-1}(\overline{x}_{k-1}) + W_k(\overline{x}_k) \\
&+ \frac{n-k+1}{1-\eta}\int_{t-d(t)}^{t}\xi_k^{4l}(s)\,ds,
\end{aligned} \tag{19}$$

where

$$W_k(\overline{x}_k) = \int_{x_k^*}^{x_k}\left(s^{\sigma/r_k} - x_k^{*\sigma/r_k}\right)^{(4l\sigma-\tau_k-r_k)/\sigma} ds. \tag{20}$$

Noting that $\sigma/r_i \geq 2$, using a similar method as in [40], V_k can be shown to be C^2, proper and positive definite. Moreover, we can obtain

$$\begin{aligned}
\frac{\partial W_k}{\partial x_k} &= \xi_k^{(4l\sigma-\tau_k-r_k)/\sigma}, \\
\frac{\partial^2 W_k}{\partial x_k^2} &= \frac{4l\sigma-\tau_k-r_k}{r_k}\xi_k^{((4l-1)\sigma-\tau_k-r_k)/\sigma} x_k^{(\sigma-r_k)/r_k}, \\
\frac{\partial^2 W_k}{\partial x_k \partial x_i} &= \frac{\partial^2 W_k}{\partial x_i \partial x_k} \\
&= \frac{4l\sigma-\tau_k-r_k}{\sigma}\xi_k^{((4l-1)\sigma-\tau_k-r_k)/\sigma}\frac{\partial x_k^{*\sigma/r_k}}{\partial x_i}, \\
\frac{\partial W_k}{\partial x_i} &= -\frac{4l\sigma-\tau_k-r_k}{\sigma}\frac{\partial x_k^{*\sigma/r_k}}{\partial x_i} \\
&\quad \times \int_{x_k^*}^{x_k}\left(s^{\sigma/r_k}-x_k^{*\sigma/r_k}\right)^{((4l-1)\sigma-\tau_k-r_k)/\sigma} ds, \\
\frac{\partial^2 W_k}{\partial x_i^2} &= \frac{4l\sigma-\tau_k-r_k}{\sigma}\cdot\frac{(4l-1)\sigma-\tau_k-r_k}{\sigma}\left(\frac{\partial x_k^{*\sigma/r_k}}{\partial x_i}\right)^2 \\
&\quad \times \int_{x_k^*}^{x_k}\left(s^{\sigma/r_k}-x_k^{*\sigma/r_k}\right)^{((4l-2)\sigma-\tau_k-r_k)/\sigma} ds \\
&\quad - \frac{4\sigma-\tau_k-r_k}{\sigma}\frac{\partial^2 x_k^{*\sigma/r_k}}{\partial x_i^2} \\
&\quad \times \int_{x_k^*}^{x_k}\left(s^{\sigma/r_k}-x_k^{*\sigma/r_k}\right)^{((4l-1)\sigma-\tau_k-r_k)/\sigma} ds, \\
\frac{\partial^2 W_k}{\partial x_i \partial x_j} &= \frac{4l\sigma-\tau_k-r_k}{\sigma}\cdot\frac{(4l-1)\sigma-\tau_k-r_k}{\sigma} \\
&\quad \times \frac{\partial x_k^{*\sigma/r_k}}{\partial x_i}\frac{\partial x_k^{*\sigma/r_k}}{\partial x_j} \\
&\quad \times \int_{x_k^*}^{x_k}\left(s^{\sigma/r_k}-x_k^{*\sigma/r_k}\right)^{((4l-2)\sigma-\tau_k-r_k)/\sigma} ds,
\end{aligned} \tag{21}$$

where $i, j = 1, \ldots, k-1$, $i \neq j$.

Using (19), (20), and (21), it follows that

$$\begin{aligned}
&\mathscr{L}V_k \\
&\leq -(n-k+2)\sum_{i=1}^{k-1}\xi_i^{4l} - (n-k+1)\sum_{i=1}^{k-1}\xi_i^{4l}(t-d(t)) \\
&\quad + h_{k-1}\xi_{k-1}^{(4l\sigma-\tau_{k-1}-r_{k-1})/\sigma}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right) \\
&\quad + h_k\xi_k^{(4l\sigma-\tau_k-r_k)/\sigma}x_{k+1}^{p_k} + \xi_k^{(4l\sigma-\tau_k-r_k)/\sigma}f_k \\
&\quad + \sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}(h_i x_{i+1} + f_i) + \frac{1}{2}\sum_{i,j=1,i\neq j}^{k-1}\left|\frac{\partial^2 W_k}{\partial x_i \partial x_j}\right|\left|g_i^T\right|\left|g_j^T\right| \\
&\quad + \frac{1}{2}\sum_{i=1}^{k-1}\left|\frac{\partial^2 W_k}{\partial x_i^2}\right|\left|g_i^T\right|^2 + \frac{1}{2}\sum_{i=1}^{k-1}\left|\frac{\partial^2 W_k}{\partial x_k \partial x_i}\right|\left|g_k^T\right|\left|g_i^T\right| \\
&\quad + \frac{1}{2}\left|\frac{\partial^2 W_k}{\partial x_k^2}\right|\left|g_k^T\right|^2 + \frac{n-k+1}{1-\eta}\xi_k^{4l} \\
&\quad - \frac{(n-k+1)\left(1-\dot{d}(t)\right)}{1-\eta}\xi_k^{4l}(t-d(t)).
\end{aligned} \tag{22}$$

In order to proceed further, an appropriate bounding estimate should be given for each term on the right-hand side of (22). This is accomplished in the following propositions whose technical proofs are given in the Appendix.

Proposition 11. *There exists a positive constant l_{k1} such that*

$$h_{k-1}\xi_{k-1}^{(4l\sigma-\tau_{k-1}-r_{k-1})/\sigma}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right) \leq \frac{1}{4}\xi_{k-1}^{4l} + \xi_k^{4l}l_{k1}. \tag{23}$$

Proposition 12. *There exists a nonnegative smooth function l_{k2} such that*

$$\xi_k^{(4l\sigma-\tau_k-r_k)/\sigma}f_k \leq \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^{4l} + \frac{1}{8}\sum_{i=1}^{k}\xi_i^{4l}(t-d(t)) + \xi_k^{4l}l_{k2}. \tag{24}$$

Proposition 13. *There exists a nonnegative smooth function l_{k3} such that*

$$\begin{aligned}
&\sum_{i=1}^{k-1}\frac{\partial W_k}{\partial x_i}(h_i x_{i+1} + f_i) \\
&\quad \leq \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^{4l} + \frac{1}{8}\sum_{i=1}^{k}\xi_i^{4l}(t-d(t)) + \xi_k^{4l}l_{k3}.
\end{aligned} \tag{25}$$

Proposition 14. *There exists a nonnegative smooth function l_{k4} such that*

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j=1,i\neq j}^{k-1}\left|\frac{\partial^2 W_k}{\partial x_i \partial x_j}\right|\left|g_i^T\right|\left|g_j^T\right| \\
&\quad \leq \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^{4l} + \frac{1}{8}\sum_{i=1}^{k}\xi_i^{4l}(t-d(t)) + \xi_k^{4l}l_{k4}.
\end{aligned} \tag{26}$$

Proposition 15. *There exists a nonnegative smooth function l_{k5} such that*

$$\begin{aligned}
&\frac{1}{2}\sum_{i=1}^{k-1}\left|\frac{\partial^2 W_k}{\partial x_i^2}\right|\left|g_i^T\right|^2 \\
&\quad \leq \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^{4l} + \frac{1}{8}\sum_{i=1}^{k}\xi_i^{4l}(t-d(t)) + \xi_k^{4l}l_{k5}.
\end{aligned} \tag{27}$$

Proposition 16. *There exists a nonnegative smooth function l_{k6} such that*

$$\frac{1}{2}\sum_{i=1}^{k-1}\left|\frac{\partial^2 W_k}{\partial x_k \partial x_i}\right|\left|g_k^T\right|\left|g_i^T\right| \le \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^{4l}+\frac{1}{8}\sum_{i=1}^{k}\xi_i^{4l}(t-d(t))+\xi_k^{4l}l_{k6}. \tag{28}$$

Proposition 17. *There exists a nonnegative smooth function l_{k7} such that*

$$\frac{1}{2}\left|\frac{\partial^2 W_k}{\partial x_k^2}\right|\left|g_k^T\right|^2 \le \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^{4l}+\frac{1}{8}\sum_{i=1}^{k}\xi_i^{4l}(t-d(t))+\xi_k^{4l}l_{k7}. \tag{29}$$

Substituting (23)–(29) into (22) yields

$$\begin{aligned}\mathscr{L}V_k \le & -(n-k+1)\sum_{i=1}^{k-1}\xi_i^{4l}-(n-k)\sum_{i=1}^{k}\xi_i^{4l}(t-d(t)) \\ & + h_k\xi_k^{(4l\sigma-\tau_k-r_k)/\sigma}\left(x_{k+1}^{p_k}-x_{k+1}^{*p_k}\right) \\ & + h_k\xi_k^{(4l\sigma-\tau_k-r_k)/\sigma}x_{k+1}^{*p_k}+\xi_k^{4l}\left(\sum_{j=1}^{7}l_{kj}+\frac{n-k+1}{1-\eta}\right).\end{aligned} \tag{30}$$

Now, it easy to see that the virtual controller

$$\begin{aligned}x_{k+1}^* & = -\frac{1}{c_{k1}}\left(n-k+1+\sum_{j=1}^{7}l_{kj}+\frac{n-k+1}{1-\eta}\right)\xi_k^{(r_k+\tau_k)/\sigma p_k} \\ & := -\beta_k^{r_{k+1}/\sigma}\xi_k^{r_{k+1}/\sigma}\end{aligned} \tag{31}$$

renders

$$\begin{aligned}\mathscr{L}V_k \le & -(n-k+1)\sum_{i=1}^{k}\xi_i^{4l} \\ & -(n-k)\sum_{i=1}^{k}\xi_i^{4l}(t-d(t)) \\ & + h_k\xi_k^{(4l\sigma-\tau_k-r_k)/\sigma}\left(x_{k+1}^{p_k}-x_{k+1}^{*p_k}\right).\end{aligned} \tag{32}$$

Finally, when $k=n$, $x_{n+1}=x_{n+1}^*=u$ is the actual control. By choosing the actual control law,

$$u=-\beta_n^{r_{n+1}/\sigma}\xi_n^{r_{n+1}/\sigma}, \tag{33}$$

we get

$$\mathscr{L}V_n \le -\sum_{i=1}^{n}\xi_i^{4l}, \tag{34}$$

where

$$V_n=\sum_{i=1}^{n}W_i+\sum_{i=1}^{n}\frac{n-i+1}{1-\eta}\int_{t-d(t)}^{t}\xi_i^{4l}(s)\,ds. \tag{35}$$

We are now ready to state the main theorem of this paper.

Theorem 18. *If Assumptions 6 and 7 hold for stochastic high-order nonlinear time-delay system* (1)*, under the state-feedback controller* (33)*, then the closed-loop system has a unique solution on $[-d,\infty)$ and the equilibrium at the origin of the closed-loop system is GAS in probability.*

Proof. We prove Theorem 18 in two steps.

Step 1. We first prove that u^{p_n} in (1) satisfies the locally Lipschitz condition.

From (17), (33), and $r_{n+1}p_n=r_n+\tau_n$, one has

$$u^{p_n}=-\left(\overline{\beta}_1x_1^{\sigma/r_1}+\cdots+\overline{\beta}_nx_n^{\sigma/r_n}\right)^{(r_n+\tau_n)/\sigma}, \tag{36}$$

where $\overline{\beta}_i=\beta_n,\ldots,\beta_i$. Then, for $i=1,\ldots,n$,

$$\frac{\partial u^{p_n}}{\partial x_i}=-A_i\left(\overline{\beta}_1x_1^{\sigma/r_1}+\cdots+\overline{\beta}_nx_n^{\sigma/r_n}\right)^{(r_n+\tau_n-\sigma)/\sigma}, \tag{37}$$

where

$$A_i=\frac{r_n+\tau_n}{r_i}\overline{\beta}_ix_i^{(\sigma-r_i)/r_i}+\sum_{j=1}^{n-1}\frac{r_j+\tau_j}{\sigma}\frac{\partial\overline{\beta}_j}{\partial x_i}x_j^{\sigma/r_j}. \tag{38}$$

By the definition of σ, we know that $r_n+\tau\ge\sigma\ge\max_{1\le i\le n}\{r_i\}>0$, which implies that

$$\frac{\sigma-r_i}{r_i}\ge 0,\qquad \frac{r_n+\tau_n-\sigma}{\sigma}\ge 0, \tag{39}$$

from which, (37), (38), and β_i's being smooth, we obtain that $\partial u^{p_n}/\partial x_i$ is continuous. Therefore, u^{p_n} is C^1.

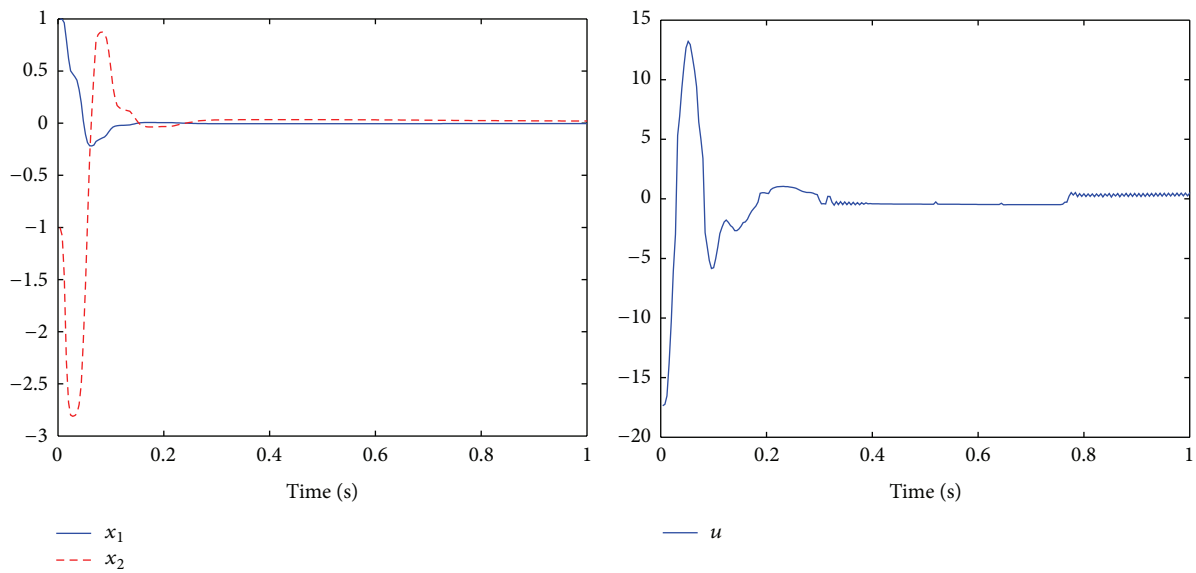
Since f_i and g_i are assumed to be locally Lipschitz, so the system consisting of (1), (17), and (33) satisfies the locally Lipschitz condition.

Step 2. Then we prove that the origin of closed-loop system is GAS in probability.

From the definitions of W_i's, we easily see that $\sum_{i=1}^{n}W_i$, as the function of x, is positive, definite, and radially unbounded. Then, by (34), (35), and Lemma 4.3 in [41], there exist K_∞ functions β_1, β_2, and β_3 such that

$$\begin{aligned}\beta_1\left(\left|\xi(t)\right|\right)\le V_n(t,\xi_t(\theta))\le\beta_2\left(\sup_{-d\le\theta\le 0}\left|\xi_t(t+\theta)\right|\right), \\ \mathscr{L}V_n\le-\beta_3\left(\left|\xi(t)\right|\right).\end{aligned} \tag{40}$$

With the help of Lemma 2, we conclude that the closed-loop system has a unique solution on $[-d,\infty)$, and the equilibrium $\xi=0$ is GAS in probability. This together with the definitions of x_i^*'s directly concludes that the origin $x=0$ of system (1) is also GAS in probability. Thus, the proof is completed. □



(a) System states

(b) Control input

Figure 1: The responses of closed-loop system (41) and (42).

Remark 19. In the deterministic work [29], strong stability theory [42] guarantees the existence of a solution; hence, the locally Lipschitz condition need not to be satisfied. While for stochastic system, according to the existing stochastic stability theory, to guarantee the existence and uniqueness of the solution, the designed controller must guarantee that the closed-loop system satisfies the locally Lipschitz condition. What should be emphasized is that how to guarantee the constructed Lyapunov functional being C^2 and the closed-loop systems satisfying the locally Lipschitz condition simultaneously is one of the difficulties in this paper.

Remark 20. Since Itô stochastic differentiation involves not only the gradient but also the higher order Hessian term in the Lyapunov design procedure, many tedious nonlinear terms will arise in the design process, especially in the case of the appearance of high order, time-varying delay and nonlinear assumption. How to deal with these terms is another difficulty of this paper.

Remark 21. From the above design procedure, we can see that the upper bound of the change rate of time delays has important impact on the control effort. To keep the control effort within the certain range, the upper bound of the change rate of time delays cannot be arbitrarily close to 1, which should be considered in practical engineering design.

4. Simulation Example

To illustrate the effectiveness of the proposed controller, we consider the following low-dimensional system

$$dx_1 = (1.5 + 0.5\cos t)\, x_2^{5/3} dt + \frac{1}{4} x_1^{5/3} (t - d(t))\, dt,$$
$$dx_2 = u^3 + \frac{1}{4} x_1 \sin x_1 dt + \frac{1}{8} x_2^{4/3} (t - d(t))\, dw, \tag{41}$$

where $d(t) = (1/3)(1 + \sin(t))$.

It is evident that, even though system (41) is simple, it could not be globally asymptotically stabilized using the design method presented in [34, 35] because of the unsatisfiability of the power order restriction. However, if we choose $\tau_1 = \tau_2 = 2/3$ which together with $r_1 = 1$ and $p_1 = 5/3$implies that $r_2 = 1$, it is easy to get $|f_1| \le (1/4)|x_1(t - d(t))|^{5/3}$, $|f_2| \le (1/4)|x_1|^{5/3}$, and $|g_2| \le (1/8)|x_2(t - d(t))|^{4/3}$. Clearly, Assumption 7 is satisfied. Moreover, noting that $1 \le 1.5 + 0.5\cos t \le 2$ and $\dot{d}(t) = (1/3)\cos t \le (1/3) < 1$, the controller proposed in this paper is applicable.

By choosing $l = 1$ and $\sigma = 1$, following the design procedure given in Section 3, we can get

$$x_2^* = -(5 + l_{11})\, x_1,$$
$$u = -\left(\frac{5}{2} + l_{21} + l_{22} + l_{23}\right)^{5/27} \xi_2^{5/27}, \tag{42}$$

where $\beta_1 = 5 + l_{11}$, $\xi_2 = x_2 + \beta_1 x_1$, $l_{11} = (7/12) \times (5/6)^{5/7} \times (1/4)^{12/7}$, $l_{21} = 54 \times (10/3)^4 + (5/12) \times (3/7)^{7/5} \times (10/3)^{12/5}$, $l_{22} = (5/12) \times (3/7)^{7/5} \times (1/4)^{12/5}$, and $l_{23} = 2 + 2\beta_1^{4/3} + (3/4) \times (1 + \beta_1^{4/3})^3$.

In the simulation, we choose the initial values $x_1(0) = 1$ and $x_2(0) = -1$. Figure 1 gives the responses of (41) and (42). From the figure, we can see that under the constructed controller, the solution process of the closed-loop system asymptotically converges to zero almost surely. We can

also see that a little larger control effort is needed at the beginning. As mentioned in [32], when there exist stochastic disturbances and time delays, the effort of a controller designed based on the backstepping method is bigger than the common case, to which attention should be paid in practical use.

5. Conclusion

This paper deals with the state-feedback stabilization problem for a class of stochastic high-order nonlinear time-delay systems under weaker conditions. The designed state-feedback controller ensures that the origin of the closed-loop system is GAS in probability. It should be noted that the proposed controller can only work well when the whole state vector is measurable. Therefore, a natural and more interesting problem is how to design output feedback stabilization controller for the systems studied in the paper if only partial state vector is being measurable, which is now under our further investigation.

Appendix

Proof of Proposition 11. There are two different cases for the proof.

If $r_k p_{k-1} < \sigma$, using (17), it follows from Lemma 3 that

$$\begin{aligned}\left|x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right| &= \left|\left(x_k^{\sigma/r_k}\right)^{p_{k-1}r_k/\sigma} - \left(x_k^{*\sigma/r_k}\right)^{p_{k-1}r_k/\sigma}\right| \\ &\le 2^{(\sigma-p_{k-1}r_k)/\sigma}\left|\xi_k\right|^{p_{k-1}r_k/\sigma}.\end{aligned} \tag{A.1}$$

Noting that $p_{k-1}r_k = r_{k-1} + \tau_{k-1}$, by (A.1), from Assumptions 6 and 7 and Lemma 5, it can be obtained that

$$\begin{aligned}&h_{k-1}\xi_{k-1}^{(4l\sigma-\tau_{k-1}-r_{k-1})/\sigma}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right) \\ &\quad\le c_{k-1,2}2^{(\sigma-p_{k-1}r_k)/\sigma} \\ &\qquad\times\left|\xi_{k-1}\right|^{(4l\sigma-\tau-r_{k-1})/\sigma}\left|\xi_k\right|^{p_{k-1}r_k/\sigma} \\ &\quad\le \frac{1}{4}\xi_{k-1}^{4l} + l_{k,1,1}\xi_k^{4l},\end{aligned} \tag{A.2}$$

where $l_{k,1,1}$ is a positive constant.

If $r_k p_{k-1} \ge \sigma$, from (17) and Lemma 4, we can get

$$\begin{aligned}&\left|x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right| \\ &\quad= \left|\left(x_k^{\sigma/r_k}\right)^{p_{k-1}r_k/\sigma} - \left(x_k^{*\sigma/r_k}\right)^{p_{k-1}r_k/\sigma}\right| \\ &\quad\le c\left|x_k^{\sigma/r_k} - x_k^{*\sigma/r_k}\right|\left|\left(x_k^{\sigma/r_k} - x_k^{*\sigma/r_k}\right)^{(p_{k-1}r_k-\sigma)/\sigma}\right. \\ &\qquad\qquad\left. - \left(x_k^{*\sigma/r_k}\right)^{(p_{k-1}r_k-\sigma)/\sigma}\right| \\ &\quad\le \bar{c}\left|\xi_k\right|\left(\left|\xi_{k-1}\right|^{(p_{k-1}r_k-\sigma)/\sigma} + \left|\xi_k\right|^{(p_{k-1}r_k-\sigma)/\sigma}\right),\end{aligned} \tag{A.3}$$

where c and $\bar{c}$ are positive constants.

By applying Lemma 5, we have

$$\begin{aligned}&h_{k-1}\xi_{k-1}^{(4l\sigma-\tau_{k-1}-r_{k-1})/\sigma}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right) \\ &\quad\le c_{k-1,2}\left|\xi_{k-1}\right|^{(4l\sigma-\tau_{k-1}-r_{k-1})/\sigma}\bar{c}\left|\xi_k\right| \\ &\qquad\times\left(\left|\xi_{k-1}\right|^{(p_{k-1}r_k-\sigma)/\sigma} + \left|\xi_k\right|^{(p_{k-1}r_k-\sigma)/\sigma}\right) \\ &\quad\le \frac{1}{4}\xi_{k-1}^{4l} + l_{k,1,2}\xi_k^{4l},\end{aligned} \tag{A.4}$$

where $l_{k,1,2}$ is a positive constant.

By choosing $l_{k1} = \max\{l_{k,1,1}, l_{k,1,2}\}$, with (A.2) and (A.4), we can obtain that

$$h_{k-1}\xi_{k-1}^{(4l\sigma-\tau-r_{k-1})/\sigma}\left(x_k^{p_{k-1}} - x_k^{*p_{k-1}}\right) \le \frac{1}{4}\xi_{k-1}^{4l} + l_{k1}\xi_k^{4l}. \tag{A.5}$$

□

Proof of Proposition 12. According to (17), Assumption 7, and Lemma 3, it follows that

$$\begin{aligned}&\left|f_k\right| \\ &\quad\le a_{k1}\sum_{i=1}^{k}\left(\left|x_i\right|^{(r_k+\tau_k)/r_i} + \left|x_i\left(t-d\left(t\right)\right)\right|^{(r_k+\tau_k)/r_i}\right) \\ &\quad= a_{k1}\sum_{i=1}^{k}\left(\left|\xi_i + \beta_{i-1}\xi_{i-1}\right|^{r_i/\sigma}\right)^{(r_k+\tau_k)/r_i} \\ &\qquad+ a_{k1}\sum_{i=1}^{k}\left(\left|\xi_i\left(t-d\left(t\right)\right) + \beta_{i-1}\xi_{i-1}\left(t-d\left(t\right)\right)\right|^{r_i/\sigma}\right)^{(r_k+\tau_k)/r_i} \\ &\quad\le a_{k1}\sum_{i=1}^{k}\left(1 + \beta_{i-1}^{(r_k+\tau_k)/\sigma}\right) \\ &\qquad\times\left(\left|\xi_i\right|^{(r_k+\tau)/\sigma} + \left|\xi_i\left(t-d\left(t\right)\right)\right|^{(r_k+\tau_k)/\sigma}\right) \\ &\quad\le \tilde{a}_{k1}\sum_{i=1}^{k}\left(\left|\xi_i\right|^{(r_k+\tau_k)/\sigma} + \left|\xi_i\left(t-d\left(t\right)\right)\right|^{(r_k+\tau_k)/\sigma}\right),\end{aligned} \tag{A.6}$$

where $\beta_0 = \xi_0 = 0$ and $\tilde{a}_{k1}$ is a nonnegative smooth function.

By using (A.6) and Lemma 5, we have

$$\begin{aligned}&\xi_k^{(4l\sigma-\tau_k-r_k)/\sigma}f_k \\ &\quad\le \left|\xi_k\right|^{(4l\sigma-\tau_k-r_k)/\sigma}\tilde{a}_{k1} \\ &\qquad\times\sum_{i=1}^{k}\left(\left|\xi_i\right|^{(r_k+\tau_k)/\sigma} + \left|\xi_i\left(t-d\left(t\right)\right)\right|^{(r_k+\tau_k)/\sigma}\right) \\ &\quad\le \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^{4l} + \frac{1}{8}\sum_{i=1}^{k}\xi_i^{4l}\left(t-d\left(t\right)\right) + l_{k2}\xi_k^{4l},\end{aligned} \tag{A.7}$$

where l_{k2} is a nonnegative smooth function. □

Proof of Proposition 13. Note that

$$x_k^{*\sigma/r_k} = -\beta_{k-1}\xi_{k-1} = -\sum_{i=1}^{k-1} B_i x_i^{\sigma/r_i}, \tag{A.8}$$

where

$$B_i = \begin{cases} \beta_{k-1}, \ldots, \beta_i, & i = 1, \ldots, k-2, \\ 1, & i = k-1. \end{cases} \tag{A.9}$$

Then, for $i = 1, \ldots, n$, we have

$$\frac{\partial x_k^{*\sigma/r_k}}{\partial x_i} = -\sum_{j=1}^{k-1} \frac{\partial B_j}{\partial x_i} x_i^{\sigma/r_i} - \frac{\sigma}{r_i} B_i x_i^{(\sigma-r_i)/r_i}. \tag{A.10}$$

By (17), (21), (A.6), (A.10), and Assumptions 6 and 7, we get

$$\begin{aligned}
&\sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i}\left(h_i x_{i+1}^{p_i} + f_i\right) \\
&= \sum_{i=1}^{k-1}\left(-\frac{4l\sigma - \tau_k - r_k}{\sigma}\frac{\partial x_k^{*\sigma/r_k}}{\partial x_i}\right. \\
&\qquad \left. \times \int_{x_k^*}^{x_k}\left(s^{\sigma/r_k} - x_k^{*\sigma/r_k}\right)^{((4l-1)\sigma-\tau_k-r_k)/\sigma} ds\right) \\
&\qquad \times \left(h_i x_{i+1}^{p_i} + f_i\right) \\
&\le \sum_{i=1}^{k-1} b_k |\xi_k|^{((4l-1)\sigma-\tau_k)/\sigma}\left|\frac{\partial x_k^{*\sigma/r_k}}{\partial x_j}\right|\left(c_{i2}\left|x_{i+1}^{p_i}\right| + |f_i|\right) \\
&\le \overline{b}_k \sum_{i=1}^{k-1} |\xi_k|^{((4l-1)\sigma-\tau_k)/\sigma}|x_i|^{(\sigma-r_i)/r_i}\left(c_{i2}\left|x_{i+1}^{p_i}\right| + |f_i|\right) \\
&\le \overline{b}_k \sum_{i=1}^{k-1} |\xi_k|^{((4l-1)\sigma-\tau_k)/\sigma}\left(|\xi_i + \beta_{i-1}\xi_{i-1}|^{r_i/\sigma}\right)^{(\sigma-r_i)/r_i} \\
&\qquad \times \left[c_{i2}\left(|\xi_{i+1} + \beta_i\xi_i|^{r_{i+1}/\sigma}\right)^{p_i}\right. \\
&\qquad \left. + \widetilde{a}\sum_{j=1}^{i}\left(|\xi_j|^{(r_i+\tau_i)/\sigma} + |\xi_j(t-d(t))|^{(r_i+\tau_i)/\sigma}\right)\right] \\
&\le \widetilde{b}_k \sum_{i=1}^{k-1} |\xi_k|^{((4l-1)\sigma-\tau_k)/\sigma}\left(|\xi_{i-1}| + |\xi_i|\right)^{(\sigma-r_i)/\sigma} \\
&\qquad \times \left[\left(|\xi_i| + |\xi_{i+1}|\right)^{p_i r_{i+1}/\sigma}\right. \\
&\qquad \left. + \sum_{j=1}^{i}\left(|\xi_j|^{(r_i+\tau_i)/\sigma} + |\xi_j(t-d(t))|^{(r_i+\tau_i)/\sigma}\right)\right] \\
&\le \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^{4l} + \frac{1}{8}\sum_{i=1}^{k}\xi_i^{4l}(t-d(t)) + l_{k3}\xi_k^{4l},
\end{aligned} \tag{A.11}$$

where b_k, $\overline{b}_k$, and $\widetilde{b}_k$ are nonnegative smooth functions.

It is noted that r_i satisfies $p_i r_{i+1} = r_i + \tau_i$ and $\tau_i \ge \tau_k$ and by Lemma 5, we have

$$\begin{aligned}
&\sum_{i=1}^{k-1} \frac{\partial W_k}{\partial x_i}\left(h_i x_{i+1}^{p_i} + f_i\right) \\
&\qquad \le \frac{1}{8}\sum_{i=1}^{k-1}\xi_i^{4l} + \frac{1}{8}\sum_{i=1}^{k}\xi_i^{4l}(t-d(t)) + l_{k3}\xi_k^{4l},
\end{aligned} \tag{A.12}$$

where l_{k3} is a nonnegative smooth function. □

Proof of Proposition 14. From (17), Assumption 7, and Lemma 3, it follows that

$$\begin{aligned}
|g_k| &\le a_{k2}\sum_{i=1}^{k}\left(|x_i|^{(2r_k+\tau_k)/2r_i} + |x_i(t-d(t))|^{(2r_k+\tau_k)/2r_i}\right) \\
&= a_{k2}\sum_{i=1}^{k}\left(|\xi_i + \beta_{i-1}\xi_{i-1}|^{r_i/\sigma}\right)^{(2r_k+\tau_k)/2r_i} \\
&\quad + a_{k2}\sum_{i=1}^{k}\left(|\xi_i(t-d(t)) \right. \\
&\qquad \left. + \beta_{i-1}\xi_{i-1}(t-d(t))|^{r_i/\sigma}\right)^{(2r_k+\tau_k)/2r_i} \\
&\le a_{k2}\sum_{i=1}^{k}\left(1 + \beta_{i-1}^{(2r_k+\tau_k)/2\sigma}\right) \\
&\qquad \times \left(|\xi_i|^{(2r_k+\tau_k)/2\sigma} + |\xi_i(t-d(t))|^{(2r_k+\tau_k)/2\sigma}\right) \\
&\le \widehat{a}_{k2}\sum_{i=1}^{k}\left(|\xi_i|^{(2r_k+\tau_k)/2\sigma} + |\xi_i(t-d(t))|^{(2r_k+\tau_k)/2\sigma}\right),
\end{aligned} \tag{A.13}$$

where $\beta_0 = \xi_0 = 0$ and $\widehat{a}_{k2}$ is a nonnegative smooth function.

According to (17) and (21), we have

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j=1,i\ne j}^{k-1}\left|\frac{\partial^2 W_k}{\partial x_i \partial x_j}\right|\left|g_i^T\right|\left|g_j^T\right| \\
&= \frac{1}{2}\sum_{i,j=1,i\ne j}^{k-1}\left|\frac{4l\sigma - \tau_k - r_k}{\sigma}\cdot\frac{(4l-1)\sigma - \tau_k - r_k}{\sigma}\right. \\
&\qquad \times \frac{\partial x_k^{*\sigma/m_i}}{\partial x_i}\frac{\partial x_k^{*\sigma/m_k}}{\partial x_j} \\
&\qquad \left. \times \int_{x_k^*}^{x_k}\left(s^{\sigma/r_k} - x_k^{*\sigma/r_k}\right)^{((4l-2)\sigma-\tau_k-r_k)/\sigma} ds\right| \\
&\qquad \times \left|g_i^T\right|\left|g_j^T\right|
\end{aligned}$$

$$
\begin{aligned}
&\le d_k \sum_{i,j=1,i\neq j}^{k-1} \left|\xi_k\right|^{((4l-2)\sigma-\tau_k)/\sigma} \left|x_i\right|^{(\sigma-r_i)/r_i} \\
&\quad \times \left|x_j\right|^{(\sigma-r_j)/r_j} \left|g_i^T\right| \left|g_j^T\right| \\
&\le \widehat{d}_k \sum_{i,j=1,i\neq j}^{k-1} \sum_{m=1}^{i} \sum_{p=1}^{j} \left|\xi_k\right|^{((4l-2)\sigma-\tau_k)/\sigma} \left|x_i\right|^{(\sigma-r_i)/r_i} \times \left|x_j\right|^{(\sigma-r_j)/r_j} \\
&\qquad \times \left(\left|x_m\right|^{(2r_i+\tau_i)/2r_m} + \left|x_m(t-d(t))\right|^{(2r_i+\tau_i)/2r_m}\right) \\
&\qquad \times \left(\left|x_p\right|^{(2r_j+\tau_j)/2r_p} + \left|x_p(t-d(t))\right|^{(2r_j+\tau_j)/2r_p}\right) \\
&\le \widetilde{d}_k \sum_{i,j=1,i\neq j}^{k-1} \sum_{m=1}^{i} \sum_{p=1}^{j} \left|\xi_k\right|^{((4l-2)\sigma-\tau_k)/\sigma} \left(\left|\xi_{i-1}\right| + \left|\xi_i\right|\right)^{(\sigma-r_i)/\sigma} \\
&\qquad \times \left(\left|\xi_{j-1}\right| + \left|\xi_j\right|\right)^{(\sigma-r_j)/\sigma} \\
&\qquad \times \left(\left|\xi_m\right|^{(2r_i+\tau_i)/2\sigma} + \left|\xi_m(t-d(t))\right|^{(2r_i+\tau_i)/2\sigma}\right) \\
&\qquad \times \left(\left|\xi_p\right|^{(2r_j+\tau_j)/2\sigma} + \left|\xi_p(t-d(t))\right|^{(2r_j+\tau_j)/2\sigma}\right),
\end{aligned}
\tag{A.14}
$$

where d_k, $\widehat{d}_k$, and $\widetilde{d}_k$ are nonnegative smooth functions.

Noting that $\tau_i \ge \tau_k$ and $\tau_j \ge \tau_k$, by applying Lemma 5 to the above inequality, we have

$$
\begin{aligned}
&\frac{1}{2} \sum_{i,j=1,i\neq j}^{k-1} \left|\frac{\partial^2 W_k}{\partial x_i \partial x_j}\right| \left|g_i^T\right| \left|g_j^T\right| \\
&\quad \le \frac{1}{8} \sum_{i=1}^{k-1} \xi_i^{4l} + \frac{1}{8} \sum_{i=1}^{k} \xi_i^{4l}(t-d(t)) + l_{k4}\xi_k^{4l},
\end{aligned}
\tag{A.15}
$$

where l_{k4} is a nonnegative smooth function.

In a similar way, Propositions 15–17 can be proved and they are omitted here. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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